

Some Coefficient Inequalities and Distortion Bounds Associated with Certain New Subclasses of Analytic Functions

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Abstract

The authors introduce and investigate two new subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ of normalized analytic functions satisfying certain coefficient inequalities in the open unit disk $\mathbb{U}$. The main results of the present paper provide various interesting properties of functions belonging to the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$. Some of these properties include (for example) several coefficient inequalities, distortion bounds and inclusion relationships for the function classes which are considered here.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* in the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* in the *open* unit disk

$$(1.2) \quad \mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are also *univalent* in $\mathbb{U}$.

Let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$(1.3) \quad \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

A function $f \in \mathcal{S}^*(\alpha)$ is said to be *starlike of order α* in $\mathbb{U}$. Furthermore, let $\mathcal{K}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy the following inequality:

$$(1.4) \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

A function $f \in \mathcal{K}(\alpha)$ is said to be *convex of order α* in $\mathbb{U}$. We note that

$$f(z) \in \mathcal{K}(\alpha) \iff zf'(z) \in \mathcal{S}^*(\alpha).$$

(See, for details, [1] and [2]; see also [3] and [6], and the references cited therein.)

About three decades ago, Silverman [5] gave the following coefficient inequalities for the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

Theorem A (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$(1.5) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then

$$(1.6) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1),$$

that is, that $f(z) \in \mathcal{S}^(\alpha)$.*

Theorem B (Silverman [5]). *If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$(1.7) \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq 1 - \alpha \quad (0 \leq \alpha < 1),$$

then

$$(1.8) \quad \left| \frac{zf''(z)}{f'(z)} \right| < 1 - \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1),$$

that is, that $f(z) \in \mathcal{K}(\alpha)$.

More recently, Sekine and Owa [4] considered the subclass of functions $f \in \mathcal{A}$ which satisfy the following inequality:

$$(1.9) \quad \left| \frac{zf'(z)}{f(z)} - a \right| < a - \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1; a > \alpha).$$

In this paper, we consider a new subclass $\mathcal{M}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ such that

$$(1.10) \quad \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in \mathbb{U}; 0 < \alpha < 1).$$

We also introduce and investigate here the subclass $\mathcal{N}(\alpha)$ of the class $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following inclusion relationship:

$$zf'(z) \in \mathcal{M}(\alpha).$$

Let us now define the function $F(z)$ by

$$F(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{M}(\alpha)).$$

Then $f(z)$ satisfies the inequality:

$$(1.11) \quad F(z) + \bar{F}(z) > 2\alpha \quad (z \in \mathbb{U}; 0 < \alpha < 1),$$

so that

$$(1.12) \quad \Re(F(z)) = \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}; 0 < \alpha < 1).$$

It follows from (1.12) that

$$\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{N}(\alpha) \subset \mathcal{K}(\alpha).$$

Example. Let us consider the function given by

$$(1.13) \quad f(z) = z + \frac{1}{k}z^2 \quad (k \geq 2).$$

Then we have

$$(1.14) \quad \frac{zf'(z)}{f(z)} - 1 = \frac{k+2z}{k+z} - 1 = \frac{z}{k+z}.$$

Since

$$(1.15) \quad \left| \frac{z}{k+z} + \frac{1}{k^2-1} \right| < \frac{k}{k^2-1} \quad (z \in \mathbb{U}),$$

we see that

$$(1.16) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1}{k-1} = 1 - \frac{k-2}{k-1}$$

which readily implies that

$$(1.17) \quad f(z) \in \mathcal{S}^*\left(\frac{k-2}{k-1}\right).$$

On the other hand, we observe that

$$(1.18) \quad \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} = \frac{k+z}{k+2z} - \frac{1}{2\alpha} = \frac{1}{2} \left(1 - \frac{1}{\alpha} + \frac{k}{k+2z} \right).$$

Noting also that

$$(1.19) \quad \left| \frac{k}{k+2z} - \frac{k^2}{k^2-4} \right| < \frac{2k}{k^2-4} \quad (z \in \mathbb{U}),$$

we have

$$(1.20) \quad \left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2}A(k, \alpha),$$

where

$$(1.21) \quad A(k, \alpha) := \max \left\{ \left| 1 - \frac{1}{\alpha} + \frac{k}{k+2} \right|, \left| 1 - \frac{1}{\alpha} + \frac{k}{k-2} \right| \right\}.$$

Thus we obtain

$$(1.22) \quad \frac{1}{\alpha} = \frac{1}{2}A(k, \alpha)$$

for $f(z) \in \mathcal{M}(\alpha)$. Let us put $\alpha = \alpha_0$. If α is given by

$$(1.23) \quad \alpha = \frac{1}{A(k, \alpha)},$$

then $f(z) \in \mathcal{M}(\alpha_0)$. By the fact that $\mathcal{M}(\alpha) \subset \mathcal{S}^*(\alpha)$, we have

$$(1.24) \quad \alpha_0 \geq \frac{k-2}{k-1}.$$

If we set

$$(1.25) \quad \alpha = \frac{k-2}{k-1},$$

then we have

$$(1.26) \quad 1 - \frac{1}{\alpha} + \frac{k}{k+2} = 1 - \frac{k-1}{k-2} + \frac{k}{k+2} = \frac{(k+1)(k-4)}{(k+2)(k-2)}$$

and

$$(1.27) \quad 1 - \frac{1}{\alpha} + \frac{k}{k-2} = 1 - \frac{k-1}{k-2} + \frac{k}{k-2} = \frac{k-1}{k-2}.$$

Therefore, in the case when $k \geq 4$, we have

$$(1.28) \quad \frac{k-1}{k-2} - \frac{(k+1)(k-4)}{(k+2)(k-2)} = \frac{2(2k+1)}{(k+2)(k-2)} \geq 0.$$

Moreover, in the case when $2 \leq k < 4$, we have

$$(1.29) \quad \frac{k-1}{k-2} - \frac{(k+1)(4-k)}{(k+2)(k-2)} = \frac{2(k^2-k-3)}{(k+2)(k-2)}.$$

Thus, if

$$2 \leq k \leq \frac{1+\sqrt{13}}{2} = 2.3027\dots,$$

then we have

$$(1.30) \quad \frac{k-1}{k-2} \leq \frac{(k+1)(4-k)}{(k+2)(k-2)}.$$

Therefore

$$(1.31) \quad A(k, \alpha) = \begin{cases} \frac{k-1}{k-2} & \left(k \geq \frac{1+\sqrt{13}}{2}\right) \\ \frac{(k+1)(4-k)}{(k+2)(k-2)} & \left(2 \leq k < \frac{1+\sqrt{13}}{2}\right). \end{cases}$$

By the condition (1.23), we have

$$(1.32) \quad \alpha_0 = \frac{k-2}{k-1} \quad \left(k \geq \frac{1+\sqrt{13}}{2}\right)$$

such that

$$(1.33) \quad f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \quad \left(k \geq \frac{1+\sqrt{13}}{2} = 2.3027\dots\right).$$

Thus we have

$$(1.34) \quad \frac{5-\sqrt{13}}{6} \leq \frac{k-2}{k-1} < 1 \quad \left(\frac{5-\sqrt{13}}{6} = 0.23241\dots\right).$$

When $0 < \alpha \leq \beta < 1$, we have the following inclusion relationship:

$$(1.35) \quad \mathcal{M}(\alpha) \supset \mathcal{M}(\beta),$$

which results from the definition of the class $\mathcal{M}(\alpha)$. Thus we conclude that

$$(1.36) \quad f(z) \in \mathcal{M}\left(\frac{k-2}{k-1}\right) \subset \mathcal{M}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{S}^*\left(\frac{5-\sqrt{13}}{6}\right).$$

We now consider the following function:

$$(1.37) \quad f(z) = z + \frac{1}{2k}z^2 \quad (k \geq 2),$$

which immediately yields

$$(1.38) \quad zf'(z) = z + \frac{1}{k}z^2 \quad (k \geq 2).$$

Since, by definition,

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha),$$

we finally obtain the following inclusion relationship:

$$(1.39) \quad f(z) \in \mathcal{N}\left(\frac{k-2}{k-1}\right) \subset \mathcal{N}\left(\frac{5-\sqrt{13}}{6}\right) \subset \mathcal{K}\left(\frac{5-\sqrt{13}}{6}\right) \\ \left(k \geq \frac{1+\sqrt{13}}{2} = 2.3027\ldots\right).$$

2 A Set of Coefficient Inequalities

Our first coefficient inequality is contained in Theorem 1 below.

Theorem 1. *Let $0 < \alpha < 1$. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:*

$$(2.1) \quad \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leq \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1 - \alpha & \left(\frac{1}{2} \leq \alpha < 1\right), \end{cases}$$

then $f(z) \in \mathcal{M}(\alpha)$.

Proof. By virtue of the condition (1.10), we have to show that

$$(2.2) \quad \left| \frac{2\alpha f(z)}{zf'(z)} - 1 \right| < 1.$$

We first observe that

$$(2.3) \quad \left| \frac{2\alpha f(z) - zf'(z)}{zf'(z)} \right| = \left| \frac{1 - 2\alpha + \sum_{n=2}^{\infty} (n - 2\alpha)a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} na_n z^{n-1}} \right|$$

$$\begin{aligned}
& \leq \frac{|1 - 2\alpha| + \sum_{n=2}^{\infty} (n - 2\alpha) |a_n| \cdot |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| \cdot |z|^{n-1}} \\
& < \frac{|1 - 2\alpha| + \sum_{n=2}^{\infty} (n - 2\alpha) |a_n|}{1 - \sum_{n=2}^{\infty} n |a_n|}.
\end{aligned}$$

Now, by using the coefficient inequality (2.1), we have

$$(2.4) \quad \frac{|1 - 2\alpha| + \sum_{n=2}^{\infty} (n - 2\alpha) |a_n|}{1 - \sum_{n=2}^{\infty} n |a_n|} \leq 1,$$

which, in conjunction with (2.2) and (2.3), completes the proof of Theorem 1. $\square$

By means of Theorem 1, we introduce the subclass $\mathcal{M}^*(\alpha)$ of the class $\mathcal{M}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.1) for some α ($0 < \alpha < 1$).

Theorem 2. Suppose that $0 < \alpha < 1$. If $f(z) \in \mathcal{A}$ satisfies the following coefficient inequality:

$$(2.5) \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq \frac{1}{2}(1 - |1 - 2\alpha|) = \begin{cases} \alpha & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1 - \alpha & \left(\frac{1}{2} \leq \alpha < 1\right), \end{cases}$$

then $f(z) \in \mathcal{N}(\alpha)$.

Proof. The proof of Theorem 2 follows from Theorem 1 and the aforementioned fact that

$$f(z) \in \mathcal{N}(\alpha) \iff zf'(z) \in \mathcal{M}(\alpha).$$

$\square$

By means of Theorem 2, we also introduce the subclass $\mathcal{N}^*(\alpha)$ of the class $\mathcal{N}(\alpha)$ consisting of all functions $f(z)$ which satisfy the coefficient inequality (2.5) for some α ($0 < \alpha < 1$).

3 Distortion Bounds

For $f \in \mathcal{A}$, we define the integro-differential operators $I_k f(z)$ given by

$$I_{-1}f(z) = f'(z), \quad I_0f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1}f(t)dt \quad (k \in \mathbb{N} := \{1, 2, 3, \dots\}).$$

Then we find from (1.1) that

$$(3.1) \quad I_k f(z) = \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k}.$$

Theorem 3. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$(3.2) \quad \begin{aligned} \frac{1}{(k+1)!} |z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)!(2-\alpha)} |z|^{k+2} &\leq |I_k f(z)| \\ &\leq \frac{1}{(k+1)!} |z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)!(2-\alpha)} |z|^{k+2} \\ (z \in \mathbb{U}; k \in \mathbb{N}_0 \cup \{-1\}; \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \end{aligned}$$

Proof. We begin by noting that

$$(3.3) \quad \begin{aligned} |I_k f(z)| &= \left| \frac{1}{(k+1)!} z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} a_n z^{n+k} \right| \\ &\leq \frac{1}{(k+1)!} |z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n| \cdot |z|^{n+k} \\ &< \frac{1}{(k+1)!} |z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n|. \end{aligned}$$

Now it is easy to see that

$$(3.4) \quad \frac{(k+2)!(2-\alpha)}{2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n| \leq \sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq \frac{1}{2} (1 - |1 - 2\alpha|),$$

which implies that

$$(3.5) \quad \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n| \leq \frac{1 - |1 - 2\alpha|}{(k+2)!(2-\alpha)}.$$

Therefore, we have

$$(3.6) \quad |I_k f(z)| \leq \frac{1}{(k+1)!} |z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)!(2-\alpha)} |z|^{k+2} \quad (z \in \mathbb{U}).$$

Also we can easily observe that

$$(3.7) \quad |I_k f(z)| \geq \frac{1}{(k+1)!} |z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)!(2-\alpha)} |z|^{k+2} \quad (z \in \mathbb{U}).$$

□

By setting $k = -1, 0, 1$ in Theorem 3, we deduce Corollary 1 below.

Corollary 1. *If $f(z) \in \mathcal{M}^*(\alpha)$, then*

$$(3.8) \quad 1 - \frac{1 - |1 - 2\alpha|}{2 - \alpha}|z| \leq |f'(z)| \leq 1 + \frac{1 - |1 - 2\alpha|}{2 - \alpha}|z| \quad (k = -1),$$

$$(3.9) \quad |z| - \frac{1 - |1 - 2\alpha|}{2(2 - \alpha)}|z|^2 \leq |f(z)| \leq |z| + \frac{1 - |1 - 2\alpha|}{2(2 - \alpha)}|z|^2 \quad (k = 0),$$

and

$$(3.10) \quad \frac{1}{2}|z|^2 - \frac{1 - |1 - 2\alpha|}{6(2 - \alpha)}|z|^3 \leq |I_1 f(z)| \leq \frac{1}{2}|z|^2 + \frac{1 - |1 - 2\alpha|}{6(2 - \alpha)}|z|^3 \quad (k = 1).$$

For $f \in \mathcal{A}$, we consider again the following integro-differential operators:

$$I_{-2}f(z) = f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}, \quad I_{-1}f(z) = f'(z), \quad I_0f(z) = f(z),$$

and

$$I_k f(z) = \int_0^z I_{k-1} f(t) dt \quad (k \in \mathbb{N}).$$

Next we state and prove the following result.

Theorem 4. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.11) \quad 2|a_2| - \frac{1 - |1 - 2\alpha|}{2}|z| \leq |I_{-2}f(z)| \leq 2|a_2| + \frac{1 - |1 - 2\alpha|}{2}|z|$$

and

$$(3.12) \quad \begin{aligned} & \frac{1}{(k+1)!}|z|^{k+1} - \frac{1 - |1 - 2\alpha|}{(k+2)!(2 - \alpha)}|z|^{k+2} \leq |I_k f(z)| \\ & \leq \frac{1}{(k+1)!}|z|^{k+1} + \frac{1 - |1 - 2\alpha|}{(k+2)!(2 - \alpha)}|z|^{k+2} \quad (z \in \mathbb{U}; k \in \mathbb{N}_0 \cup \{-1\}). \end{aligned}$$

Proof. We note that, for $k \in \mathbb{N}_0 \cup \{-1\}$,

$$(3.13) \quad \begin{aligned} |I_k f(z)| &= \left| \frac{1}{(k+1)!}z^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}a_n z^{n+k} \right| \\ &\leq \frac{1}{(k+1)!}|z|^{k+1} + \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n| \cdot |z|^{n+k} \\ &< \frac{1}{(k+1)!}|z|^{k+1} + |z|^{k+2} \sum_{n=2}^{\infty} \frac{n!}{(n+k)!}|a_n|. \end{aligned}$$

Since, for $f(z) \in \mathcal{N}^*(\alpha)$,

$$(3.14) \quad (k+2)!(2-\alpha) \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n| \leq \sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq \frac{1}{2}(1 - |1-2\alpha|),$$

we find that

$$(3.15) \quad \sum_{n=2}^{\infty} \frac{n!}{(n+k)!} |a_n| \leq \frac{1 - |1-2\alpha|}{2(k+2)!(2-\alpha)}.$$

Therefore, we have

$$(3.16) \quad \begin{aligned} \frac{1}{(k+1)!} |z|^{k+1} - \frac{1 - |1-2\alpha|}{2(k+2)!(2-\alpha)} |z|^{k+2} &\leq |I_k f(z)| \\ &\leq \frac{1}{(k+1)!} |z|^{k+1} + \frac{1 - |1-2\alpha|}{2(k+2)!(2-\alpha)} |z|^{k+2} \\ &\quad (z \in \mathbb{U}; k \in \mathbb{N}_0 \cup \{-1\}). \end{aligned}$$

In the exceptional case of (3.16) when $k = -2$, we have

$$(3.17) \quad 2|a_2| - \frac{1 - |1-2\alpha|}{2} |z| \leq |I_{-2} f(z)| \leq 2|a_2| + \frac{1 - |1-2\alpha|}{2} |z|$$

$$(z \in \mathbb{U}).$$

□

By setting $k = -1, 0, 1$ in Theorem 4, we deduce the following corollary.

Corollary 2. *If $f(z) \in \mathcal{N}^*(\alpha)$, then*

$$(3.18) \quad 1 - \frac{1 - |1-2\alpha|}{2(2-\alpha)} |z| \leq |f'(z)| \leq 1 + \frac{1 - |1-2\alpha|}{2(2-\alpha)} |z| \quad (k = -1),$$

$$(3.19) \quad |z| - \frac{1 - |1-2\alpha|}{4(2-\alpha)} |z|^2 \leq |f(z)| \leq |z| + \frac{1 - |1-2\alpha|}{4(2-\alpha)} |z|^2 \quad (k = 0),$$

and

$$(3.20) \quad \frac{1}{2} |z|^2 - \frac{1 - |1-2\alpha|}{12(2-\alpha)} |z|^3 \leq |I_1 f(z)| \leq \frac{1}{2} |z|^2 + \frac{1 - |1-2\alpha|}{12(2-\alpha)} |z|^3 \quad (k = 1).$$

4 Inclusion Relationships Between the Function Classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$

Using the coefficient inequalities for the classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$, we now derive Theorem 5 below.

Theorem 5. *The following inclusion relationships hold true for the class $\mathcal{M}^*(\alpha)$:*

- (A) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \quad \left(0 < \alpha \leq \frac{1}{2}\right).$
- (B) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*\left(1 - \frac{1}{3-2\alpha}\right) \quad \left(\frac{1}{2} \leq \alpha < 1\right).$
- (C) $\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(\beta) \quad \left(0 < \alpha \leq \beta \leq \frac{1}{2}\right).$
- (D) $\mathcal{M}^*(\beta) \subset \mathcal{M}^*(\alpha) \quad \left(\frac{1}{2} \leq \alpha \leq \beta < 1\right).$

Proof. (A) For

$$0 < \alpha \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq \beta < 1,$$

we consider the maximum value of β such that

$$(4.1) \quad \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha} |a_n| \leq 1.$$

Thus we need to find the maximum value of β such that

$$(4.2) \quad \beta \leq \frac{n(1-\alpha)-\alpha}{n-2\alpha} \quad (n \in \mathbb{N} \setminus \{1\}).$$

By taking the derivative of the right-hand side of (4.2) with respect to n , it is easily seen that the right-hand side of (4.2) is monotonically decreasing for n . Thus, upon letting $n \rightarrow \infty$, we have $\beta = 1 - \alpha$. Noting also that

$$\frac{1}{2} \leq \beta < 1 \quad \text{for} \quad 0 < \alpha \leq \frac{1}{2},$$

we have

$$\mathcal{M}^*(\alpha) \subset \mathcal{M}^*(1-\alpha) \quad \left(0 < \alpha \leq \frac{1}{2}\right),$$

which evidently completes the proof of (A).

The proofs of (B), (C), and (D) are much akin to the proof of (A). □

Finally, we consider some relationships between the function classes $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$.

Theorem 6. *Each of the following assertions holds true:*

- (A) If $f(z) \in \mathcal{N}^*(\alpha)$ for $0 < \alpha \leq \frac{1}{2}$, then $f(z) \in \mathcal{M}^*\left(\frac{4-4\alpha}{4-3\alpha}\right)$.
- (B) If $f(z) \in \mathcal{N}^*(\alpha)$ for $\frac{1}{2} \leq \alpha < 1$, then $f(z) \in \mathcal{M}^*\left(\frac{2-2\alpha}{5-3\alpha}\right)$.
- (C) If $f(z) \in \mathcal{N}^*(\alpha)$ for $0 < \alpha \leq \frac{1}{2}$, then $f(z) \in \mathcal{M}^*\left(\frac{2\alpha}{4-\alpha}\right)$.
- (D) If $f(z) \in \mathcal{N}^*(\alpha)$ for $\frac{1}{2} \leq \alpha < 1$, then $f(z) \in \mathcal{M}^*\left(\frac{2}{3-\alpha}\right)$.

Proof. (A) Let

$$0 < \alpha \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq \beta < 1.$$

We consider the maximum value of β such that

$$(4.3) \quad \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha} |a_n| \leq 1.$$

This means that

$$(4.4) \quad \beta \leq \frac{n^2 - 2n\alpha}{n^2 - n\alpha - \alpha} \quad (n \in \mathbb{N} \setminus \{1\}).$$

If we take the derivative of the right-hand side of (4.4) with respect to n , then the numerator becomes

$$(4.5) \quad n^2\alpha - 2n\alpha + 2\alpha^2 \geq 0 \quad \left(0 < \alpha \leq \frac{1}{2}; n \in \mathbb{N} \setminus \{1\}\right).$$

Therefore, the right-hand side of (4.4) is monotonically increasing for n . Thus, by setting $n = 2$, we have

$$\beta = \frac{4-4\alpha}{4-3\alpha}.$$

It is easy to see that

$$\frac{1}{2} \leq \beta < 1 \quad \text{for} \quad 0 < \alpha \leq \frac{1}{2},$$

which obviously completes the proof of (A).

The proofs of (B), (C), and (D) would run parallel to the proof of (A).

□

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