

Geometric Properties of Generalized Fractional Integral Operator

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Abstract

Let $\mathcal{A}$ be the class of normalized analytic functions in the unit disk Δ and define the class

$$\mathcal{P}(\beta) = \{f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ such that } \operatorname{Re}[e^{i\varphi}(f'(z) - \beta)] > 0, z \in \Delta\}.$$

In this paper we find conditions on the number β and the nonnegative weight function $\lambda(t)$ such that the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt$$

is convex of order γ ($0 \leq \gamma \leq 1/2$) when $f \in \mathcal{P}(\beta)$. Some interesting further consequences are also considered.

Key Words. Gaussian hypergeometric function, integral transform, convex function, starlike function, fractional integral

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1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent, starlike of order γ and convex of order γ in Δ , respectively. In particular, the classes $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the familiar ones of starlike and convex functions in Δ , respectively. We note that for $0 \leq \gamma < 1$,

$$f(z) \in \mathcal{K}(\gamma) \iff zf'(z) \in \mathcal{S}^*(\gamma)$$

and $f \in \mathcal{S}^*(\gamma)$ if and only if $\operatorname{Re}(zf(z)/f(z)) > \gamma$ for $z \in \Delta$.

Let a, b and c be complex numbers with $c \neq 0, -1, -2, \dots$. Then the *Gaussian/classical hypergeometric function* ${}_2F_1(a, b; c; z) \equiv F(a, b; c; z)$ is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

The hypergeometric function $F(a, b; c; z)$ is analytic in Δ and if a or b is a negative integer, then it reduces to a polynomial. For functions $f_j(z)$ ($j = 1, 2$) of the forms

$$f_j(z) := \sum_{n=1}^{\infty} a_{j,n} z^n \quad (a_{j,1} := 1; j = 1, 2),$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) := \sum_{n=1}^{\infty} a_{1,n} a_{2,n} z^n \quad (a_{j,1} := 1; j = 1, 2).$$

For $f \in \mathcal{A}$, the following special case gives rise to a natural convolution operator $H_{a,b,c}$ defined by

$$H_{a,b,c}(f)(z) := zF(a, b; c; z) * f(z).$$

Note that this is a three-parameter family of operators and contains as special cases several of the known linear integral or differential operators studied by a number of authors. In fact, this operator was considered first time in this form by Hoholov [7] and has been studied extensively by Ponnusamy [11], Ponnusamy and Rønning [14] and many others [2, 8, 5]. For example, by letting $H_{1,b,c}(f) \equiv \mathcal{L}(b, c)(f)$, we get the operator $\mathcal{L}(b, c)(f)$ discussed by Carlson and Shaffer [4]. Clearly, $\mathcal{L}(b, c)$ maps $\mathcal{A}$ onto itself, and $\mathcal{L}(c, b)$ is the inverse of $\mathcal{L}(b, c)$, provided that $b \neq 0, -1, -2, \dots$. Furthermore, $\mathcal{L}(b, b)$ is the unit operator and

$$(1.1) \quad \mathcal{L}(b, c) = \mathcal{L}(b, e)\mathcal{L}(e, c) = \mathcal{L}(e, c)\mathcal{L}(b, e) \quad (c, e \neq 0, -1, -2, \dots).$$

Also, we note that $\mathcal{L}(b, b)f(z) = f(z)$, $\mathcal{L}(2, 1)f(z) = zf'(z)$,

$$\mathcal{K}(\gamma) = \mathcal{L}(1, 2)\mathcal{S}^*(\gamma) \quad (0 \leq \gamma < 1),$$

$$(1.2) \quad \mathcal{S}^*(\gamma) = \mathcal{L}(2, 1)\mathcal{K}(\gamma) \quad (0 \leq \gamma < 1)$$

and the Ruscheweyh derivatives [16] of $f(z)$ are $\mathcal{L}(n+1, 1)f(z)$, $n \in \mathbb{N} \cup \{0\}$. For $\beta < 1$, we define

$$\mathcal{P}(\beta) = \{f \in \mathcal{A} : \exists \varphi \in \mathbb{R} \text{ such that } \operatorname{Re}[e^{i\varphi}(f'(z) - \beta)] > 0, z \in \Delta\}.$$

Throughout this paper we let $\lambda : [0, 1] \rightarrow \mathbb{R}$ be a nonnegative function with the normalization $\int_0^1 \lambda(t) dt = 1$. For certain specific subclasses of $f \in \mathcal{A}$, many authors considered the geometric properties of the integral transform of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.$$

More recently, starlikeness of this general operator $V_\lambda(f)$ was discussed by Fournier and Ruscheweyh [6] by assuming that $f \in \mathcal{P}(\beta)$. The method of proof is the duality principle developed mainly by Ruscheweyh [17]. This result was later extended by Ponnusamy and Rønning [15] by means of finding conditions such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into starlike functions of order γ , $0 \leq \gamma \leq 1/2$ and was further generalized in [3].

In this paper, we find conditions on β , γ and the function $\lambda(t)$ such that $V_\lambda(f)$ carries $\mathcal{P}(\beta)$ into $\mathcal{K}(\gamma)$. As a consequence of this investigation, a number of new results are established. The following lemma is the key for the proof of our main results.

1.3. Lemma. Let $\Lambda(t)$ be a real valued monotone decreasing function on $[0, 1]$ satisfying $\Lambda(1) = 0$, $t\Lambda(t) \rightarrow 0$ for $t \rightarrow 0^+$ and

$$-\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}} = \frac{\lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0, 1)$ where

$$\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds.$$

If $\beta = \beta(\lambda, \gamma)$ is given by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) \frac{1 - \gamma(1+t)}{(1-\gamma)(1+t)^2} dt$$

then $V_\lambda(\mathcal{P}_\beta) \subset \mathcal{K}(\gamma)$, $0 \leq \gamma \leq \frac{1}{2}$, where $V_\lambda(f)$ is defined above.

Proof. Proof of this lemma quickly follows from the work of Ponnusamy and Rønning [15] and therefore, we omit the details. $\square$

2. Main Results

In order to apply Lemma 1.3 with $\gamma \in [0, \frac{1}{2}]$ it suffices to show that

$$u(t) = -\frac{t\Lambda'(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on the interval $(0, 1)$ where $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds$. Taking the logarithmic derivative of $u(t)$ and using the fact that $\Lambda'(t) = -\frac{\lambda(t)}{t}$, we have

$$\frac{u'(t)}{u(t)} = \frac{\lambda'(t)}{\lambda(t)} + \frac{2(\gamma + (1+\gamma)t)}{1-t^2}$$

and therefore, $u(t)$ is decreasing on $(0, 1)$ if and only if

$$(2.1) \quad (1-t^2)\lambda'(t) + 2(\gamma + (1+\gamma)t)\lambda(t) \leq 0.$$

From now on, we define

$$(2.2) \quad \varphi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n \quad (b_n \geq 0)$$

and

$$(2.3) \quad \lambda(t) = Ct^{b-1}(1-t)^{c-a-b}\varphi(1-t)$$

where C is a normalized constant so that $\int_0^1 \lambda(t) dt = 1$. For $f \in \mathcal{A}$, Balasubramanian *et al.* [2] defined the operator $P_{a,b,c}$ by

$$P_{a,b,c}(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,$$

where $\lambda(t)$ is given by (2.3). Special choices of $\varphi(1-t)$ led to various interesting geometric properties concerning certain well-known operators. Observe that $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds$ is monotone decreasing on $[0, 1]$, $\lim_{t \rightarrow 0+} t\Lambda(t) = 0$ and (2.1) is equivalent to

$$(c-a-3-2\gamma)t^2 + (c-a-b-2\gamma)t + 1-b \geq -t(1-t^2) \frac{\varphi'(1-t)}{\varphi(1-t)}$$

and this inequality may be rewritten in a convenient form as

$$(2.4) \quad D(t^2+t) + (1-b)(1-t^2) + t(1-t) \geq -t(1-t^2) \frac{\varphi'(1-t)}{\varphi(1-t)}$$

where $D = c-a-b-1-2\gamma$. In view of (2.2), $\varphi(1-t) > 0$ and $\varphi'(1-t) \geq 0$ on $(0, 1)$, so that the right hand side of the inequality (2.4) is nonpositive for all $t \in (0, 1)$. If we assume that $0 \leq \gamma \leq 1/2$, $a > 0$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$, then the left hand side of the inequality (2.4) clearly is nonnegative for all $t \in (0, 1)$. Thus, the inequality (2.4) holds for all $t \in (0, 1)$. In conclusion, from Lemma 1.3, we have the following theorem and techniques as in the proofs of [5, Theorem 1] and [13, 15, 8] show that the value β in Theorem 2.5 is sharp.

2.5. Theorem. Let $0 \leq \gamma \leq 1/2$, $a > 0$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$, and let $\lambda(t)$ be given by (2.3). Define $\beta = \beta(a, b, c, \gamma)$ by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \lambda(t) \frac{1 - \gamma(1+t)}{(1-\gamma)(1+t)^2} dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $P_{a,b,c}(f)(z) \in \mathcal{K}(\gamma)$. The value of β is sharp.

2.6. Corollary. Let $0 \leq \gamma \leq 1/2$, $0 < a \leq 1$, $0 < b \leq 1$ and $c \geq a+b+2\gamma+1$. Suppose that $\varphi(1-t)$ and C are defined by

$$(2.7) \quad \varphi(1-t) = F(c-a, 1-a; c-a-b+1; 1-t)$$

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and

$$(2.8) \quad C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)},$$

respectively. Define $\beta = \beta(a, b, c, \gamma)$ by

$$(2.9) \quad \frac{\beta - \frac{1}{2}}{1 - \beta} = -C \int_0^1 (1-t)^{c-a-b} t^{b-1} \left(\frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2} \right) \varphi(1-t) dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $H_{a,b,c}(f)(z)$ defined by

$$H_{a,b,c}(f)(z) := C \int_0^1 (1-t)^{c-a-b} t^{b-2} \varphi(1-t) f(tz) dt.$$

belongs to $\mathcal{K}(\gamma)$. The value of β is sharp.

Proof. The integral representation for $H_{a,b,c}(f)(z)$ has been obtained in [2, 8]. By (2.7) and (2.8), it follows that the corresponding operator $P_{a,b,c}(f)(z)$ equals $H_{a,b,c}(f)(z)$. Note that the assumption implies that $0 < a \leq 1$ and $c-a > 0$ and $c-a-b+1 > 0$ from which the nonnegativity of $\varphi(1-t)$ on $(0, 1)$ is clear. Now, the desired result follows from Theorem 2.5. $\square$

Setting $a = 1$ in Corollary 2.6, we obtain

2.10. Corollary. Let $0 \leq \gamma \leq 1/2$, $0 < b \leq 1$ and $c \geq b + 2\gamma + 2$. Also let

$$(2.11) \quad \beta(1, b, c, \gamma) = 1 - \frac{1-\gamma}{2[1 - F(2, b; c; -1) - \gamma(1 - F(1, b; c; -1))]}.$$

If $\beta(1, b, c, \gamma) \leq \beta < 1$ and $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{L}(b, c)f(z) \in \mathcal{K}(\gamma)$.

Proof. Putting $a = 1$ in (2.9) it follows that

$$\begin{aligned} \frac{\beta - \frac{1}{2}}{1 - \beta} &= -\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2} dt \\ &= \frac{\Gamma(c)}{(1-\gamma)\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \left\{ \frac{\gamma}{1+t} - \frac{1}{(1+t)^2} \right\} dt \\ &= \frac{1}{1-\gamma} [\gamma F(1, b; c; -1) - F(2, b; c; -1)] \end{aligned}$$

where the last step follows from the Euler integral representation. Solving the last equation gives the number $\beta(1, b, c, \gamma)$ given by (2.11). The desired conclusion follows from Corollary 2.6. $\square$

2.12. Theorem. Let $-1 < a \leq 2$, $0 \leq \gamma \leq 1/2$ and $p \geq 2(1+\gamma)$. Suppose that $\beta = \beta(a, p, \gamma)$ is given by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a (\log(1/t))^{p-1} \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2} dt.$$

Then, for $f \in \mathcal{P}(\beta)$, the Hadamard product function $\Phi_p(a; z) * f(z)$ defined by

$$\Phi_p(a; z) * f(z) = \left(\sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n \right) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{K}(\gamma)$. The value of β is sharp.

Proof. To obtain this theorem, we choose $\phi(1-t)$ and $\lambda(t)$ in Theorem 2.5 as

$$\phi(1-t) = \left(\frac{\log(1/t)}{1-t} \right)^{p-1} = \left(\frac{-\log(1-(1-t))}{1-t} \right)^{p-1},$$

and

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a (1-t)^{p-1} \phi(1-t),$$

respectively. The desired conclusion follows from Theorem 2.5 and the hypotheses. $\square$

Our final application concerns the integral operator studied by Ponnusamy [12], Ponnusamy and Rønning [13] and later by Balasubramanian, Ponnusamy and Vuorinen [2]. Define

$$(2.13) \quad \lambda(t) = \begin{cases} (a+1)(b+1) \left(\frac{t^a(1-t^{b-a})}{b-a} \right) & \text{for } b \neq a, a > -1, b > -1, \\ (a+1)^2 t^a \log(1/t) & \text{for } b = a, a > -1. \end{cases}$$

With this $\lambda(t)$, we have an integral transform

$$G_f(a, b; z) := \left(\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n \right) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.$$

In view of symmetry between a and b , without loss of generality, we assume that $b > a$ in the case $b \neq a$. Note that in the limiting case $b \rightarrow \infty$ ($b \neq a$), $G_f(a, b; z)$ reduces to a well-known Bernadi operator given by

$$G_f(a, \infty; z) := \left(\sum_{n=1}^{\infty} \frac{1+a}{n+a} z^n \right) * f(z) = \frac{1+a}{z^a} \int_0^z t^{a-1} f(t) dt \equiv \mathcal{L}(a+1, a+2)f(z).$$

2.14. Theorem. Let $b > -1$, $a > -1$ be such that any one of the following conditions holds:

- (i) $-1 < a \leq 0$ and $a = b$
- (ii) $-1 < a \leq 0$ and $b > a$ with $-1 < b \leq 2$.

Suppose that $\lambda(t)$ is defined by (2.13) and β is given by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = - \int_0^1 \frac{\lambda(t)}{(1+t)^2} dt.$$

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If $f \in \mathcal{P}(\beta)$, then the function $G_f(a, b; z)$ is convex in Δ . The value of β is sharp.

Proof. Clearly, as in the proof of Theorem 2.5, it suffices to verify the inequality (2.1) for the $\lambda(t)$ defined by (2.13). Now, for the $\lambda(t)$ given by (2.13), we have

$$\lambda'(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a} t^{a-1} (a - bt^{b-a}) & \text{for } b > a > -1, \\ (a+1)^2 (-1 + a \log(1/t)) t^{a-1} & \text{for } b = a > -1. \end{cases}$$

Case (i): Let $b = a > -1$. If we substitute the $\lambda(t)$ and the $t\lambda'(t)$ expression in (2.1), the inequality (2.1) is seen to be equivalent to

$$(2.15) \quad -a(1-t^2) \log(1/t) + 1 - t^2 - 2t^2 \log(1/t) \geq 0, \quad t \in (0, 1).$$

Clearly, as $-1 < a \leq 0$, this inequality holds if it holds for $a = 0$. Substituting $a = 0$, this becomes

$$1 - t^2 - 2t^2 \log(1/t) \geq 0, \quad t \in (0, 1),$$

which, for $t = e^{-x}$, is equivalent to

$$e^{2x} \geq 1 + 2x, \quad x \geq 0.$$

Since this inequality holds for all $x \geq 0$, the inequality (2.15) holds for all $t \in (0, 1)$ and the desired conclusion holds in this case.

Case (ii): Let $b > a > -1$. If we substitute the $\lambda(t)$ given by (2.13) and the corresponding $t\lambda'(t)$ expression in (2.1), the inequality (2.1) is seen to be equivalent to

$$(2.16) \quad (1-t^2)(at^{a-1} - bt^{b-1}) + 2(t^{a+1} - t^{b+1}) \leq 0$$

which may be rewritten as

$$\psi_t(a) - \psi_t(b) \leq 0, \quad t \in (0, 1),$$

where

$$\psi_t(a) = a(1-t^2)t^{a-1} + 2t^{a+1}.$$

For each fixed $t \in (0, 1)$, we first claim that $\psi_t(a)$ is an increasing function of a . Differentiating $\psi_t(a)$ with respect to a , we find that

$$\psi'_t(a) = t^{a-1} [1 - t^2 - 2t^2 \log(1/t) - a(1-t^2) \log(1/t)].$$

Using the previous case, namely the inequality (2.15), it follows that $\psi'_t(a) \geq 0$ for all $a \in (-1, 0)$ and for $t \in (0, 1)$. In particular, for $b > a$ with $b \in (-1, 0)$ and $a \in (-1, 0)$, the inequality (2.16) holds.

When $b > a$ with $0 \leq b \leq 2$ and $a \in (-1, 0]$, we have

$$\psi_t(a) \leq \psi_t(0) = 2t \quad \text{for } t \in (0, 1).$$

Now, we claim that for $b > a$ with $0 \leq b \leq 2$ and $a \in (-1, 0]$, the inequality

$$2t \leq \psi_t(b) = b(1 - t^2)t^{b-1} + 2t^{b+1}$$

holds for all $t \in (0, 1)$. To verify this inequality, we rewrite it as

$$2(t^{-b} - 1) \leq b(t^{-2} - 1) \quad \text{for } t \in (0, 1)$$

which, for $t = 1 - x$, is equivalent to the inequality

$$(2.17) \quad 2((1 - x)^{-b} - 1) \leq b((1 - x)^{-2} - 1) \quad \text{for } x \in (0, 1).$$

Since

$$2(b)_n \leq b(2)_n \quad \text{for all } n \geq 1,$$

a comparison of the coefficients of x^n on both sides of the inequality (2.17) implies that (2.17) clearly holds. Thus, for $0 \leq b \leq 2$ and $a \in (-1, 0]$ with $b > a$, we have

$$\psi_t(a) \leq 2t \leq \psi_t(b) \quad \text{for } t \in (0, 1)$$

and the proof is now complete. $\square$

3. The Fractional Integral Operator

There are a number of definitions for fractional calculus operators in the literature. We use here the following definition due to Saigo [18] (see also [10, 19]).

For $\lambda > 0$, $\mu, \nu \in \mathbb{R}$, the fractional integral operator $\mathcal{I}^{\lambda, \mu, \nu}$ is defined by

$$\mathcal{I}^{\lambda, \mu, \nu} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z - \zeta)^{\lambda-1} F(\lambda + \mu, -\nu; \lambda; 1 - \frac{\zeta}{z}) f(\zeta) d\zeta,$$

where $f(z)$ is taken to be an analytic function in a simply-connected region of the z -plane containing the origin with the order

$$f(z) = \mathcal{O}(|z|^\epsilon) \quad (z \rightarrow 0)$$

for $\epsilon > \max\{0, \mu - \nu\} - 1$, and the multivaluedness of $(z - \zeta)^{\lambda-1}$ is removed by requiring that $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

In [10], Owa *et al.* considered the normalized fractional integral operator by defining $\mathcal{J}^{\lambda, \mu, \nu}$ by

$$\mathcal{J}^{\lambda, \mu, \nu} f(z) = \frac{\Gamma(2 - \mu)\Gamma(2 + \lambda + \nu)}{\Gamma(2 - \mu + \nu)} z^\mu \mathcal{I}^{\lambda, \mu, \nu} f(z), \quad \min\{\lambda + \nu, -\mu + \nu, -\mu\} > -2.$$

Clearly, $\mathcal{J}^{\lambda, \mu, \nu}$ maps $\mathcal{A}$ onto itself and for $f \in \mathcal{A}$

$$(3.1) \quad \mathcal{J}^{\lambda, \mu, \nu} f(z) = \mathcal{L}(2, 2 - \mu) \mathcal{L}(2 - \mu + \nu, 2 + \lambda + \nu) f(z).$$

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A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha, \gamma)$ if

$$(f * s_\alpha)(z) \in \mathcal{S}^*(\gamma) \quad (0 \leq \alpha < 1; 0 \leq \gamma < 1).$$

Here $s_\alpha(z) = z/(1-z)^{-2(1-\alpha)}$ ($0 \leq \alpha < 1$) denotes the well-known extremal function for the class $\mathcal{S}^*(\alpha)$. Note that

$$(3.2) \quad \mathcal{R}(\alpha, \gamma) = \mathcal{L}(1, 2-2\alpha)\mathcal{S}^*(\gamma)$$

and $\mathcal{R}(\alpha, \alpha) \equiv \mathcal{R}(\alpha)$ is the subclass of $\mathcal{A}$ consisting of *prestarlike functions of order α* which was introduced by Suffridge [21]. In [20], it is shown that $\mathcal{R}(\alpha) \subset \mathcal{S}$ if and only if $\alpha \leq 1/2$.

Our result in this section is to obtain a univalence criterion for the operator $\mathcal{J}^{\lambda, \mu, \nu}$.

3.3. Theorem. Let $0 \leq \gamma \leq 1/2$, $0 \leq \mu < 2$, $\lambda \geq 2(1+\gamma) - \mu$ and $\mu - 2 < \nu \leq \mu - 1$. Define $\beta = \beta(\lambda, \mu, \nu, \gamma)$ by

$$\beta = 1 - \frac{1-\gamma}{2[1 - F(2, 2-\mu+\nu; 2+\lambda+\nu; -1) - \gamma(1 - F(1, 2-\mu+\nu; 2+\lambda+\nu; -1))]}.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}^{\lambda, \mu, \nu} f(z) \in \mathcal{R}(\mu/2, \gamma)$.

Proof. Making use of (1.1) and (3.1), we note that

$$(3.4) \quad \begin{aligned} \mathcal{J}^{\lambda, \mu, \nu} f(z) &= \mathcal{L}(2, 2-\mu)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu)f(z) \\ &= \mathcal{L}(1, 2-\mu)\mathcal{L}(2, 1)\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu)f(z). \end{aligned}$$

By using Corollary 2.10, we obtain

$$\mathcal{L}(2-\mu+\nu, 2+\lambda+\nu)f(z) \in \mathcal{K}(\gamma).$$

Since $0 \leq \mu < 2$, from (1.2), (3.2) and (3.4), we have $\mathcal{J}^{\lambda, \mu, \nu} f(z) \in \mathcal{R}(\mu/2, \gamma)$ and we complete the proof. $\square$

Taking $\mu = 2\gamma$ in Theorem 3.3, we get

3.5. Corollary. Let $0 \leq \gamma \leq 1/2$, $\lambda \geq 2$ and $2(\gamma-1) < \nu \leq 2\gamma-1$. Define $\beta = \beta(\lambda, \nu, \gamma)$ by

$$\beta = 1 - \frac{1-\gamma}{2[1 - F(2, 2-2\gamma+\nu; 2+\lambda+\nu; -1) - \gamma(1 - F(1, 2-2\gamma+\nu; 2+\lambda+\nu; -1))]}.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}_{0,z}^{\lambda, 2\gamma, \nu} f(z) \in \mathcal{R}(\gamma) \subset \mathcal{S}$.

Proof. If we put $\mu = 2\gamma$ in Theorem 3.3, then

$$\mathcal{J}_{0,z}^{\lambda, 2\gamma, \nu} f(z) \in \mathcal{R}(\gamma, \gamma) = \mathcal{R}(\gamma).$$

Since $\gamma \leq 1/2$, we have $\mathcal{R}(\gamma) \subset \mathcal{S}$ and therefore, the proof is completed. $\square$

3.6. Remark. In [2], Balasubramanian *et al.* found the conditions on the number β and the function $\lambda(t)$ such that $P_{a,b,c}(f)(z) \in \mathcal{S}^*(\gamma)$ ($0 \leq \gamma \leq 1/2$). Since

$$\mathcal{J}^{\lambda,\mu,\nu} f(z) = P_{1-\mu,2,\lambda-\nu+2}(f)(z)$$

with

$$\phi(1-t) = F(\lambda + \mu, -\nu; \lambda; 1-t)$$

and

$$C = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)},$$

it is easy to find that the condition on β and $\lambda(t)$ such that $\mathcal{J}^{\lambda,\mu,\nu} f(z) \in \mathcal{S}^*(\gamma)$.

Finally, by using Lemma 1.3 again, we investigate convexity of the operator $\mathcal{J}^{\lambda,\mu,\nu}$.

3.7. Theorem. Let $0 \leq \gamma \leq 1/2$, $0 < \lambda \leq 1 + 2\gamma$, $2 < \mu < 3$ and $\nu > \mu - 2$. Define $\beta = \beta(\lambda, \mu, \nu, \gamma)$ by

$$\frac{\beta - \frac{1}{2}}{1 - \beta} = -\frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} \int_0^1 \frac{t(1-t)^{\lambda-1}(1-\gamma(1+t))}{(1-\gamma)(1+t)^2} F(\lambda + \mu, -\nu; \lambda; 1-t) dt.$$

If $f(z) \in \mathcal{P}(\beta)$, then $\mathcal{J}^{\lambda,\mu,\nu} f(z) \in \mathcal{K}(\gamma)$. The value of β is sharp.

Proof. Let $0 \leq \gamma \leq 1/2$, $0 < \lambda \leq 1 + 2\gamma$, $2 < \mu < 3$, $\nu > \mu - 2$, and let

$$(3.8) \quad \lambda(t) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\nu)}{\Gamma(\lambda)\Gamma(2-\mu+\nu)} t(1-t)^{\lambda-1} F(\lambda + \mu, -\nu; \lambda; 1-t).$$

Then we can easily see that $\int_0^1 \lambda(t) dt = 1$, $\Lambda(t) = \int_t^1 \lambda(s) ds/s$ is monotone decreasing on $[0, 1]$ and $\lim_{t \rightarrow 0+} t\Lambda(t) = 0$. Also we find that the function $u(t) = \lambda(t)/(1+t)(1-t)^{1+2\gamma}$ is decreasing on $(0, 1)$, where $\lambda(t)$ is given by (3.8). Hence, $t\Lambda'(t)/(1+t)(1-t)^{1+2\gamma} = -u(t)$ is increasing on $(0, 1)$. From Lemma 1.3, we obtain the desired result. $\square$

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