

On the state numbers of a virtual knot

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1 Introduction

This note is mainly a summary of our studies for the “state numbers” of a virtual knot obtained in [5] and [6].

In knot theory, there are many “minimal-type” numerical invariants of a knot K . For example, the crossing number $c(K)$ for K is the minimal number of crossings in any diagrams of K , and the unknotting number $u(K)$ is the minimal number of crossing changes in any diagrams of K needed to create a diagram of the unknot. Those invariants measure a certain complexity of the knot.

In [5], we define the n -state number $s_n(K)$ of a virtual knot K . A *state* S of a virtual knot diagram D is a union of circles obtained from D by splicing all real crossings in D . Let $s_n(D)$ be the number of states of D consisting of n circles. The n -state number of a virtual knot K is defined to be the minimal number of $s_n(D)$ for all possible diagram of K . In this note, we show some properties of the n -state numbers of a virtual knot. First, we give upper and lower bounds for the n -state numbers of a virtual knot K for $n = 1, 2, 3$ in terms of the real crossing number of K . Second, we consider a set of virtual knots whose n -state number is equal to i for each non-negative integer i and study the finiteness of the set. Finally, we give lower bounds for the 1-state number $s_1(K)$ of a virtual knot K in terms of a special value of the Jones polynomial and the Miyazawa polynomial of K .

2 The state numbers for a virtual knot

A *virtual knot diagram* D is an immersed circle in the plane $\mathbb{R}^2$ whose double points are ordinary crossings, which are called *real crossings*  and *virtual crossings* .

A *virtual knot* K is an equivalence class of virtual knot diagrams under *generalized Reidemeister moves* as in Figure 1 (cf. [3]).

A *state* S of a virtual knot diagram D is a union of circles possibly with virtual crossings obtained from D by splicing all real crossings as in Figure 2. A state S is said to be an n -state if S consists of n circles. We denote by $s_n(D)$ the number of n -states of D .

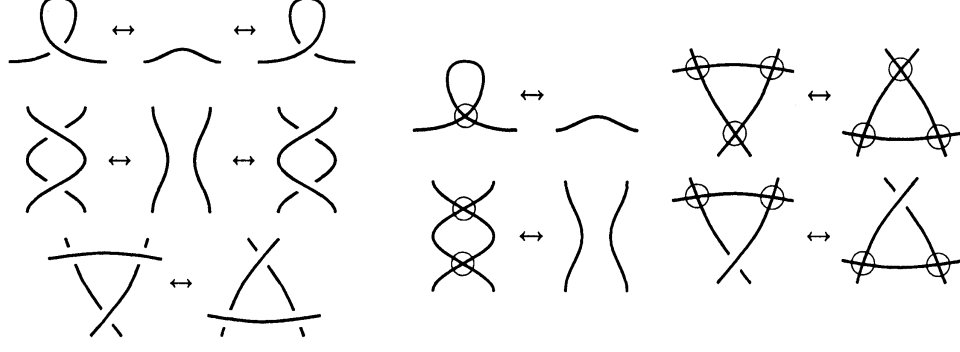


Figure 1:

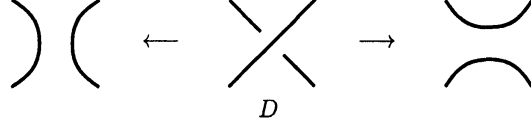


Figure 2:

Example 2.1. Let D be the virtual knot diagram as in Figure 3 (a), which presents the virtual knot labeled by 2.1 in Green's table [1]. By splicing the real crossings in D , we obtained four states as in Figure 3 (b). Three of them are 1-states and the other is a 2-state. Hence we have $s_1(D) = 3$, $s_2(D) = 1$, $s_i(D) = 0$ ($i \geq 3$).

Let K be a virtual knot. The n -state number of K , denoted by $s_n(K)$, is defined to be the minimal number of $s_n(D)$ for all possible virtual knot diagrams D of K (cf. [5]). For example, we can easily see that $s_1(K) = 1$ and $s_i(K) = 0$ ($i \geq 2$) if K is trivial.

Let D be an oriented virtual knot diagram. We regard D as the image of an immersion of a circle into $\mathbb{R}^2$ with crossing information at each double point. The *Gauss diagram* of D is an oriented circle regarded as the preimage of the immersed circle with chords, each of which connects the preimages of each double point corresponding to a real crossing. A chord is oriented from the preimage of the over-crossing-point to that of the under-crossing-point in the circle, and labeled by the sign of the corresponding real crossing. Figure 4 illustrates an example of a virtual knot diagram and its Gauss diagram.

Two chords of a Gauss diagram G is *linked* if their end-points appear along the circle of G alternately. A chord is *free* if it is not linked with any other chords.

Lemma 2.2 ([5]). *Let D and D' be virtual knot diagrams with the same Gauss diagram by ignoring the orientation of the circle and the orientation and sign of each chord. Then $s_n(D) = s_n(D')$ holds for any natural number n .*

By Lemma 2.2, $s_n(D)$ is determined by its unoriented and unsigned Gauss diagram G . In this sense, we denote $s_n(D)$ by $s_n(G)$.

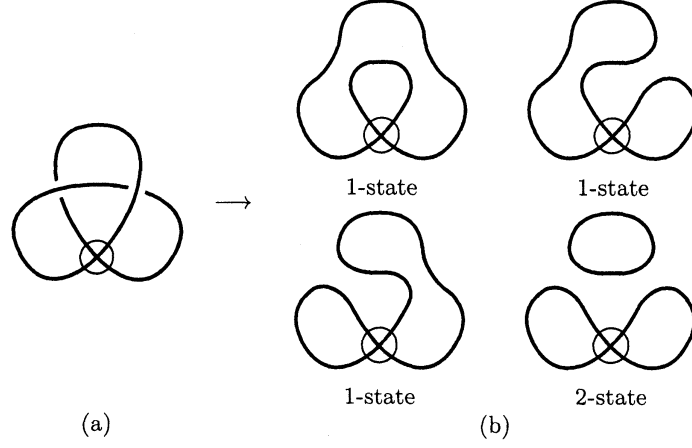


Figure 3:

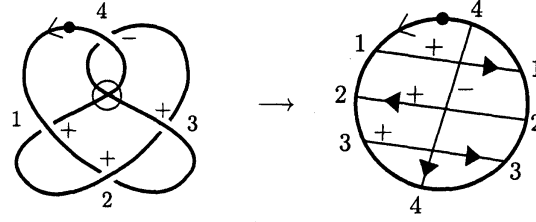


Figure 4:

3 Bounds for 1-, 2-, and 3-state numbers

In this section, we give upper bounds and lower bounds for the n -state number of a virtual knot K for $n = 1, 2, 3$. Let $c(K)$ be the *minimal real crossing number* for K , that is, the minimal number of real crossings for all possible virtual knot diagrams of K .

Theorem 3.1 ([5]). *Any virtual knot K satisfies (1) $1 \leq s_1(K) \leq \frac{2 \cdot 2^{c(K)} + (-1)^{c(K)}}{3}$, (2) $0 \leq s_2(K) \leq \frac{1}{2} \cdot 2^{c(K)}$, and (3) $0 \leq s_3(K) \leq \frac{3}{8} \cdot 2^{c(K)}$.*

The lower bounds of (2) and (3) in Theorem 3.1 are obvious. For any virtual knot diagram D , we see that there is at least one sequence of virtual knot diagrams $D = D_0, D_1, D_2, \dots, D_m$ such that D_i is obtained from D_{i-1} by splicing a real crossing in D_{i-1} ($i = 1, 2, \dots, m$) and D_m has no real crossing. This gives the lower bound of (1) in Theorem 3.1. The upper bounds in Theorem 3.1 are given by the following lemma.

Lemma 3.2 ([5]). *Let G be a Gauss diagram of one circle with r chords. Then we have (1) $s_1(G) \leq \frac{2 \cdot 2^r + (-1)^r}{3}$, (2) $s_2(G) \leq \frac{1}{2} \cdot 2^r$, and (3) $s_3(G) \leq \frac{3}{8} \cdot 2^r$.*

We remark that G produces 2^r states. The following examples of Gauss diagrams realizes upper bounds in Lemma 3.2.

Example 3.3. Let F_r be a Gauss diagram as in Figure 5. Then we have $s_1(F_r) = \frac{2 \cdot 2^r + (-1)^r}{3}$.

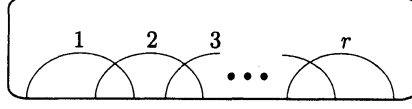


Figure 5:

For $r \leq 3$, there exists a virtual knot K presented by F_r with $s_1(K) = \frac{2 \cdot 2^r + (-1)^r}{3}$. It is an open question whether there exists a virtual knot K presented by F_r realizing $s_1(K) = \frac{2 \cdot 2^r + (-1)^r}{3}$ for $r \geq 4$.

Example 3.4. Let F'_r ($r \geq 3$) be a Gauss diagram as in Figure 6 (a) and $F_{r-1} + 1$ a Gauss diagram as in Figure 6 (b). A Gauss diagram $F_{r-1} + 1$ is obtained from F_{r-1} by adding a free chord in any place. Then we have $s_2(F'_r) = s_2(F_{r-1} + 1) = \frac{1}{2} \cdot 2^r$.

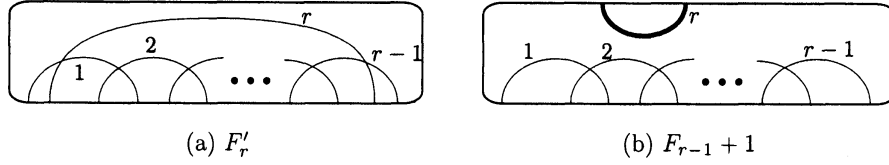


Figure 6:

There exists a virtual knot presented by F'_3 with $s_2(K) = 4$. It is an open question whether there exists a virtual knot K presented by F'_r realizing $s_2(K) = \frac{1}{2} \cdot 2^r$ for $r \geq 4$.

Example 3.5. A Gauss diagram $F'_{r-2} + 2$ as in Figure 7 (a) is obtained from F'_{r-2} by adding two free chords and a Gauss diagram $F'_{r-3} + 3$ as in Figure 7 (b) is obtained from F'_{r-3} by adding three free chords in any place. Then we have $s_3(F'_{r-2} + 2) = s_3(F'_{r-3} + 3) = \frac{3}{8} \cdot 2^r$.

4 The number of virtual knots with a given state number

Let $\mathcal{S}_n(i)$ be the set of virtual knots with $s_n(K) = i$ for a non-negative integer i . In this section, we consider the finiteness of $\mathcal{S}_n(i)$.

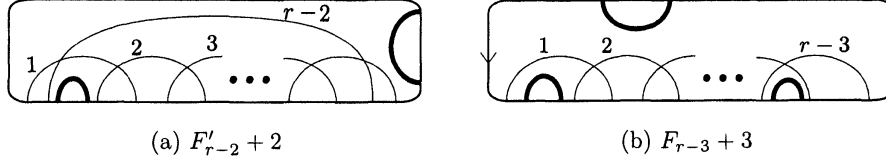


Figure 7:

Proposition 4.1 ([5]). (1) Both of $\mathcal{S}_1(1)$ and $\mathcal{S}_2(0)$ consist of the trivial knot. (2) $\mathcal{S}_1(2k)$ is the empty set for $k \geq 0$.

It is showed in [5] that $c(K) \leq s_2(K)$ for any virtual knot K with $c(K) \geq 3$ and $s_2(K) \leq 1$ for any virtual knot $c(K) \leq 2$.

Theorem 4.2. $\mathcal{S}_2(i)$ is finite for any non-negative integer i .

On the other hand, the cases of $n \geq 3$ are different from that of $n = 2$. It is showed that $\mathcal{S}_n(0)$ is a subset of $\mathcal{S}_{n+1}(0)$ ([5]) and $\mathcal{S}_3(0)$ contains infinitely many virtual knots ([6]).

Theorem 4.3. $\mathcal{S}_n(0)$ is infinite for $n \geq 3$.

In addition, it is showed in [6] that any non-trivial virtual knot contained in $\mathcal{S}_3(0)$ is *non-classical*, that is, it has no diagram without virtual crossings.

In [5], for any Gauss diagram G of a virtual knot diagram D it is showed that $s_1(G) \geq r(G) - f(G)$, where $r(G)$ is the number of chords and $f(G)$ is the number of free chords in G . For any Gauss diagram G whose $s_1(G)$ is an odd prime, we see that there exists at least one free chord in G corresponding to a real crossing in D which can be eliminated by a Reidemeister move I.

Theorem 4.4. (1) $\mathcal{S}_1(i)$ is finite for any odd prime i . In particular, $\mathcal{S}_1(3)$ consists of four virtual knots 2.1, 3.5, 3.6, and 3.7 in Green's table.

(2) $\mathcal{S}_1(9)$ is infinite.

The outline of a proof of Theorem 4.4 (2) is the following. Let D_m be a virtual knot diagram as in Figure 8 (a) and K_m the virtual knot presented by D_m . The Gauss diagram G_m of D_m is illustrated in Figure 8 (b). We see that $s_1(D_m) = 9$ for any non-negative integer m . Thus we have $s_1(K_m) \leq 9$. We can show that K_m is not equivalent to $K_{m'}$ if $m \neq m'$ by the Miyazawa polynomial. Since $\mathcal{S}_1(i)$ is finite for $i \leq 7$ by Proposition 4.1 (1) and Theorem 4.4 (1), $\{K_m\}_{m \geq 0}$ contains infinitely many virtual knots whose 1-state number is equal to 9.

5 Lower bounds for the 1-state number by polynomial invariants

The Jones polynomial of an ordinary knot is naturally generalized to that of a virtual knot through Kauffman's f -polynomial. Let K be an oriented virtual knot and D an

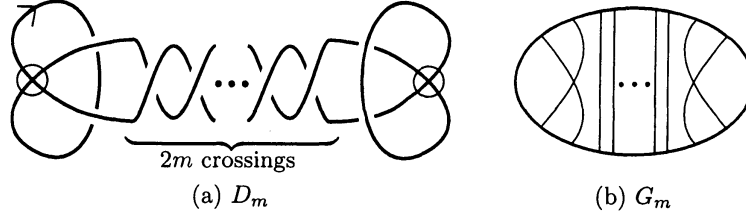


Figure 8:

oriented virtual knot diagram of K with the writhe $w(D)$. Let S be a state of the unoriented D . Let $a(S)$ (resp. $b(S)$) be the number of A-splices (resp. B-splices) to obtain S from D as in Figure 9. We denote by $|S|$ the number of circles in S . Then Kauffman's f -polynomial is defined as

$$f_K(A) = (-A^{-3})^{w(D)} \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{|S|-1} \in \mathbb{Z}[A, A^{-1}].$$

By substituting $A = t^{-\frac{1}{4}}$, we obtain the Jones polynomial $V_K(t)$ of K .

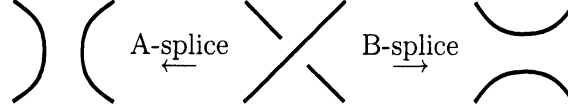


Figure 9:

Proposition 5.1 ([5]). *For any virtual knot K , we have $s_1(K) \geq |f_K(\xi)| = |V_K(-1)|$, where $\xi = e^{\frac{\pi}{4}i}$.*

We review the definition of the Miyazawa polynomial of an oriented virtual knot K ([4]) as the state-sum of *poled states* ([2]).

Let K be an oriented virtual knot and D an oriented virtual knot diagram of K with the writhe $w(D)$. There are two ways of splicing at a real crossing c in D with respect to the orientation of arcs. One of which is coherent, otherwise is non-coherent. For a non-coherent splicing, we set up a pair of poles on spliced arcs as in Figure 10. If c is positive, then A-splicing at c is coherent and B-splicing is non-coherent. By splicing all real crossing of D , we obtained a state with poles. A state with poles is called a poled state.

Let S be a poled state of D and $a(S)$ (resp. $b(S)$) the number of A-splices (resp. B-splices) to obtain S from D . Let $\mathcal{C}(S)$ be the set of circles in a poled state S .

We note that the number of poles on $C \in \mathcal{C}(S)$ is always even. We reduce the number of poles on C by the following moves: A pole can slide along C and pass a virtual crossing. If there exist two successive poles on the same side of C , then they are canceled. See Figures 11 (a) and (b). Let $\tilde{C}$ be the circle with poles obtained from C after reducing poles on C .

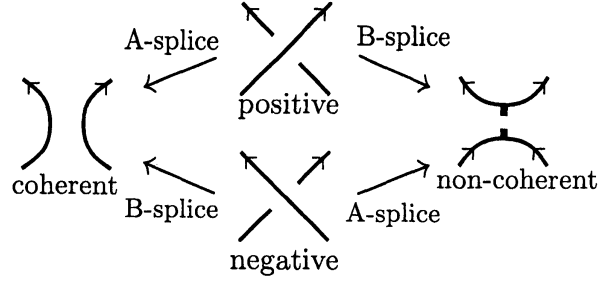


Figure 10:

as possible. Then the poles on $\tilde{C}$ stand on the left and right side of $\tilde{C}$ alternately as in Figure 11 (c).

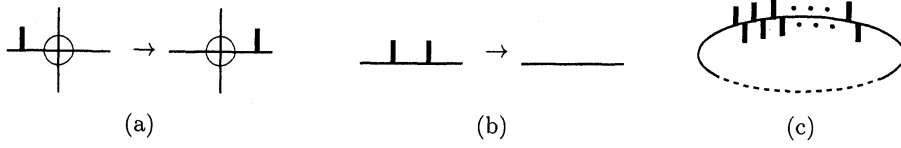


Figure 11:

Let $\lambda(C)$ be the half of the number of poles on $\tilde{C}$ and $c_i(S)$ the number of circles in S with $\lambda(C) = i$. Then the Miyazawa polynomial $R_K(A, \vec{x}) \in \mathbb{Z}[A, A^{-1}, x_1, x_2, \dots]$ of K is defined by

$$R_K(A, \vec{x}) = (-A^3)^{-w(D)} \sum_S A^{a(S)-b(S)} (-A^2 - A^{-2})^{c_0(S)} x_1^{c_1(S)} x_2^{c_2(S)} \dots$$

Example 5.2. Let D be the virtual knot diagram as indicated in Figure 12, which presents the virtual knot labeled by 4.81 in Green's table. Since the number of real crossings of D is equal to four, D has 16 poled states. Each poled state S is assigned by $A^{a(S)-b(S)} (-A^2 - A^{-2})^{c_0(S)} x_1^{c_1(S)} x_2^{c_2(S)} \dots$ according to the information of the number of A- or B-splices, the number of poles on each circles in S as in Figure 12, where $d = -A^2 - A^{-2}$. Then the Miyazawa polynomial of K is given by

$$R_K(A, \vec{x}) = (-A^2 - A^{-2})(A^8 - A^4 + 1) + (A^6 - A^2 + A^{-2})x_1 + A^{10}x_3 + A^8x_1x_2.$$

We describe the Miyazawa polynomial $R_K(A, \vec{x})$ of K as

$$R_K(A, \vec{x}) = F_0(A) + \sum_I F_I(A) x_1^{c_1} x_2^{c_2} \dots x_k^{c_k},$$

where I is a finite sequence of integers such that $I = (c_1, c_2, \dots, c_k)$ with $c_i \geq 0$ ($i = 1, 2, \dots, k-1$) and $c_k > 0$. It is known that $F_0(A)$ is divisible by $-A^2 - A^{-2}$ ([4]). Let $\tilde{F}_0(A) = F_0(A)/(-A^2 - A^{-2})$.

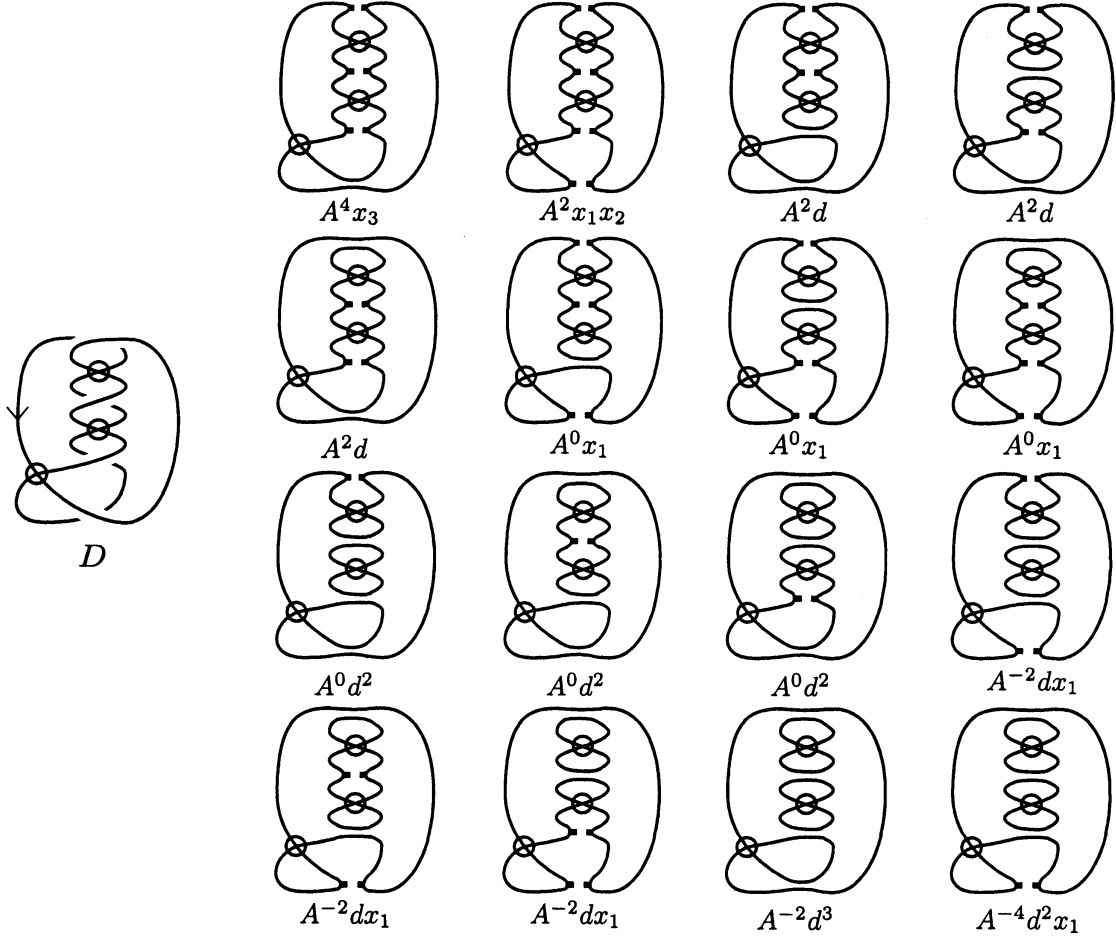


Figure 12:

Theorem 5.3 ([5]). *For any virtual knot K , we have $s_1(K) \geq \left| \tilde{F}_0(\xi) \right| + \sum_{k=1}^{\infty} |F_{I_k}(\xi)| \geq |f_A(\xi)|$, where $I_k = (\underbrace{0, 0, \dots, 0}_{k-1}, 1)$ and $\xi = e^{\frac{\pi}{4}i}$.*

Example 5.4. Let K be the virtual knot 4.81 in Example 5.2. We see that there are seven 1-states obtained from D as in Figure 12. Hence we have $s_1(K) \leq 7$. By direct calculation, we have $f_K(A) = A^8 - A^4 - A^2 + 1 + A^{-2}$, and hence $|f_K(\xi)| = \sqrt{13}$. By Proposition 5.1, $\sqrt{13} < 4 \leq s_1(K) \leq 7$.

Since $R_K(A, \vec{x}) = (-A^2 - A^{-2})(A^8 - A^4 + 1) + (A^6 - A^2 + A^{-2})x_1 + A^{10}x_3 + A^8x_1x_2$, we have $\tilde{F}_0(A) = A^8 - A^4 + 1$, $F_{(1)} = A^6 - A^2 + A^{-2}$, and $F_{(0,0,1)}(A) = A^{10}$. Then we see

that

$$\left| \widetilde{F}_0(\xi) \right| + \sum_{k=1}^{\infty} |F_{I_k}(\xi)| = \left| \widetilde{F}_0(\xi) \right| + |F_{(1)}(\xi)| + |F_{(0,0,1)}(\xi)| = 7.$$

Therefore we have $s_1(K) = 7$.

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