

Facial structure of convex sets and some applications

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§1 INTRODUCTION

Let Ω be a measure space and let $S(\Omega)$ be the space of all measurable functions f on Ω such that $f(t) < \infty$ (a.e. $t \in \Omega$). An operator $F : X \supset D(F) \rightarrow S(\Omega)$ is called a convex operator if $D(F)$ is a convex set in a real vector space X , and for each $x, y \in D(F)$ and $0 < \alpha < 1$,

$$F((1 - \alpha)x + \alpha y)(t) \leq (1 - \alpha)F(x)(t) + \alpha F(y)(t) \quad (\text{a.e. } t \in \Omega).$$

On the other hand, a function $f : X \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ is called a convex integrand if for each $t \in \Omega$ the function $f(\cdot, t)$ is convex on $\mathbb{R}^d$. The convex integrand theory is well known and there are many applications. (See [7] for example.) We say that a convex integrand f represents a convex operator F if

$$(1) \quad f(x, t) = \begin{cases} F(x)(t) & \text{for a.e. } t \in \Omega \quad x \in D(F) \\ \infty & x \notin D(F) \end{cases}$$

In two of the author's previous paper [3, 4], many applications of integrand representations of convex operators were demonstrated. However, the existence of integrand representation is nontrivial, and it is known only in some special cases. When X is the d -dimensional Euclidean space $\mathbb{R}^d$, the representation theorem has been proved in [3]. In this note, we apply the theory of the faces of convex sets, and give a new method of the proof which is expected to have an advantage in extending the representation theorem to infinite dimensional cases.

§2 FACES OF CONVEX SETS

Let $\mathbb{R}^d$ be the d -dimensional Euclidean space. When $x, y \in \mathbb{R}^d$ are distinct points, then the set $[x, y] = \{(1 - t)x + ty \mid 0 \leq t \leq 1\}$ is called the closed segment between x and y . Half open segments $(x, y]$, $[x, y)$ and open segments (x, y) are defined analogously. Through this section, we fix a nonempty closed convex set D in $\mathbb{R}^d$. A convex subset C of D is called a face of D if

$$(2) \quad \left\{ \begin{array}{l} x, y \in D \\ (x, y) \cap C \neq \emptyset \end{array} \right\} \text{ implies } [x, y] \subset C.$$

By $\mathfrak{F}(D)$, we denote the set of all faces of D . For $C \in \mathfrak{F}(D)$, $\dim C$ is defined to be the dimension of $\text{aff } C$ (the affine hull of C). It is clear that $x \in D$ is an extreme point of D if and only if $\{x\}$ is a 0-dimensional face of D . For preparation, we will state some fundamental properties of faces in the following propositions whose proofs are given in [1].

Proposition 1. *If $C_\lambda \in \mathfrak{F}(D)$, ($\lambda \in \Lambda$), then $\cap_{\lambda \in \Lambda} C_\lambda \in \mathfrak{F}(D)$, and also there exists a smallest face of D containing $\cup_{\lambda \in \Lambda} C_\lambda$. Hence $(\mathfrak{F}(D), \subset)$ forms a complete lattice.*

Proposition 2. *Let C_1 be a face of D and suppose that $C_2 \subset C_1$. Then $C_2 \in \mathfrak{F}(D)$ if and only if $C_2 \in \mathfrak{F}(C_1)$.*

For a convex set C in $\mathbb{R}^d$, $\overset{\circ}{C}$ denotes the relative interior of C , which means the interior of C with respect to the relative topology of $\text{aff } C$. It is easy to see that every face of D is a closed set. Indeed, if x is a point of the closure of a face C and $x_0 \in \overset{\circ}{C}$, the convexity of C yields $[x_0, x) \subset \overset{\circ}{C} \subset C$. Since C is a face of D , x must be in C .

Proposition 3. *If $C_1, C_2 \in \mathfrak{F}(D)$, and $C_1 \subsetneq C_2$, then $C_1 \cap \overset{\circ}{C}_2 = \emptyset$.*

Proposition 4. *Let x be a point of D and let C be a face of D . Then C is the smallest face of D containing x if and only if $x \in \overset{\circ}{C}$.*

Proposition 5. *Let C_1 be a face of D and let x be a relative boundary point of C_1 . If C_2 is the smallest face of D containing x , then C_2 is contained by the relative boundary of C_1 .*

From these propositions we obtain the following decomposition of a convex set by its faces.

Proposition 6. *For a closed convex set D in $\mathbb{R}^d$,*

$$D = \cup \{ \overset{\circ}{C}_\lambda \mid C_\lambda \in \mathfrak{F}(D) \}$$

and the union is disjoint.

We say that a collection $\{C_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{F}(D)$ is normal if $\lambda \in \Lambda$ and $C_\lambda \subset C_\mu \in \mathfrak{F}(D)$ imply $\mu \in \Lambda$. Now we define

$$\mathfrak{A} = \{A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda \mid \{C_\lambda\}_{\lambda \in \Lambda} \text{ is normal}\}.$$

Since $\{\overset{\circ}{D}\}$ is normal and $\overset{\circ}{D} \in \mathfrak{A}$, $\mathfrak{A}$ is at least nonempty. It is easy to see that if each A_λ ($\lambda \in \Lambda$) is a member of $\mathfrak{A}$, then so are $\bigcup_{\lambda \in \Lambda} A_\lambda$ and $\bigcap_{\lambda \in \Lambda} A_\lambda$, and therefore $(\mathfrak{A}, \subset)$ is a complete lattice.

Lemma 1. *If $A \in \mathfrak{A}$, then A is a convex set.*

proof. We write $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$ and let x, y be arbitrary points of A . Then there exist λ and μ such that $x \in \overset{\circ}{C}_\lambda$ and $y \in \overset{\circ}{C}_\mu$. Let z be an arbitrary point of the open segment (x, y) , and let C_ν be the smallest face containing z . Since C_ν is a face, we have $[x, y] \subset C_\nu$. By Proposition 4, C_λ is the smallest face containing x , and it follows that $C_\lambda \subset C_\nu$. Since the collection $\{C_\lambda\}_{\lambda \in \Lambda}$ is normal, we obtain $\overset{\circ}{C}_\nu \subset A$. This means that $z \in A$, and thus A is convex.

§3 REPRESENTATION OF CONVEX OPERATORS

In this section, we prove a representation theorem of convex operators. Let $D(F)$ be a convex set in $\mathbb{R}^d$ and let $F : D(F) \rightarrow S(\Omega)$ be a convex operator. We can assume without loosing generality that the interior of $D(F)$ is nonempty. Through this section, $\overline{D}$ denotes the closure of $D(F)$. First we state the main theorem.

Theorem 1. *Every convex operator $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$ has at least a representation. That is, there exists a convex integrand $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ such that (1) holds.*

For $D = \overline{D(F)}$, we define $\mathfrak{A}$ as in §2. For $A \in \mathfrak{A}$, a convex integrand $f : A \times \Omega \rightarrow \mathbb{R} \cup \{\infty\}$ is said to represent F on A , if

$$f(x, t) = \begin{cases} F(x)(t) & \text{for a.e. } t \in \Omega \quad x \in A \cap D(F) \\ \infty & x \in A \setminus D(F). \end{cases}$$

Definition. For a convex operator F , we define

$$\tilde{\mathfrak{A}} = \{(A, f) \mid A \in \mathfrak{A}, \text{ and } f \text{ represents } F \text{ on } A\}.$$

Moreover, for $(A_1, f_1), (A_2, f_2) \in \tilde{\mathfrak{A}}$, we write $(A_1, f_1) \leq (A_2, f_2)$ when $A_1 \subset A_2$ and f_2 is an extension of f_1 to A_2 .

Lemma 2. $(\tilde{\mathfrak{A}}, \leq)$ is inductively ordered.

proof. Let $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\tilde{\mathfrak{A}}$. Then $A = \bigcup_{\lambda \in \Lambda} A_\lambda$ is an element of $\mathfrak{A}$. Moreover we can define a convex integrand f on $A \times \Omega$ satisfying $f = f_\lambda$ on $A_\lambda \times \Omega$ for every $\lambda \in \Lambda$. Clearly, $(A, f) \in \tilde{\mathfrak{A}}$ and it is an upper bound of $\{(A_\lambda, f_\lambda)\}_{\lambda \in \Lambda}$.

Lemma 3. For $A \in \mathfrak{A}$ such that $A \neq D$, we define $\mathfrak{S}_A = \{C \in \mathfrak{F}(D) \mid C \cap A = \emptyset\}$. Then $(\mathfrak{S}_A, \subset)$ is inductively ordered.

proof. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a totally ordered subset of $\mathfrak{S}_A$. If we put $C = \bigcup_{\lambda \in \Lambda} C_\lambda$, then C is a convex set and $C \cap A \neq \emptyset$. Moreover $C \in \mathfrak{F}(D)$. Indeed, if we assume $(x, y) \cap C \neq \emptyset$, then there exists $\lambda \in \Lambda$ such that $(x, y) \cap C_\lambda \neq \emptyset$. Hence it follows that $[x, y] \subset C_\lambda \subset C$. Thus $C \in \mathfrak{S}_A$ and it is an upper bound of $\{C_\lambda\}_{\lambda \in \Lambda}$.

Lemma 4. Let A be an element of $\mathfrak{A}$, and assume that $A \neq D$. Then there exists $C \in \mathfrak{S}_A$ such that $A \cup \overset{\circ}{C} \in \mathfrak{A}$.

proof. By Lemma 3 and Zorn's lemma, $\mathfrak{S}_A$ has at least a maximal element C . It is sufficient to show that $A \cup \overset{\circ}{C} \in \mathfrak{A}$. Put $A = \bigcup_{\lambda \in \Lambda} \overset{\circ}{C}_\lambda$, and take $C_1 \in \mathfrak{F}(D)$, such that $C_1 \supset C$. Since C is a maximal element of $\mathfrak{S}_A$, we have $C_1 \notin \mathfrak{S}_A$ and hence $C_1 \cap A \neq \emptyset$. Therefore we can choose $\lambda \in \Lambda$ such that $\overset{\circ}{C}_\lambda \cap C_1 \neq \emptyset$. It follows from Proposition 3 that, $C_\lambda \subset C_1$ holds. Since the collection $\{C_\lambda\}_{\lambda \in \Lambda}$ is normal, $\overset{\circ}{C}_1 \subset A \subset A \cup \overset{\circ}{C}$. This shows that the collection $\{C_\lambda\}_{\lambda \in \Lambda} \cup \{C\}$ is also normal, and $A \cup \overset{\circ}{C} \in \mathfrak{A}$.

Lemma 5. $\tilde{\mathfrak{A}}$ is not empty. In other words, there exists $A \in \mathfrak{A}$ such that F has a representation f on A .

The proof can be done by constructing a convex integrand f which represents F on $\overset{\circ}{D}$. The method of construction is an analogy of that in [4].

Lemma 6. *Suppose that $(A, f) \in \tilde{\mathfrak{A}}$ and $A \neq D$. Let $C \in \mathfrak{S}_A$ is a face such that $A \cup \overset{\circ}{C} \in \mathfrak{A}$ as in Lemma 4. Then f has an extension f_1 defined on $(A \cup \overset{\circ}{C}) \times \Omega$ such that $(A \cup \overset{\circ}{C}, f_1) \in \tilde{\mathfrak{A}}$.*

The proof of this lemma is an analogy of one provide in a previous paper by the author [3].

proof of Theorem 1. By Lemma 3, Lemma 5 and Zorn's lemma, $\tilde{\mathfrak{A}}$ has at least a maximal element (A_0, f_0) . Moreover, Lemma 6 shows that $A_0 = D$, and this means that f_0 represents F on D . Defining $f_0 = \infty$ on $D^c \times \Omega$, we complete the construction of a representation of F .

§4 NORMAL REPRESENTATIONS

A convex integrand $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ is said to be normal if $f(\cdot, t)$ is lower semicontinuous for every $t \in \Omega$ and there exists a countable family of measurable functions $\xi_n : \Omega \longrightarrow \mathbb{R}^d$ ($n = 1, 2, \dots$) such that

- (1) for each n , $f(\xi_n(t), t)$ is measurable in $t \in \Omega$,
- (2) for each $t \in \Omega$, $\{\xi_n(t)\}_{n=1}^\infty$ is dense in $D(f(\cdot, t))$,

where $D(f(\cdot, t)) = \{x \in \mathbb{R}^d \mid f(x, t) < \infty\}$. If a convex integrand f is normal, then $f(\xi(t), t)$ is measurable in $t \in \Omega$ whenever $\xi : \Omega \longrightarrow \mathbb{R}^d$ is measurable. A convex operator F is said to have a normal representation if there exists a normal convex integrand which represents F . We will find some conditions under which a convex operator has a normal representation. By the conjugate of a convex integrand f , we mean the convex integrand $f^* : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(\xi, t) = \sup_{x \in \mathbb{R}^d} \{ \langle x, \xi \rangle - f(x, t) \}.$$

Also the biconjugate $f^{**} : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \infty$ is given by

$$f^{**}(x, t) = \sup_{\xi \in \mathbb{R}^d} \{ \langle x, \xi \rangle - f^*(\xi, t) \}.$$

If a convex integrand f is normal, then so are f^* and f^{**} . We note that if a convex integrand f represents a convex operator F then $D(f(\cdot, t))$ does not depend on $t \in \Omega$.

Lemma 7. Let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of some onvex operator. Then f is normal if and only if $f(\cdot, t)$ is lower semicontinuous, in other words, $f^{**} = f$ on $\mathbb{R}^d \times \Omega$.

proof. Let $D = D(f(\cdot, t))$ and take a conountable subset $\{a_n\}$ of D . If we put $\xi_n(t) = a_n$ for all $t \in \Omega$ and $n = 1, 2, \dots$, then the family $\{\xi_n\}$ satisfies the definition of nomality.

Remark. If a convex integrand f satisfies

(1) for each $x \in \mathbb{R}^d$, $f(x, \cdot)$ is measurable, and

(2) $\overline{D(\cdot, t)}$ does not depend on $t \in \Omega$,

the conclusion of Lemma 7 is also valid.

Let $L(\mathbb{R}^d, S(\Omega))$ denotes the space of all linear mapping from $\mathbb{R}^d$ to $S(\Omega)$. We identify $L(\mathbb{R}^d, S(\Omega))$ with the set $S(\Omega)^d = \{\xi = (\xi_1, \dots, \xi_d) \mid \xi_i \in \xi(\Omega), i = 1, \dots, d\}$ by corresponding $S(\Omega)^d \ni (\xi_1, \dots, \xi_d)$ to the mapping $\varphi : \mathbb{R}^d \ni (x_1, \dots, x_d) \longrightarrow \langle x, \xi \rangle = x_1\xi_1 + \dots + x_d\xi_d \in S(\Omega)$. The conjugate operator $F^* : L(\mathbb{R}^d, S(\Omega)) \supset D(F^*) \longrightarrow S(\Omega)$ of F is defined by

$$F^*(\xi) = \bigvee_{x \in D(F^*)} (\langle x, \xi \rangle - F(x))$$

where $\bigvee$ means the lattice supremum in the space $S(\Omega)$, and $D(F^*)$ is the set of all $\xi \in S(\Omega)^d$ such that the supremum F^* exists. The bi-conjugate operator F^{**} is defined on the space $L(L(\mathbb{R}^d, S(\Omega)), S(\Omega)) = L(S(\Omega)^d, S(\Omega))$, and we regard $S(\Omega)^d$ and $\mathbb{R}^d$ as the subspaces of this by corresponding $\eta \in S(\Omega)^d$ and $x \in \mathbb{R}^d$ to $\langle \eta, \cdot \rangle$ and $\langle x, \cdot \rangle \in L(S(\Omega)^d, S(\Omega))$ respectively. For $x \in \mathbb{R}^d$ and $\eta \in S(\Omega)$, F^{**} is defined by

$$F^{**}(x) = \bigvee_{\xi \in D(F^*)} (\langle x, \xi \rangle - F^*(\xi)),$$

$$F^{**}(\eta) = \bigvee_{\xi \in D(F^*)} (\langle \eta, \xi \rangle - F^*(\xi)).$$

They are only defined on the domain $D(F^{**})$ where these suprema exist.

Theoem 2. Let $F : \mathbb{R}^d \supset D(F) \longrightarrow S(\Omega)$ be a convex operator and let $f : \mathbb{R}^d \times \Omega \longrightarrow \mathbb{R} \cup \{\infty\}$ be a representation of F . Then the convex integrand f^* and f^{**} are normal representations of F^* and F^{**} respectively. Moreover for $\xi \in D(F^*)$ and $\eta \in D(F^{**})$,

$$(F^*(\xi))(t) = f^*(\xi(t), t)$$

$$(F^{**}(\eta))(t) = f^{**}(\eta(t), t)$$

holds for almost every $t \in \Omega$.

This theorem is due to the following lemma.

Lemma 8. *Let $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$ be a convex operator, and let $f : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \cup \{\infty\}$ be a representation of F . Let U be a convex subset of $D(F)$ and suppose that $\inf_{x \in U} f(x, t) > -\infty$ for almost every $t \in \Omega$. Then $\bigwedge_{x \in U} F(x) \in S(\Omega)$ exists and*

$$\left(\bigwedge_{x \in U} F(x) \right)(t) = \inf_{x \in U} f(x, t).$$

proof. Let E be a countable dense set in U . Then we have

$$\inf_{x \in U} f(x, t) = \inf_{x \in E} f(x, t)$$

for a.e. $t \in \Omega$. Hence $\inf_{x \in U} f(x, t)$ is measurable in t and

$$\begin{aligned} \left(\bigwedge_{x \in U} F(x) \right)(t) &\leq \left(\bigwedge_{x \in E} F(x) \right)(t) \\ &= \inf_{x \in E} f(x, t) \\ &= \inf_{x \in U} f(x, t) \\ &= \left(\bigwedge_{x \in U} F(x) \right)(t) \end{aligned}$$

for a.e. $t \in \Omega$, and the lemma is proved.

proof of Theorem 2. By Lemma 8 we have

$$\begin{aligned} (F^*(\xi))(t) &= \bigvee_{x \in D(F)} (\langle \xi, x \rangle - F(x))(t) \\ &= \sup_{x \in D(F)} (\langle \xi(t), x \rangle - f(x, t)) \\ &= f^*(\xi(t), t) \quad (a.e. t \in \Omega), \end{aligned}$$

for every $\xi \in D(F^*) \subset S(\Omega)^d$. The latter statement can be obtained by analogy.

Combining Lemma 7 and Theorem 2, we obtain the following result.

Theorem 3. *A convex operator $F : \mathbb{R}^d \supset D(F) \rightarrow S(\Omega)$ satisfies*

$$F^{**}(x) = F(x)$$

for every $x \in D(F)$, if and only if F has a normal representation.

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