数理解析研究所講究録
1163 巻 2000 年 99-102

Notes on discrete subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg translations II

Shigeyasu KAMIYA

神谷茂保　(岡山理科大学)

In a previous paper [8] we have seen that under some conditions Parker's theorem yields the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. We show that we can obtain the same result as in [8] without the assumption on r.

1. First we recall some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V=V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*,w^*)=-(\overline{z_0^*}w_1^*+\overline{z_1^*}w_0^*)+\overline{z_2^*}w_2^*$$

for $z^*=(z_0^*,z_1^*,z_2^*), w^*=(w_0^*,w_1^*,w_2^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*),g(w^*))=\widetilde{\Phi}(z^*,w^*)$ for z^*,w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbf{C})$. Let $V_0=\{w^*\in V|\quad \widetilde{\Phi}(w^*,w^*)=0\}$ and $V_-=\{w^*\in V|\quad \widetilde{\Phi}(w^*,w^*)<0\}$. It is clear that V_0 and V_- are invariant under $U(1,2;\mathbf{C})$. We denote $U(1,2;\mathbf{C})/(center)$ by $PU(1,2;\mathbf{C})$. Set $V^*=V_-\cup V_0-\{0\}$. Let $\pi:V^*\longrightarrow\pi(V^*)$ be the projection map defined by $\pi(w_0^*,w_1^*,w_2^*)=(w_1,w_2)$, where $w_1=w_1^*/w_0^*$ and $w_2=w_2^*/w_0^*$. We write ∞ for $\pi(0,1,0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^2=\{w=(w_1,w_2)\in\mathbf{C}^2\quad|\quad Re(w_1)>\frac{1}{2}|w_2|^2\}.$$

We can regard an element of $PU(1,2;\mathbf{C})$ as a transformation acting on H^2 and its boundary ∂H^2 (see [6]). Denote $H^2\cup\partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2}-\{\infty\}$. Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The $H-coordinates$ of a point $(w_1,w_2)\in\overline{H^2}-\{\infty\}$ are defined by $(k,t,w_2)_H\in(\mathbf{R}^+\cup\{\mathbf{0}\})\times\mathbf{R}\times\mathbf{C}$ such that $k=Re(w_1)-\frac{1}{2}|w_2|^2$ and $t=Im(w_1)$. For simplicity, we write $(t_1,w')_H$ for $(0,t_1,w')_H$. The *Cygan metric* $\rho(p,q)$ for $p=(k_1,t_1,w')_H$ and $q=(k_2,t_2,W')_H$ is given by

$$\rho(p,q)=|\{\frac{1}{2}|W'-w'|^2+|k_2-k_1|\}+i\{t_1-t_2+Im(\overline{w'}W')\}|^{\frac{1}{2}}.$$

We note that the Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^2 (see [7]).

Let $f = (a_{ij})_{1\leq i,j\leq 3}$ be an element of $PU(1,2;\mathbf{C})$ with $f(\infty) \neq \infty$. We define the *isometric sphere* I_f of f by

$$I_f = \{w = (w_1, w_2) \in \overline{H}^2 \mid |\tilde{\Phi}(W,Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0,1,0)$, $W = (1, w_1, w_2)$ in V^* (see [4]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k,t,w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

2. We shall give a modified version of the stable basin theorem of Basmajian and Miner in [1]. Let

$$B_r = \{z \in \partial H^2 \mid \delta(z, 0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 - \overline{B}_s$. Given r and s with $r < s$, the pair of open sets $(B_r, \overline{B}_s^c)$ is said to be *stable* with respect to a set S of elements in $PU(1,2;\mathbf{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s^c.$$

A loxodromic element f has a unique complex dilation factor $\lambda(f)$ such that $|\lambda(f)| > 1$. Let $S(r,\varepsilon)$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon$. For positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\varepsilon(r,r')$ by

$$\varepsilon(r,r') = \sup\{|\lambda(f) - 1|\}, \tag{2.1}$$

where $|\lambda(f) - 1|$ satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{2 + \left(\frac{1-(3+|\lambda(f)-1|)r^2)}{1-2r^2}\right)^2 \left(\frac{1-3r^2}{1-r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - \sqrt{2}. \tag{2.2}$$

A triple of non-negative numbers (r, r', ε) is said to be a *basin point* provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\varepsilon < \varepsilon(r,r')$. In particular, if $r' \leq r$, we call (r, r', ε) a *stable basin point.* Call the set of all such points the *stable basin region.* For simplicity, we abbreviate (r, r, ε) to (r, ε)

Theorem 2.1 ([8; Theorem 3.1]). *Given positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}, \overline{B}_{1/r'}^c)$ is stable with respect to the family $S(r, \varepsilon(r,r'))$, where $\varepsilon(r,r')$ is given by (2.1).*

3. We begin with introducing Parker's theorem on the discreteness of subgroups of $PU(1,2;\mathbf{C})$.

Theorem 3.1 ([9; Theorem 2.1]). *Let g be a Heisenberg translation with the form*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a} \\ a & 0 & 1 \end{pmatrix},$$

where $Re(s) = \frac{1}{2}|a|^2$. Let f be any element of $PU(1,2;\mathbf{C})$ with isometric sphere of radius R_f. If

$$R_f^2 > \delta(gf^{-1}(\infty), f^{-1}(\infty))\delta(gf(\infty), f(\infty)) + 2|a|^2,$$

then the group $< f, g >$ generated by f and g is not discrete.

Remark 3.2. Suppose that g is a vertical Heisenberg translation. As $a = 0$, this theorem is equivalent to results in [5] and [6].

In Theorem 4.5 of [8] we have shown that if $r < 0.484$, then Theorem 3.1 leads to the discreteness condition of Basmajian and Miner for groups with a Heisenberg translation. By using a more precise estimate on the Heisenberg distance between fixed points of f in terms of R_f and $\lambda(f)$, we have the following same result without the assumption on r.

Theorem 3. 3. *Fix a stable basin point (r, ε). Let g be the same element as in Theorem 3.1. Let f be a loxodromic element with fixed point 0 and q, and satisfying $|\lambda(f) - 1| < \varepsilon$. If $\delta(0, q) > \frac{\delta(0, g(0))}{r^2}(1 + r^2 + \sqrt{1 + r^2})$, then the group $< f, g >$ generated by f and g is not discrete.*

References

1. A. Basmajian and R. Miner, Discrete subgroups of complex hyperbolic motions, Invent. Math. 131, 85-136 (1998).
2. A.F. Beardon, The Geometry of Discrete Groups, Springer-Verlag, New York, 1983.
3. L. R. Ford, Automorphic Functions (Second Edition), Chelsea, New York, 1951.
4. W. M. Goldman, Complex hyperbolic geometry, Oxford University Press, 1999.
5. S. Kamiya, Notes on non-discrete subgroups of $\tilde{U}(1, n; F)$, Hiroshima Math. J. 13, 501-506, (1983).
6. S. Kamiya, Notes on elements of $U(1, n; C)$, Hiroshima Math. J. 21, 23-45, (1991).

7. S. Kamiya, Parabolic elements of $U(1, n; C)$, Rev. Romaine Math. Pures et Appl. 40, 55-64, (1995).
8. S. Kamiya, On discrete subgroups of $PU(1, 2; \mathbf{C})$ with Heisenberg translations, to appear in J. London Math. Soc.
9. J. Parker, Uniform discreteness and Heisenberg translations, Math. Z. 225, 485-505 (1997).

Okayama University of Science
1-1 Ridai-cho, Okayama 700-0005 JAPAN
e-mail:kamiya@mech.ous.ac.jp