数理解析研究所講究録
334 巻 1978 年 27-30

A THEOREM ON BINARY DIGITS

UNIVERSITY OF COLORADO

WOLFGANG M. SCHMIDT

Let $B(n)$ denote the number of digits 1 in the representation of a natural number n in the binary scale. It is well known that for most n, the number $B(n)$ is about half the total number of digits, so that $B(n)$ is roughly equal to $\frac{1}{2}\nu$, where $\nu = \nu(n) = \log_2 n$, with $\log_2$ denoting the logarithm to the base 2. In fact it follows from the Central Limit Theorem of probability theory that the numbers n with

$$\frac{B(n) - \frac{1}{2}\nu}{\sqrt{\nu}} \leqq \xi$$

have density

$$\varphi(\xi) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\xi} e^{-2t^2}\, dt.$$

Here we say that a set S of natural numbers has <u>density</u> φ if the number $S(x)$ of elements $n \in S$, $n \leqq x$ satisfies the asymptotic relation $S(x) \sim \varphi x$ as $x \to \infty$.

Stolarsky was the first to compare $B(n)$ and $B(kn)$ where $k > 1$ is a fixed odd integer. He called a number k-<u>sturdy</u> if $B(n) \leqq B(kn)$,

and simply sturdy if it is k-sturdy for every k. Stolarsky proved that the 3-sturdy numbers have density $\frac{1}{2}$. Here we are going to sketch a proof that for any odd $k>1$, the k-sturdy numbers have density $\frac{1}{2}$. The interest in the proof lies in the fact that it uses Markov theorems. The main result is as follows.

THEOREM. *Let $k_1, \dots, k_s$ be distinct odd integers. The matrix $M=(\mu_{ij})$ with entries $\mu_{ij}=k_i^{-1}k_j^{-1}\left(\gcd(k_i,k_j)\right)^2$ $(1\le i,j\le s)$ has an inverse $Q=(q_{ij})$, and the quadratic form $Q(\underline{t})=Q(t_1,\dots,t_s)=\sum_{i,j=1}^{s} q_{ij}t_it_j$ is positive definite. Hence*

$$\varphi(\xi_1,\dots,\xi_s)=(2/\pi)^{s/2}(\det M)^{-1/2}\int_{-\infty}^{\xi_1}\cdots\int_{-\infty}^{\xi_s} dt_1\dots dt_s$$

is well defined for $(\xi_1,\dots,\xi_s)\in\mathbb{R}^s$. The main assertion now is that the natural numbers n having simultaneously

$$\frac{B(k_in)-\frac{1}{2}n}{\sqrt{n}}\le\xi_i \qquad (i=1,\dots,s)$$

have density $\varphi(\xi_1,\dots,\xi_s)$.

A corollary is that for distinct odd integers k_1, k_2, the numbers n with $B(k_1n)-B(k_2n)\le\eta\sqrt{n}$ have density

$$\psi(\eta)=\left(\pi(1-\mu)\right)^{-1/2}\int_{-\infty}^{\eta} e^{-t^2/(1-\mu)}\,dt$$

with $\mu=k_1^{-1}k_2^{-1}\left(\gcd(k_1,k_2)\right)^2$. In particular, numbers n with $B(k_1n)\le B(k_2n)$ have density $\psi(0)=\frac{1}{2}$. Another corollary is that the sturdy numbers have density 0.

Let Ω be the ring of 2-adic integers

$$n = a_1 + 2a_2 + 2^2 a_3 + \cdots ,$$

with each "digit" a_t either 0 or 1. The triple $(\Omega, \mathcal{F}, \underline{\underline{P}})$, where $\underline{\underline{P}}$ is the Haar measure on Ω, and $\mathcal{F}$ consists of $\underline{\underline{P}}$-measurable subsets of Ω, is a probability triple. Write $B_h(n)$ for the number of digits 1 among $a_1, \ldots, a_h$. Given distinct odd $k_1, \ldots, k_s$, put $S_h^{(i)} = B_h(k_i n) - \frac{1}{2}h \quad (i = 1, \ldots, s)$, and write

$$\underline{\underline{R}}_h(\xi_1, \ldots, \xi_s) = \underline{\underline{P}}\left\{ h^{-1/2} S_h^{(i)} \leq \xi_i \qquad (i = 1, \ldots, s) \right\}.$$

The theorem can be shown to be a consequence of the

PROPOSITION. $\lim_{h \to \infty} \underline{\underline{R}}_h(\xi_1, \ldots, \xi_s) = \varphi(\xi_1, \ldots, \xi_s)$.

Write $k_i n = b_1^{(i)} + 2 b_2^{(i)} + \cdots$, and put $\gamma_t^{(i)} = \begin{cases} \frac{1}{2} & \text{if } b_t^{(i)} = 1 \\ -\frac{1}{2} & \text{if } b_t^{(i)} = 0. \end{cases}$

Then $S_h^{(i)} = \gamma_1^{(i)} + \cdots + \gamma_h^{(i)}$. For given i, the random variables $\gamma_1^{(i)}, \gamma_2^{(i)}, \ldots$ are independent. But the random vectors $\underline{\underline{\gamma}}_1, \underline{\underline{\gamma}}_2, \ldots$, where $\underline{\underline{\gamma}}_t = (\gamma_t^{(1)}, \ldots, \gamma_t^{(s)})$, are <u>not</u> independent. Hence we cannot use the Central Limit Theorem for sums of independent random variables.

Call a vector $\underline{\underline{x}} = (x_1, \ldots, x_s)$ <u>admissible</u>, if there is a real number α in $0 \leq \alpha < 2$ with $x_i = [k_i \alpha] \quad (i = 1, \ldots, s)$, where $[\ ]$ denotes the integer part. There are finitely many admissible vectors. The vector $\underline{\underline{0}} = (0, \ldots, 0)$ is admissible, and if $\underline{\underline{x}}$ is admissible, then both $\underline{\underline{x}}^0 = \left([x_1/2], \ldots, [x_s/2]\right)$ and $\underline{\underline{x}}^1 = \left([x_1/2] + k_1, \ldots, [x_s/2] + k_s\right)$ are admissible.

Given n as above, put $n_t = a_1 + 2a_2 + \cdots + 2^{t-1} a_t$. Put $\underline{\underline{x}}_0 = \underline{\underline{0}}$

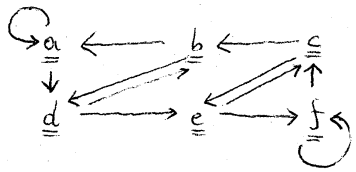
and $\underline{x}_t = \underline{x}_t(n) = \left([2^{-t+1} k_1 \eta_t], \ldots, [2^{-t+1} k_s \eta_t] \right) \quad (t = 1, 2 \ldots)$.

Then $\underline{x}_0, \underline{x}_1, \ldots$ are (vector valued) random variables. Clearly $\underline{x}_t(n)$ is always admissible, and conversely if $\underline{x}$ is admissible, then there is an n and a t with $\underline{x}_t(n) = \underline{x}$. It is now easy to prove the

LEMMA. The random variables $\underline{x}_0, \underline{x}_1, \ldots$ form a Markov Chain. The transition probabilities are given by the rule that for given $\underline{x}_t$, we have $\underline{x}_{t+1}$ either equal to $\underline{x}_t^0$, or to $\underline{x}_t^1$, each with probability $\frac{1}{2}$.

We now observe that the random variable $\underline{y}_t$ is a "functional" of $\underline{x}_t$: We have $y_t^{(i)} = f^{(i)}(\underline{x}_t)$, where $f^{(i)}(\underline{x}) = \frac{1}{2}$ if x_i is odd, $f^{(i)}(\underline{x}) = -\frac{1}{2}$ if x_i is even. Hence the machinery of Markov Chains can be used to complete the proof of our theorem.

In the special case when $s = 2$, $k_1 = 1$, $k_2 = 3$, there are 6 admissible vectors : $\underline{a} = (0, 0)$, $\underline{b} = (0, 1)$, $\underline{c} = (0, 2)$, $\underline{d} = (1, 3)$, $\underline{e} = (1, 4)$, $\underline{f} = (1, 5)$. The transition probabilities are given by the following diagram, where each arrow represents a probability of $\frac{1}{2}$:



References

W. M. Schmidt. The Joint Distribution of the digits of certain integer s-tuples. Acta Math. [Hung. (To appear)

K.B. Stolarsky. Integers whose multiples have anomalous digital frequencies. Acta Arith. (To appear).