

SELF-ADJOINTNESS AND SYMMETRICITY OF OPERATOR MEANS

HIROYUKI OSAKA*

1. INTRODUCTION

We recall that a n -monotone function on $[0, \infty)$ is a function which preserves the order on the set of all $n \times n$ positive semi-definite matrices. Moreover, if f is n -monotone for all $n \in \mathbb{N}$, then f is called operator monotone.

In the theory of operator connections by Kubo and Ando it is well-known that there is an affine order isomorphism from the class of operator connections σ onto the class of nonnegative operator monotone functions f on $(0, \infty)$ by $f(t) = I\sigma t$. A connection σ is called mean if it satisfies the normalization condition $I\sigma I = I$, which is equivalent to that the representing function f of σ satisfies $f(1) = 1$. This theory has found a number of applications in operator theory and quantum information theory. Restricting the definition of operator connections on the set of positive semi-definite matrices of order n , we can consider matrix connections of positive matrices of order n (or matrix connections of order n).

Definition 1.1. A binary operation σ on M_n^+ , $(A, B) \mapsto A\sigma B$ is called a *matrix connection of order n* (or *n -connection*) if it satisfies the following properties:

- (I) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.
- (II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
- (III) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \dots$ and A_n converges strongly to A .

A *mean* is a normalized connection, i.e. $1\sigma 1 = 1$. An *operator connection* means a connection of every order. A *n -semi-connection* is a binary operation on M_n^+ satisfying the conditions (II) and (III).

Recall that a n -monotone function f is symmetric if $f(t) = tf\left(\frac{1}{t}\right)$ and f is self-adjoint if $f(t) = \frac{1}{f\left(\frac{1}{t}\right)}$.

A function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called an *interpolation function of order n* ([1]) if for any $T, A \in M_n$ with $A > 0$ and $T^*T \leq 1$

$$T^*AT \leq A \implies T^*f(A)T \leq f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbb{R}_+$.

Date: 15 Jan., 2015.

2000 *Mathematics Subject Classification.* Primary 46L30; Secondary 15A45.

Remark 1.2. Let $P(\mathbb{R}_+)$ be a set of all Pick functions on $\mathbb{R}_+$, P' the set of all positive Pick functions on $\mathbb{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{1+ts} d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0, \infty]$. For $n \in \mathbb{N}$ denote by P'_n the set of all strictly positive n -monotone functions. The following properties can be found in [1], [2],[3], [12], [17] or [4], :

- (i) $P' = \cap_{n=1}^{\infty} P'_n$, $P' = \cap_{n=1}^{\infty} C_n$;
- (ii) $C_{n+1} \subseteq C_n$;
- (iii) $P'_{n+1} \subseteq C_{2n+1} \subseteq C_{2n} \subseteq P'_n$, $P'_n \subsetneq C_n$
- (iv) $C_{2n} \subsetneq P'_n$ [20];
- (v) A function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to C_n if and only if $\frac{t}{f(t)}$ belongs to C_n [4, Proposition 3.5].

The following useful characterization of a function in C_n is due to Donoghue (see [10], [9]), and to Ameer (see [1]).

Theorem 1.3. [4, Corollary 2.4] A function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to C_n if and only if for every n -set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a positive Pick function h on $\mathbb{R}$, such that

$$f(\lambda_i) = h(\lambda_i) \quad \text{for } i = 1, \dots, n.$$

As a consequence, Ameer gave a 'local' integral representation of every function in C_n as follows.

Theorem 1.4. [2, Theorem 7.1] Let A be a positive definite matrix in M_n and $f \in C_n$. Then there exists a positive Radon measure $\rho_{\sigma(A)}$ on $[0, \infty]$ such that

$$(1) \quad f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho_{\sigma(A)}(s),$$

where $\sigma(A)$ is the set of eigenvalues of A .

Applying this representation, we give a 'local' integral formula for a connection of order n corresponding to a n -monotone function on $(0, \infty)$ Furthermore, this 'local' formula also establishes, for each interpolation function f of order $2n$, a connection σ of order n corresponding to the given interpolation function f . Therefore, it shows that the map from the n -connections to the interpolation functions of order n is injective with the range containing the interpolation functions of order $2n$.

In this note we present two topics as follows:

- (1) For each $n \in \mathbb{N}$ there is an affine isomorphism from the set of matrix symmetric connections of order n onto the class of matrix symmetric n -monotone functions, which is based on [D. T. Hoa, T. M. Ho, H. Osaka, Interpolation classes and matrix means, Banach Journal of Mathematical Analysis, 9(2015), no. 3, 140-152].
- (2) We characterize a class of non-selfadjoint operator means and a class of non-symmetric operator means between the harmonic mean ! and the arithmetic mean ∇ which is based on the joint work with Shuhei Wada.

2. FROM n -CONNECTIONS TO P'_n

For any n -connection σ , the matrix $I_n\sigma(tI_n)$ is a scalar by [13, Theorem 3.2], and so we can define a function f on $(0, \infty)$ by

$$f(t)I_n = I_n\sigma(tI_n),$$

where I_n is the identity in M_n . Then $f \in P'_n \subsetneq \mathcal{C}_n$. Moreover, this correspondence is injective.

Let f be a function belonging to $\mathcal{C}_n$. We can define a binary operation σ on positive definite matrices in M_n by:

$$(2) \quad A\sigma B = A^{\frac{1}{2}}f[A^{-\frac{1}{2}}BA^{-\frac{1}{2}}]A^{\frac{1}{2}}, \quad \forall A, B > 0.$$

This operation satisfies the property (III) of the definition of connection.

Lemma 2.1. Let f be a positive function on $(0, \infty)$ belonging to $\mathcal{C}_n$. Then there is a semi-connection of order n , σ , such that $f(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. (i.e., a binary operation σ satisfying the axiom (II) and (III) in Definition 1.1).

Proof. We can define a binary σ by the formula (2). Because of the continuity of f (see Remark 2.2 below), we imply that σ has the property (III) in the definition. By Theorem 1.4, there exists a Radon measure ρ such that

$$A\sigma B = \int_{[0, \infty]} \frac{1+s}{s} \{(sA) : B\} d\rho(s)$$

For any positive definite matrix C of order n ,

$$\begin{aligned} C(A\sigma B)C &= \int_{[0, \infty]} \frac{1+s}{s} C\{(sA) : B\} C d\rho(s) \\ &= \int_{[0, \infty]} \frac{1+s}{s} \{(sCAC) : CBC\} d\rho(s) \\ &= (CAC)\sigma(CBC). \end{aligned}$$

■

In the proof above, we need the continuity of $f \in \mathcal{C}_n$. Actually, we follow the definition of interpolation function in [4] and the continuity is the prior assumption for any function. However, even if we did not assume the continuity of the functions under consideration, we have

Remark 2.2. If $f \in \mathcal{C}_n(I)$ for $n > 2$ then f is continuous on I .

Now we can state the main theorem of this section.

Theorem 2.3. For any natural number n there is an injective map Σ from the set of matrix connections of order n to $P'_n \supset \mathcal{C}_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains $\mathcal{C}_{2n}$.

Proof. We have only to prove that the range of the map Σ contains $\mathcal{C}_{2n}$. For any $f \in \mathcal{C}_{2n}$, since $\mathcal{C}_{2n} \subset \mathcal{C}_n$, by Lemma 2.1 there is a semi-connection σ_f defined by the formula (2) and $f(t)I_n = I_n\sigma_f(tI_n)$ on $(0, \infty)$. Since $f \in \mathcal{C}_{2n}$, by Theorem 1.4

we have that for any $0 < A \leq C$ and $0 < B \leq D$ there exists a Radon measure ρ on $\sigma(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) \cup \sigma(C^{\frac{-1}{2}}DC^{\frac{-1}{2}})$ such that

$$\begin{aligned} A\sigma_f B &= \int_{[0,\infty]} \frac{1+s}{s} \{(sA) : B\} d\rho(s), \\ C\sigma_f D &= \int_{[0,\infty]} \frac{1+s}{s} \{(sC) : D\} d\rho(s). \end{aligned}$$

Since $\{(sA) : B\} \leq \{(sC) : D\}$, the condition (I) satisfies. Hence σ_f is a connection of order n . Since $\Sigma(\sigma_f)(t)I_n = I_n\sigma_f(tI_n) = f(t)I_n$ for any $t \in \mathbb{R}^+$, we are done. $\blacksquare$

3. SYMMETRIC CONNECTIONS

As the same in [13], we can recall some notations and properties of connections as follows. Let σ be a n -connection. The *transpose* σ' , the *adjoint* σ^* and the *dual* $\sigma^\perp$ of σ are defined by

$$A\sigma' B = B\sigma A, \quad A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}, \quad \sigma^\perp = \sigma'^*.$$

A connection is called *symmetric* if it equals to its transpose. Denoted by Σ_n^{sym} the set of n -monotone representing functions of symmetric n -connections, i.e., Σ_n^{sym} is the image of the set of all symmetric n -connections via the canonical map in Theorem 2.3. Then, using the same argument as in [13], we can state the following properties for any n -connection:

- (1) $\sigma + \sigma'$ and $\sigma(\cdot)\sigma'$ are symmetric.
- (2) $\omega_l(\sigma)\omega_r = \sigma$; $\omega_r(\sigma)\omega_l = \sigma'$, where $A\omega_l B = A$ and $A\omega_r B = B$.
- (3) The n -monotone representing function of the n -connection $\sigma(\tau)\rho$ is $f(x)g[h(x)/f(x)]$, where f, g, h are the representing functions of σ, τ, ρ in Theorem 2.3, respectively.
- (4) σ is symmetric if and only if its n -monotone representing function f is *symmetric*, that is, $f(x) = xf(x^{-1})$.

Each n -connection corresponds to a positive n -monotone function belonging to Σ_n by Theorem 2.3. Therefore, combining with the observation above, we get the following.

Proposition 3.1. Let $f(x), g(x), h(x)$ belong to Σ_n . Then the following statements hold true:

- (i) $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $f(x)g[h(x)/f(x)]$, $af(x) + bg(x)$ all belong to Σ_n ;
- (ii) $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ all belong to Σ_n^{sym} .

Corollary 3.2.

$$\mathcal{C}_{2n} \subseteq \Sigma_n \subsetneq P'_n.$$

But if we restrict our attention to the class of the symmetric, we get the following equality.

Theorem 3.3.

$$\Sigma_n^{sym} = P_n'^{sym},$$

where $P_n'^{sym}$ is the set of all symmetric functions in P'_n .

Proof. The inclusion $\Sigma_n^{sym} \subset P_n'^{sym}$ is trivial by Theorem 2.3.

Let f be a symmetric function in P_n' . We can define a binary operation on positive definite matrices of order n by

$$A\sigma B = A^{\frac{1}{2}}f[A^{-\frac{1}{2}}BA^{-\frac{1}{2}}]A^{\frac{1}{2}}.$$

For any $B \leq D$, then $A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq A^{-\frac{1}{2}}DA^{-\frac{1}{2}}$. Since f is n -monotone and the conjugate action preserves the order on self-adjoint matrices, we obtain

$$A^{\frac{1}{2}}f[A^{-\frac{1}{2}}BA^{-\frac{1}{2}}]A^{\frac{1}{2}} \leq A^{\frac{1}{2}}f[A^{-\frac{1}{2}}DA^{-\frac{1}{2}}]A^{\frac{1}{2}}.$$

This means $A\sigma B \leq A\sigma D$. Since f is symmetric, we also have

$$A\sigma D = D^{\frac{1}{2}}f[D^{-\frac{1}{2}}AD^{-\frac{1}{2}}]D^{\frac{1}{2}}.$$

Using this identity, we can also show that $A\sigma D \leq C\sigma D$ whenever $A \leq C$. Thus, $A\sigma B \leq A\sigma D \leq C\sigma D$ for any positive matrices A, B, C, D with $A \leq C$ and $B \leq D$. ■

Remark 3.4. We would like to mention that even $P_{n+1}' \subsetneq P_n'$, but we still do not know whether $P_{n+1}'^{sym} \subsetneq P_n'^{sym}$ holds or not. As the first thought, we can obtain a symmetric function from the polynomial in P_{n+1}' but not in P_n' and such a function is a candidate to show $P_{n+1}'^{sym} \subsetneq P_n'^{sym}$. Unfortunately, this is not true as the following example.

4. NON-SYMMETRIC OPERATOR MEANS

In [13] any symmetric operator mean σ satisfies $! \leq \sigma \leq \nabla$. In this section we show that there are many non-symmetric operator means σ such that $! \leq \sigma \leq \nabla$.

4.1. Barbour transform. In [14] for any strictly positive continuous functions on $(0, \infty)$ the Barbour path function $\phi_{\alpha, \beta, \gamma} : [0, 1] \rightarrow OM_+^1$ introduced by

$$\phi_{\alpha, \beta, \gamma}(x) = \frac{\alpha x + \beta(1-x)}{x + \gamma(1-x)}$$

and the basic properties are studied in [14], [18]. In [7] Barbour studied a function $F_x(1, t) = \phi_{t, \sqrt{t}, \sqrt{t}}(x)$ which is an approximation of the exponential function t^x . We will denote a Barbour path $\phi_{\alpha, \beta, \gamma}(= \phi)$ such that $\phi(0) = f$, $\phi(\frac{1}{2}) = g$, $\phi(1) = h$ by the triple $[f, g, h]$.

Proposition 4.1. ([14]) For $f \in OM_+$ the Barbour path $[1, \frac{t+f}{1+f}, t]$ exists on OM_+^1 .

The transform $\hat{\cdot} : OM_+ \rightarrow OM_+^1$ by $f \mapsto \frac{t+f}{1+f}$ plays an important role in the analysis of OM_+ and we call this transform the Barbour transform.

Proposition 4.2. ([14])

- (1) The Barbour transform is injective and $\widehat{OM_+} = OM_+^1 \setminus \{1, t\}$.
- (2) $\{f \in OM_+^1 \mid ! \leq f \leq \nabla\} = \widehat{OM_+^1}$, where $! \leq f$ means that $! \leq \sigma_f$, that is, for any positive operators A and B $A!B \leq A\sigma_f B$.

For $g \in OM_+^1$ we can define the inverse map $\check{\cdot}$ of the Barbour transform by

$$\check{g}(t) = \frac{t - g}{g - 1},$$

then $\check{g} \in OM_+$.

Using the Barbour transform we can characterize the self-adjointness and the symmetricity in OM_+ .

Theorem 4.3. Let f be a positive continuous function on $(0, \infty)$. The followings are equivalent.

- (1) $f \in OM_+^1 \setminus \{1, t\}$ and $f = f^*$.
- (2) There exists an operator monotone function $g \in OM_+$ such that $f = \sqrt{gg^*}$.
- (3) There exists an operator monotone function $g \in OM_+$ such that

$$f = \frac{t + g + g'}{1 + g + g'}.$$

Remark 4.4. In [13] they asked existence of self-adjoint operator means except trivial means ω_l , ω_r , the geometric mean $\sharp$, and σ_{tp} ($p \in [0, 1]$), where $A\omega_l B = A$, $A\omega_r B = B$, $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ for any positive operators A and B . Using Theorem 4.3 we can construct many examples. For example, if $g(t) = \log(t+1)$, then corresponding operator means of functions $\sqrt{\log(t+1)/\log(t^{-1}+1)}$ and $\frac{t + \log(t+1) + t\log(t^{-1}+1)}{1 + \log(t+1) + t\log(t^{-1}+1)}$ are self-adjoint. ■

Theorem 4.5. Let f be a positive continuous function on $(0, \infty)$. The followings are equivalent.

- (1) $f \in OM_+^1 \setminus \{1, t\}$ and $f = f'$.
- (2) There exists an operator function $g \in OM_+$ such that

$$f = g + g'.$$

- (3) There exists an operator monotone functions $g \in OM_+$ such that

$$f = \frac{t - \sqrt{gg^*}}{\sqrt{gg^*} - 1}.$$

Proposition 4.6. Let f be a positive continuous function on $(0, \infty)$. The followings are equivalent.

- (1) $f \in OM_+^1 \setminus \{1, t\}$ and $f = f'$.
- (2) There exists an operator monotone function $g \in OM_+$ such that

$$f = \frac{t + \sqrt{gg^*}}{1 + \sqrt{gg^*}}$$

Proof. This follows from the same argument in Theorem 4.3 using the formula $(\hat{h})' = \hat{h}^*$ for $h \in OM_+$. ■

5. NON SELF-ADJOINT OPERATOR MEANS

In [13] any symmetric operator mean σ satisfies $! \leq \sigma \leq \nabla$. In this section we consider the converse problem and show that there are many non self-adjoint operator means σ such that $! \leq \sigma \leq \nabla$.

Lemma 5.1. Let $f: (0, \infty) \rightarrow (0, \infty)$ be a continuous function. The followings are equivalent.

- (1) $f \in OM_+$ and $f \geq f_\nabla$, that is $f(t) \geq \frac{1+t}{2}$ for $t \in (0, \infty)$.
- (2) There exists an operator monotone $g \in OM_+$ and nonnegative real number $a, b \geq \frac{1}{2}$ such that $\lim_{t \rightarrow 0} g(t) = 0$, $\lim_{n \rightarrow \infty} \frac{g(t)}{t} = 0$, and

$$f(t) = a + bt + g(t) \quad (t \in (0, \infty)).$$

Lemma 5.2. Let $f: (0, \infty) \rightarrow (0, \infty)$ be a continuous function. The followings are equivalent.

- (1) $f \in OM_+$ and $f \leq f_!$, that is, $f(t) \leq \frac{2t}{1+t}$ ($t \in (0, \infty)$).
- (2) There exists an operator monotone $g \in OM_+$ and nonnegative real number $a, b \geq \frac{1}{2}$ such that $\lim_{t \rightarrow 0} g(t) = 0$, $\lim_{n \rightarrow \infty} \frac{g(t)}{t} = 0$, and

$$f(t) = \frac{t}{a + bt + g(t)} \quad (t \in (0, \infty)).$$

Corollary 5.3. If $f \in OM_+^1$ and $f \leq f_!$, then $f = f_!$.

Corollary 5.4. If $f \in OM_+^1$ and $f \geq f_\nabla$, then $f = f_\nabla$.

Proposition 5.5. Suppose that $f \in OM_+$ and $f < f_!$. Then $f_! \leq \hat{f} \leq f_\nabla$ and $\hat{f}$ is not self-adjoint.

Corollary 5.6. Let a, b be nonnegative real number greater than $\frac{1}{2}$ and $g \in OM_+$ satisfying the condition (2) in Lemma 5.2. Define a function $f: (0, \infty) \rightarrow (0, \infty)$ by $f(t) = \frac{t}{a+bt+g(t)}$ ($t \in (0, \infty)$). Then $f \in OM_+$, $f_! \leq \hat{f} \leq f_\nabla$, and $\hat{f}$ is not self-adjoint.

Lemma 5.7. If a symmetric operator mean is self-adjoint, then $\sigma = \sharp$.

Proof. Let f be a corresponding operator monotone function to σ . Then

$$f(t) = tf\left(\frac{1}{t}\right) = \frac{1}{f\left(\frac{1}{t}\right)}.$$

Hence, $f(t) = \sqrt{t}$, and $\sigma = \sharp$. ■

Remark 5.8. From Lemma 5.7 we know that all operator means of Arithmetic mean, logarithmic mean, Harmonic mean, Heinz mean, Petz-Hasegawa mean, Lehmer mean, and Power difference mean, are non-self-adjoint.

5.1. Non-symmetric operator means. In this section we present an algorithm for making non-symmetric means σ such that $! \leq \sigma \leq \nabla$.

Lemma 5.9. Let f be a positive operator monotone function on $(0, \infty)$ with $f(1) = 1$. The followings are equivalent:

- (1) $\sigma_{\hat{f}}$ is non-symmetric and $! \leq \sigma_{\hat{f}} \leq \nabla$,
- (2) f is non-self-adjoint.

Proof. (2) $\rightarrow$ (1): Since $(\hat{f})' = \hat{f}^*$, if f is non-self-adjoint operator monotone, $\hat{f}$ is non-symmetric, that is, $\sigma_{\hat{f}}$ is non-symmetric. We have, then, $! \leq \sigma_{\hat{f}} \leq \nabla$ by Proposition 4.2 (2).

(1) $\rightarrow$ (2): If f is self-adjoint, then $\hat{f}$ is symmetric, and a contradiction. ■

Hence we have the following result.

Proposition 5.10.

$$\begin{aligned} & \{f \mid f : \text{non-symmetric}, f_! \leq f \leq f_{\nabla}\} \\ &= \{\hat{f} \mid f : \text{non-self-adjoint}\} \\ &= \{\hat{\hat{f}} \mid f : \text{non-symmetric}\} \\ &\supset \{\hat{f} \mid f : \text{symmetric}\} \setminus \{\# \} \end{aligned}$$

Remark 5.11. From Proposition 5.10 a non-self-adjoint positive monotone functions f with $f(1) = 1$ give non-symmetric operator mean such that $! \leq \sigma_f \leq \nabla$. For examples, let $-1 \leq p \leq 2$ and ALG_p be the corresponding function to the power difference mean defined by

$$ALG_p(t) = \begin{cases} \frac{p-1}{p} \frac{1-t^p}{1-t^{p-1}} & t \neq 1 \\ 1 & t = 1 \end{cases}$$

and the Petz-Hasegawa function f_p which is defined by

$$f_p(t) = p(p-1) \frac{(t-1)^2}{(t^p-1)(t^{1-p}-1)}$$

are non-self-adjoint. Hence, $\sigma_{\widehat{ALG_p}}$ and $\sigma_{\widehat{f_p}}$ are non-symmetric operator means between $!$ and ∇ .

Using Lemmas 5.1 and 5.2 we can give non-symmetric operator means between $!$ and ∇ .

The following should be well-known.

Corollary 5.12. Let $f \in OM_+$ such that $\sigma_f \geq \nabla$ and let $g \in OM_+$ such that $f(t) = a + bt + g(t)$ in Lemma 5.1 ($a, b \geq \frac{1}{2}$). Suppose that g is symmetric and $a \neq b$. Then $\hat{\hat{f}}$ is not symmetric and $! \leq \sigma_{\hat{\hat{f}}} \leq \nabla$.

Proof. Since g is symmetric,

$$\begin{aligned} tf\left(\frac{1}{t}\right) &= t\left(a + b\frac{1}{t} + g\left(\frac{1}{t}\right)\right) \\ &= ta + b + tg\left(\frac{1}{t}\right) \\ &= ta + b + g(t). \end{aligned}$$

Hence we know that f is not symmetric because that $a \neq b$.

Therefore, by Proposition 5.10 $\hat{f}$ is not symmetric and $! \leq \sigma_{\hat{f}} \leq \nabla$. ■

REFERENCES

- [1] Y. Ameer, *Interpolation of Hilbert spaces*, Uppsala Dissertations in Mathematics **20** (2001).
- [2] Y. Ameer, *The Calderon problem for Hilbert couples*, Ark. Math. **41** (2003) 203-231.
- [3] Y. Ameer, *A new proof of Donoghue's interpolation theorem*, J. Funct. Spaces Appl. **2** (2004), no. 3, 253-265.
- [4] Y. Ameer, S. Kaijser, S. Silvestrov, *Interpolation class and matrix monotone functions*, J. Operator Theory. **52** (2007) 409-427.
- [5] W. N. Jr. Anderson, R. J. Duffin, *Series and parallel addition of matrices*, J. Math. Anal. Appl. **26** (1969) 576-594.
- [6] W. N. Jr. Anderson, G. E. Trapp, *Shorted Operators II*, Siam J. Appl. Math. **28** (1975) 60-71.
- [7] J. M. Barbour, *A geometrical approximation to the roots of numbers*, Amer. Math. Monthly **64** (1957), 1-9.
- [8] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York (1986).
- [9] W. F. Donoghue, *The theorems of Loewner and Pick*, Israel J. Math. **4** (1966) 153-170.
- [10] W. F. Donoghue, *The interpolation of quadratic norms*, Acta Math. **118** (1967) 251-270.
- [11] W. F. Donoghue, *Monotone matrix function and analytic continuation*, Springer 1974.
- [12] C. Foias, J. L. Lions, *Sur certains theoremes d'interpolation*, Acta Sci. Math (Szeged). **22** (1961) 269-282.
- [13] F. Kubo, T. Ando, *Means of positive linear operators*, Math. Ann. **246** (1980) 205-224.
- [14] F. Kubo, N. Nakamura, K. Ohno, and S. Wada, *Barbour path of operator monotone functions*, Far East J. Math. Sci. (FJMS) **57**(2) (2011), 181-192.
- [15] F. Hansen, G. Ji, J. Tomiyama, *Gaps between classes of matrix monotone functions*, Bull. London Math. Soc. **36** (2004) 53-58.
- [16] D. T. Hoa, T. M. Ho, H. Osaka, *Interpolation classes and matrix means*, Banach Journal of Mathematical Analysis, 9(2015), no. 3, 140-152
- [17] K. Löwner, *Über monotone matrixfunktionen*, Math. Z. **38** (1934) 177-216.
- [18] N. Nakamura, *Barbour path functions and related operator means*, Linear Alg. Appl. **439** (2013), 2434-2441.
- [19] C. Niculescu and L.-E. Persson, *Convex functions and their applications*, CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC, 23. Springer, New York, 2006.
- [20] H. Osaka, J. Tomiyama, *Note on the structure of matrix monotone functions*, Analysis for Sciences, Engineering and Beyond, The tribute workshop in honor of Gunnar Sparr held in Lund, May 8-9, Spring Proceedings in Mathematics, **6** (2008) 319-324.
- [21] H. Osaka, Y. Tsurumi, S. Wada, preprint.
- [22] D. Petz, *Monotone metric on matrix spaces*, Linear Algebra Appl. **244** (1996) 81-96.

DEPARTMENT OF MATHEMATICAL SCIENCES, RITSUMEIKAN UNIVERSITY, KUSATSU, SHIGA, 525-8577 JAPAN

E-mail address: osaka@se.ritsumei.ac.jp