

Some Doubly Infinite, Finite and Mixed Infinite Sums derived from The N- Fractional Calculus of A Power Function (with Some Examinations)

Katsuyuki Nishimoto and Susana S. de Romero

Abstract

In this article theorems for some doubly infinite, finite and mixed infinite sums derived from the N-fractional calculus of a power function are reported.

Moreover some numerical examinations for the theorems are reported too.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\text{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\text{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D(z \in D)$,

$$f_\nu(z) = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}^-), \quad (1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (2)$$

where $-\pi \leq \arg(\xi-z) \leq \pi$ for C_- , $0 \leq \arg(\xi-z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in \mathbb{C}$, $\nu \in \mathbb{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. Let fractional calculus operator (Nishimoto's Operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}), \quad [\text{Refer to (1)}] \quad (3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbb{Z}^+), \quad (4)$$

and define the binary operation $\circ$ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbb{R}\} \quad (6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbb{R}\}$, where $f = f(z)$ and $z \in \mathbb{C}$. (vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G. ; Fractional calculus operator group)

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \mathbb{R}). \quad (7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \quad (8)$$

holds. [5]

(III) **Lemma.** We have [1]

$$(i) \quad ((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad \left(\left| \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} \right| < \infty \right),$$

$$(ii) \quad (\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

$$(iii) \quad ((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha} \frac{1}{\Gamma(\alpha)} \log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ in (i), and $z-c \neq 0, 1$ in (ii) and (iii). (Γ ; Gamma function),

$$(iv) \quad (u \cdot v)_\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \begin{pmatrix} u = u(z), \\ v = v(z) \end{pmatrix}.$$

§ 1. Doubly Infinite, Finite and Mixed Sums

In the following $\alpha, \beta, \gamma \in \mathbb{R}$.

Theorem 1. *Let*

$$G(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}. \quad (1)$$

(i) When $\alpha, \beta, \gamma \notin \mathbb{Z}$, we have the following doubly infinite sums ;

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(\alpha, \beta, \gamma; k, m) \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k \subseteq \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z} \right)^{\gamma-\alpha}, \quad (2)$$

where

$$|(z-c)/z| < 1, \quad |c/(z-c)| < 1,$$

and

$$\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right|, \quad \left| \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} \right| < \infty.$$

The identity (notation =) holds for $(\gamma-\alpha) \in \mathbb{Z}$.

(ii) When $\alpha, \gamma \notin \mathbb{Z}^+$, we have the following mixed infinite sums ;

$$\sum_{k=0}^{\infty} \sum_{m=0}^s G(\alpha, s, \gamma; k, m) \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k \subseteq \frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)} \left(\frac{z-c}{z} \right)^{\gamma-\alpha}, \quad (3)$$

for $s \in \mathbb{Z}^+$ where

$$|c/(z-c)| < 1, \quad |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(\gamma-\alpha-s)}{\Gamma(-\alpha-s)} \right|, \quad \left| \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} \right| < \infty.$$

The identity (notation =) holds for $(\gamma-\alpha) \in \mathbb{Z}$.

Proof of (i). We have

$$z^\alpha = (z-c)^\alpha \left(1 - \frac{c}{c-z} \right)^\alpha \quad (4)$$

$$= (z-c)^\alpha \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (c-z)^{-k} \quad (|c-z| > |c|) \quad (5)$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} (z-c)^{\alpha-k} . \quad (6)$$

Next make $(6) \times z^\beta$, then operate N- fractional calculus operator N^γ to its both sides, we obtain

$$(z^{\alpha+\beta})_\gamma = \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \left((z-c)^{\alpha-k} \cdot z^\beta \right)_\gamma \quad (7)$$

$$= \sum_{k=0}^{\infty} \frac{c^k \Gamma(\alpha+1)}{k! \Gamma(\alpha+1-k)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma+1)}{m! \Gamma(\gamma+1-m)} \left((z-c)^{\alpha-k} \right)_{\gamma-m} (z^\beta)_m , \quad (8)$$

by Lemma (i i).

Now we have

$$(z^{\alpha+\beta})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} z^{\alpha+\beta-\gamma} \left(\left| \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \right| < \infty \right) , \quad (9)$$

$$\left((z-c)^{\alpha-k} \right)_{\gamma-m} = e^{-i\pi(\gamma-m)} \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} (z-c)^{\alpha-k-\gamma+m} , \quad (10)$$

$$\left(\left| \frac{\Gamma(k-\alpha+\gamma-m)}{\Gamma(k-\alpha)} \right| < \infty \right)$$

and

$$(z^\beta)_m = e^{-i\pi m} \frac{\Gamma(m-\beta)}{\Gamma(-\beta)} z^{\beta-m} \quad (11)$$

by Lemma (i), respectively.

Therefore, substituting (9), (10) and (11) into (8) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(\alpha, \beta, \gamma ; k, m) \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = \frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(-\alpha-\beta)} \left(\frac{z-c}{z} \right)^{\gamma-\alpha} . \quad (12)$$

However the LHS (left hand side) of (12) is always one valued function, on the contrary the RHS (right hand side) of (12) is many valued function for $(\gamma-\alpha) \notin \mathbb{Z}$ and one valued one for $(\gamma-\alpha) \in \mathbb{Z}$.

Hence we must calculate as

$$\left(\frac{z-c}{z} \right)^{\gamma-\alpha} = \left(e^{i2n\pi} \frac{z-c}{z} \right)^{\gamma-\alpha} \left(\begin{matrix} (\gamma-\alpha) \notin \mathbb{Z} \\ n \in \mathbb{Z} \end{matrix} \right) , \quad (13)$$

because we are now being in the field of complex analysis.

Moreover when $(\gamma - \alpha) \in \mathbb{Z}$ both of the LHS and the RHS of (12) are one valued functions respectively. In this case we have (12) strictly.

Therefore, we have (2) from (12), considering (13) for $(\gamma - \alpha) \notin \mathbb{Z}$.

Proof of (i i). Set $\beta = s \in \mathbb{Z}^+$ in (2), we have then (3) clearly, under the conditions.

Corollary 1. When $r, p \in \mathbb{Z}^+$ we have the doubly finite sums ;

$$\sum_{k=0}^r \sum_{m=0}^p G(r, \beta, p ; k, m) \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = \frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)} \left(\frac{z-c}{z} \right)^{p-r}, \quad (14)$$

for $(p-r) \in \mathbb{Z}$, where

$$|c/(z-c)|, |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(p-r-\beta)}{\Gamma(-r-\beta)} \right|, \left| \frac{\Gamma(k-r+p-m)}{\Gamma(k-r)} \right| < \infty.$$

Proof. Set $\alpha = r$ and $\gamma = p$ in (2), we have then this corollary clearly.

Now both of the LHS and RHS of (14) are one valued functions respectively, hence the identity (notation =) holds in (2).

Corollary 2. When $r, p \in \mathbb{Z}^+$ we have the doubly finite sums ;

$$\sum_{k=0}^r \sum_{m=0}^s G(r, s, p ; k, m) \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = \frac{\Gamma(p-r-s)}{\Gamma(-r-s)} \left(\frac{z-c}{z} \right)^{p-r}, \quad (15)$$

where

$$|c/(z-c)|, |(z-c)/z| < \infty,$$

and

$$\left| \frac{\Gamma(p-r-s)}{\Gamma(-r-s)} \right|, \left| \frac{\Gamma(k-r+p-m)}{\Gamma(k-r)} \right| < \infty.$$

Proof. Set $\alpha = r$ and $\gamma = p$ in (3), we have then this corollary under the conditins, clearly.

Now both of the LHS and RHS of (15) are one valued functions respectively, hence the identity (notation =) holds in (3).

§ 2. Direct calculation of the doubly infinite sums

The direct calculation (without the use of N-fractional calculus) of the doubly infinite sum in LHS of § 1. (2) is shown as follows.

Theorem 2. Let

$$G = G(\alpha, \beta, \gamma; k, m) := \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(m-\beta)\Gamma(k-m-\alpha+\gamma)}{k! \cdot m! \Gamma(\alpha+1-k)\Gamma(\gamma+1-m)\Gamma(-\beta)\Gamma(k-\alpha)}, \quad (1)$$

and

$$P = P(\alpha, \beta, \gamma) := \frac{\sin \pi \alpha \cdot \sin \pi(\gamma - \alpha - \beta)}{\sin \pi(\alpha + \beta) \cdot \sin \pi(\gamma - \alpha)}. \quad (2)$$

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = P \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left(\frac{z-c}{z} \right)^{\gamma - \alpha}, \quad (3)$$

where

$$|c/(z-c)| < 1,$$

and

$$(\alpha + \beta), (\gamma - \alpha), (\gamma - \alpha - \beta) \notin \mathbb{Z}.$$

Proof. Now we have

$$\begin{aligned} & G \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k \\ &= \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \cdot \frac{[-\beta]_m [-\gamma]_m [\gamma - \alpha - m]_k}{k! \cdot m! [1 + \alpha - \gamma]_m} \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k \end{aligned} \quad (4)$$

using the identity

$$\Gamma(\lambda + 1 - k) = (-1)^{-k} \frac{\Gamma(\lambda + 1)\Gamma(-\lambda)}{\Gamma(k - \lambda)} \quad (5)$$

and

$$\Gamma(k + \gamma - \alpha - m) = (-1)^{-m} \Gamma(\gamma - \alpha) \frac{[\gamma - \alpha - m]_k}{[1 + \alpha - \gamma]_m}, \quad (6)$$

where

$$[\lambda]_k = \lambda(\lambda + 1) \cdots (\lambda + k - 1) = \Gamma(\lambda + k) / \Gamma(\lambda), \text{ with } [\lambda]_0 = 1$$

(notation of Pochhammer).

We have then

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \left(\frac{z-c}{z} \right)^m \left(\frac{c}{z-c} \right)^k = \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \\ & \times \sum_{m=0}^{\infty} \frac{[-\beta]_m [-\gamma]_m}{m! [1 + \alpha - \gamma]_m} \left(\frac{z-c}{z} \right)^m \sum_{k=0}^{\infty} \frac{[\gamma - \alpha - m]_k}{k!} \left(\frac{-c}{z-c} \right)^k \end{aligned} \quad (7)$$

$$= \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c} \right)^{\alpha-\gamma} \sum_{m=0}^{\infty} \frac{[-\beta]_m [-\gamma]_m}{m! [1 + \alpha - \gamma]_m} \quad (8)$$

$$= \frac{\Gamma(\gamma - \alpha)}{\Gamma(-\alpha)} \left(\frac{z}{z-c} \right)^{\alpha-\gamma} {}_2F_1(-\beta, -\gamma; 1 + \alpha - \gamma; 1) \quad (9)$$

$$= \frac{\Gamma(\gamma - \alpha) \Gamma(1 + \alpha - \gamma) \Gamma(1 + \alpha + \beta)}{\Gamma(-\alpha) \Gamma(1 + \alpha) \Gamma(1 + \alpha + \beta - \gamma)} \left(\frac{z-c}{z} \right)^{\gamma-\alpha}, \quad (10)$$

where

$$\left| \frac{c}{z-c} \right| < 1, \quad \left| \frac{z-c}{z} \right| < 1, \quad \operatorname{Re}(\alpha + \beta) > -1.$$

Because we have

$$\sum_{k=0}^{\infty} \frac{[\gamma - \alpha - m]_k}{k!} \left(\frac{-c}{z-c} \right)^k = \left(\frac{z}{z-c} \right)^{\alpha-\gamma} \left(\frac{z}{z-c} \right)^m \quad (11)$$

since

$$\sum_{k=0}^{\infty} \frac{[\lambda]_k}{k!} z^k = (1-z)^{-\lambda}, \quad (12)$$

and

$${}_2F_1(a, b; c; 1) = \sum_{m=0}^{\infty} \frac{[a]_m [b]_m}{m! [c]_m} = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad \left(\begin{array}{l} \operatorname{Re}(c-a-b) > 0 \\ c \notin \mathbf{Z}_0^- \end{array} \right). \quad (13)$$

Moreover we have the identity

$$\Gamma(\lambda) \Gamma(1-\lambda) = \frac{\pi}{\sin \pi \lambda} \quad (\lambda \notin \mathbf{Z}), \quad (14)$$

then applying (14) to (10) we obtain (3) under the conditions.

§ 3. Some Numerical Examinations for Theorem 1

[I] Examination of Theorem 1. (2) (Doubly infinite sum)

Set

$$c = 1, \quad z = 3, \quad \alpha = 1/4 \quad \text{and} \quad \beta = \gamma = 1/2$$

in Theorem 1. (2) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4, 1/2, 1/2; k, m) \left(\frac{1}{2} \right)^k \left(\frac{2}{3} \right)^m \subseteq \frac{\Gamma(-1/4)}{\Gamma(-3/4)} \left(\frac{2}{3} \right)^{1/4} \quad (1)$$

$$= 3 \cdot \frac{\Gamma(3/4)}{\Gamma(1/4)} \left(e^{i 2n \pi} \frac{2}{3} \right)^{1/4} \quad (n \in \mathbb{Z}) \quad (2)$$

$$= \begin{cases} 0.91622 \dots & (\text{for } n = 0) & (3) \\ i \cdot 0.91622 \dots & (\text{for } n = 1) & (4) \\ -0.91622 \dots & (\text{for } n = 2) & (5) \\ -i \cdot 0.91622 \dots & (\text{for } n = 3) & (6) \end{cases}$$

Now the LHS of (1) is real, then we must choose (3) and (5) from the set $\{(3), (4), (5), (6)\}$.

And now we have

$$\begin{aligned} G(1/4, 1/2, 1/2; 0, 0) \left(\frac{1}{2} \right)^0 \left(\frac{2}{3} \right)^0 & \quad \left(\begin{array}{l} \text{first term of} \\ \text{the LHS of (1)} \end{array} \right) \\ = \frac{\Gamma(1/4)}{\Gamma(-1/4)} = -\frac{1}{4} \cdot \frac{\Gamma(1/4)}{\Gamma(3/4)} < 0. \end{aligned} \quad (7)$$

Then choosing (5) from the set $\{(3), (5)\}$, since the sign of the double infinite sum of the LHS of (1) is decided by the sign of its first term (with $k = m = 0$), when

$$\left| G_{k,m} \left(\frac{1}{3} \right)^k \left(\frac{2}{3} \right)^m \right| > \left| G_{k+1,m+1} \left(\frac{1}{3} \right)^{k+1} \left(\frac{2}{3} \right)^{m+1} \right|, \quad G_{k,m} = G,$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4, 1/2, 1/2; k, m) \left(\frac{1}{2} \right)^k \left(\frac{2}{3} \right)^m = -0.91622 \dots \quad (5)$$

from (1), considering (7).

Indeed we have

$$\begin{aligned} \text{LHS of (5)} &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{5}{4})}{k! \Gamma(\frac{5}{4} - k)} \left(\frac{1}{2} \right)^k \left\{ \frac{\Gamma(k + \frac{1}{4})}{\Gamma(k - \frac{1}{4})} \right. \\ &+ \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2}) \Gamma(k - \frac{3}{4})}{\Gamma(\frac{1}{2}) \Gamma(-\frac{1}{2}) \Gamma(k - \frac{1}{4})} \left(\frac{2}{3} \right) + \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{3}{2}) \Gamma(k - \frac{7}{4})}{2! \Gamma(-\frac{1}{2}) \Gamma(-\frac{1}{2}) \Gamma(k - \frac{1}{4})} \left(\frac{2}{3} \right)^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})\Gamma(k-\frac{11}{4})}{3!\Gamma(-\frac{3}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^3 + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})\Gamma(k-\frac{15}{4})}{4!\Gamma(-\frac{5}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^4 \\
& + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{9}{2})\Gamma(k-\frac{19}{4})}{5!\Gamma(-\frac{7}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^5 + \frac{\Gamma(\frac{3}{2})\Gamma(\frac{11}{2})\Gamma(k-\frac{23}{4})}{6!\Gamma(-\frac{9}{2})\Gamma(-\frac{1}{2})\Gamma(k-\frac{1}{4})}\left(\frac{2}{3}\right)^6 + \dots - \} \quad (8) \\
& = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \left\{ -\frac{1}{4} + \frac{1}{2 \cdot 4^2} - \frac{5}{2! \cdot 2^2 \cdot 4^3} + \frac{5 \cdot 9}{3! \cdot 2^3 \cdot 4^4} - \frac{5 \cdot 9 \cdot 13}{4! \cdot 2^4 \cdot 4^5} + \frac{5 \cdot 9 \cdot 13 \cdot 17}{5! \cdot 2^5 \cdot 4^6} - \dots \right\} \\
& + \frac{\Gamma(\frac{1}{4})}{3 \Gamma(\frac{3}{4})} \left\{ -\frac{1}{2 \cdot 3} - \frac{1}{2^2 \cdot 4} + \frac{1}{2! \cdot 2^3 \cdot 4^2} - \frac{5}{3! \cdot 2^4 \cdot 4^3} + \frac{5 \cdot 9}{4! \cdot 2^5 \cdot 4^4} - \frac{5 \cdot 9 \cdot 13}{5! \cdot 2^6 \cdot 4^5} - \dots \right\} \\
& + \frac{\Gamma(\frac{1}{4})}{2! \cdot 2^2 \cdot 3^2 \Gamma(\frac{3}{4})} \left\{ -\frac{4}{3 \cdot 7} - \frac{1}{2 \cdot 3} - \frac{1}{2! \cdot 2^2 \cdot 4} + \frac{1}{3! \cdot 2^3 \cdot 4^2} - \frac{5}{4! \cdot 2^4 \cdot 4^3} + \frac{5 \cdot 9}{5! \cdot 2^5 \cdot 4^4} - \dots \right\} \\
& - \frac{\Gamma(\frac{1}{4})}{3! \cdot 2^3 \cdot 3 \Gamma(\frac{3}{4})} \left\{ \frac{4^2}{3 \cdot 7 \cdot 11} + \frac{4}{2 \cdot 3 \cdot 7} + \frac{1}{2! \cdot 3} + \frac{1}{3! \cdot 2^3 \cdot 4} \right. \\
& \quad \left. - \frac{1}{4! \cdot 2^4 \cdot 4^2} + \frac{5}{5! \cdot 2^5 \cdot 4^3} - \frac{5 \cdot 9}{6! \cdot 2^6 \cdot 4^4} + \dots \right\} \\
& + \frac{5^2 \Gamma(\frac{1}{4})}{4! \cdot 2^4 \cdot 3^2 \Gamma(\frac{3}{4})} \left\{ -\frac{4^3}{3 \cdot 7 \cdot 11 \cdot 15} - \frac{4^2}{2 \cdot 3 \cdot 7 \cdot 11} - \frac{4}{2! \cdot 2^2 \cdot 3 \cdot 7} - \frac{1}{3! \cdot 2^3 \cdot 3} \right. \\
& \quad \left. - \frac{1}{4! \cdot 2^4 \cdot 4} + \frac{1}{5! \cdot 2^5 \cdot 4^2} - \dots \right\} \\
& - \frac{5^2 \cdot 7^2 \Gamma(\frac{1}{4})}{5! \cdot 2^5 \cdot 3^3 \Gamma(\frac{3}{4})} \left\{ \frac{4^4}{3 \cdot 7 \cdot 11 \cdot 15 \cdot 19} + \frac{4^3}{2 \cdot 3 \cdot 7 \cdot 11 \cdot 15} + \frac{4^2}{2! \cdot 2^2 \cdot 3 \cdot 7 \cdot 11} \right. \\
& \quad \left. + \frac{4}{3! \cdot 2^3 \cdot 3 \cdot 7} + \frac{1}{4! \cdot 2^4 \cdot 3} + \frac{1}{5! \cdot 2^5 \cdot 4} + \dots \right\} + \dots \quad (9) \\
& = - (0.66861\dots) - (0.22272\dots) - (0.01591\dots) - (0.00690\dots) \\
& \quad - (0.00180\dots) - (0.00093\dots) - \dots \quad (10) \\
& = - 0.9168\dots \quad (11)
\end{aligned}$$

[I I] Examination of Theorem 1. (3) (Mixed infinite sum)

Set

$$c = 1, \quad z = 3, \quad \alpha = 1/4, \quad \gamma = 1/2 \quad \text{and} \quad s = 1$$

in Theorem 1. (3) we obtain

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G(1/4, 1, 1/2; k, m) \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m \subseteq \frac{\Gamma(-3/4)}{\Gamma(-5/4)} \left(\frac{2}{3}\right)^{1/4} \quad (12)$$

$$= -\frac{5}{12} \cdot \frac{\Gamma(1/4)}{\Gamma(3/4)} \left(e^{i2n\pi} \frac{2}{3} \right)^{1/4} \quad (n \in \mathbb{Z}) \quad (13)$$

$$= \begin{cases} -1.113943\dots & (\text{for } n=0) \end{cases} \quad (14)$$

$$= \begin{cases} -i \cdot 1.113943\dots & (\text{for } n=1) \end{cases} \quad (15)$$

$$= \begin{cases} 1.113943\dots & (\text{for } n=2) \end{cases} \quad (16)$$

$$= \begin{cases} i \cdot 1.113943\dots & (\text{for } n=3) \end{cases} \quad (17).$$

Now the LHS of (1) is real, then we must choose (14) and (16) from the set { (14), (15), (16), (17) }.

And now we have

$$\begin{aligned} G(1/4, 1, 1/2; 0, 0) \left(\frac{1}{2}\right)^0 \left(\frac{2}{3}\right)^0 & \quad \left(\begin{array}{l} \text{first term of} \\ \text{the LHS of (12)} \end{array} \right) \\ &= \frac{\Gamma(1/4)}{\Gamma(-1/4)} = -\frac{1}{4} \cdot \frac{\Gamma(1/4)}{\Gamma(3/4)} < 0. \end{aligned} \quad (18)$$

Then choosimg (14) from the set { (14), (16) }, since the sign of the double infinite sum of the LHS of (12) is decided by the sign of its first term (with $k = m = 0$), when

$$\left| G_{k,m} \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m \right| > \left| G_{k+1,m+1} \left(\frac{1}{2}\right)^{k+1} \left(\frac{2}{3}\right)^{m+1} \right|, \quad G_{k,m} = G(\alpha, s, \gamma; k, m),$$

we have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^1 G(1/4, 1, 1/2; k, m) \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m = -1.113943\dots, \quad (19)$$

from (13), considering (18).

Indeed we have

$$\begin{aligned} \text{LHS of (12)} &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{5}{4})}{k! \Gamma(\frac{5}{4} - k) \Gamma(k - \frac{1}{4})} \left(\frac{1}{2}\right)^k \\ &\quad \times \sum_{m=0}^1 \frac{\Gamma(\frac{3}{2}) \Gamma(m-1) \Gamma(k-m+\frac{1}{4})}{m! \Gamma(\frac{3}{2} - m) \Gamma(-1)} \left(\frac{2}{3}\right)^m \end{aligned} \quad (20)$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{5}{4}) \Gamma(k+\frac{1}{4})}{k! \Gamma(\frac{5}{4} - k) \Gamma(k - \frac{1}{4})} \left(\frac{1}{2}\right)^k \\ &\quad + \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}) \Gamma(\frac{5}{4}) \Gamma(k - \frac{3}{4})}{k! \Gamma(\frac{5}{4} - k) \Gamma(k - \frac{1}{4})} \left(\frac{2}{3}\right) \left(\frac{1}{2}\right)^k \end{aligned} \quad (21)$$

$$\begin{aligned} &= \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \left\{ -\frac{1}{4} + \frac{1}{2 \cdot 4^2} - \frac{5}{2! \cdot 2^2 \cdot 4^3} + \frac{5 \cdot 9}{3! \cdot 2^3 \cdot 4^4} - \frac{5 \cdot 9 \cdot 13}{4! \cdot 2^4 \cdot 4^5} + \frac{5 \cdot 9 \cdot 13 \cdot 17}{5! \cdot 2^5 \cdot 4^6} - \dots \right\} \\ &\quad - \frac{\Gamma(\frac{1}{4})}{3 \Gamma(\frac{3}{4})} \left\{ \frac{1}{3} + \frac{1}{8} - \frac{1}{2! \cdot 8^2} + \frac{5}{3! \cdot 8^3} - \frac{5 \cdot 9}{4! \cdot 8^4} + \frac{5 \cdot 9 \cdot 13}{5! \cdot 8^5} - \dots \right\} \end{aligned} \quad (22)$$

$$\begin{aligned} &= \frac{\Gamma(1/4)}{\Gamma(3/4)} \{ -0.25 + 0.03125 - (0.009765\dots) + (0.003662\dots) \\ &\quad - (0.00487\dots) + (0.000632\dots) - \dots \} \\ &\quad - \frac{1}{3} \cdot \frac{\Gamma(1/4)}{\Gamma(3/4)} \{ (0.333333\dots) + 0.125 - (0.007812\dots) + (0.001627\dots) \\ &\quad - (0.000457\dots) + (0.000148\dots) - \dots \} \end{aligned} \quad (23)$$

$$= (-0.667796\dots) + (-0.445583\dots) \quad (24)$$

$$= -1.113379\dots \quad (25)$$

[I I I] Examination of Corollary 1 (Doubly finite sum)

Set

$$c = 1, \quad z = 3, \quad r = 3, \quad \beta = -1/2 \quad \text{and} \quad p = 2$$

in Corollary 1, we obtain

$$\sum_{k=0}^3 \sum_{m=0}^2 G(3, -1/2, 2; k, m) \left(\frac{1}{2}\right)^k \left(\frac{2}{3}\right)^m = \frac{\Gamma(-1/2)}{\Gamma(-5/2)} \left(\frac{2}{3}\right)^{-1} = \frac{45}{8} \quad (26)$$

Indeed we have

$$\text{LHS of (26)} = \sum_{k=0}^3 \frac{3! \cdot 2!}{k! \Gamma(4-k)} \left(\frac{1}{2}\right)^k \sum_{m=0}^2 \frac{\Gamma(m+\frac{1}{2}) \Gamma(k-m-1)}{m! \Gamma(3-m) \Gamma(\frac{1}{2}) \Gamma(k-3)} \left(\frac{2}{3}\right)^m \quad (27)$$

$$= \sum_{k=0}^3 \frac{6\Gamma(k-1)}{k! \Gamma(4-k) \Gamma(k-3)} \left(\frac{1}{2}\right)^k + \sum_{k=0}^3 \frac{4\Gamma(k-2)}{k! \Gamma(4-k) \Gamma(k-3)} \left(\frac{1}{2}\right)^k \\ + \sum_{k=0}^3 \frac{2}{k! \Gamma(4-k)} \left(\frac{1}{2}\right)^k \quad (28)$$

$$= \left\{ \frac{\Gamma(-1)}{\Gamma(-3)} + \frac{3}{2} \cdot \frac{\Gamma(0)}{\Gamma(-2)} + \frac{3}{4} \cdot \frac{1}{\Gamma(-1)} + \frac{1}{8} \cdot \frac{1}{\Gamma(0)} \right\} \\ + \left\{ \frac{2}{3} \cdot \frac{\Gamma(-2)}{\Gamma(-3)} + \frac{\Gamma(-1)}{\Gamma(-2)} + \frac{1}{2} \cdot \frac{\Gamma(0)}{\Gamma(-1)} + \frac{1}{12} \cdot \frac{1}{\Gamma(0)} \right\} \\ + \left\{ \frac{1}{3} + \frac{1}{2} + \frac{1}{4} + \frac{1}{24} \right\} \quad (29)$$

$$= \{6+3+0+0\} + \{-2-2-(1/2)+0\} + (9/8) = \frac{45}{8} . \quad (30)$$

In this example both of the LHS and RHS of (26) are one valued functions respectively, then we have the identity (notation =), without the ad hoc shown in [I] and [II].

§ 4. Commentary

[I] Applying N- fractional calculus to some power functions we obtain the result shown by § 1. (2).

On the other hand we obtain the identity § 2. (3), by the direct calculation of its LHS. However

the LHS of § 2. (3) is always one valued function,
on the contrary

the RHS of § 2. (3) is many valued function for $(\gamma - \alpha) \notin \mathbb{Z}$,
and one valued one for $(\gamma - \alpha) \in \mathbb{Z}$.

We have then

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G \cdot \left(\frac{z-c}{z}\right)^m \left(\frac{c}{z-c}\right)^k \subseteq P \cdot \frac{\Gamma(\gamma - \alpha - \beta)}{\Gamma(-\alpha - \beta)} \left(\frac{z-c}{z}\right)^{\gamma - \alpha} \quad (1)$$

from § 2. (3), considering § 1. (13), in complex analysis.

[I I] Therefore, when

$$|P| = |P(\alpha, \beta, \gamma)| = 1, \quad (2)$$

§ 1. (2) and (1) overlaps each other, because

$$\left\{ \text{Set of the elements of } \left(\frac{z-c}{z} \right)^{\gamma-\alpha} \right\} = \left\{ \text{Set of the elements of } - \left(\frac{z-c}{z} \right)^{\gamma-\alpha} \right\} \quad (3)$$

$$((\gamma - \alpha) \notin \mathbb{Z}).$$

That is, the result § 1. (2) which is obtained by the use of N- fractional calculus is a special case of (1) which is the result obtained by the direct calculation of its LHS.

Namely the space $|P| = |P(\alpha, \beta, \gamma)| = 1$ in which the result § 1. (2) holds is a subspace of the space such that $|P| = |P(\alpha, \beta, \gamma)| = M < \infty$ in which the result (1) holds good.

Note, When

$$c=1, \quad z=3, \quad \gamma=1/2, \quad \alpha=1/4,$$

we have

$$\text{LHS of (3)} = \left\{ (2/3)^{1/4}, i(2/3)^{1/4}, -(2/3)^{1/4}, -i(2/3)^{1/4} \right\}$$

and

$$\text{RHS of (3)} = \left\{ -(2/3)^{1/4}, -i(2/3)^{1/4}, (2/3)^{1/4}, i(2/3)^{1/4} \right\},$$

for example.

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Katsuyuki Nishimoto
 Institute of Applied Mathematics
 Descartes Press Co.
 2 - 13 - 10 Kaguike, Koriyama
 963 - 8833 Japan