

On numerical range and norm of the generalized Aluthge transform

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ABSTRACT

In this report, we have attempted to reveal some relationships between a bounded linear operator T acting on a Hilbert space and its generalized Aluthge transformation $T(s, t)$ in terms of their numerical ranges and norms.

1. INTRODUCTION

Let $B(\mathcal{H})$ denote the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. By the polar decomposition of $T \in B(\mathcal{H})$, we mean the expression $T = U|T|$, where U is a partial isometry and $|T|$ is the positive square root of T^*T such that $\ker U = \ker |T|$. In [1], Aluthge introduced the class of p -hyponormal operators that generalizes the widely studied class of hyponormal operators. In order to reveal some important features of p -hyponormal operators, he exploited the operator $\tilde{T}$ which is now popularly known as the *Aluthge Transformation* and which is defined as

$$\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}.$$

Motivated by this article [1], several authors explored and studied new classes of operators closely connected to p -hyponormal operators with the help of the Aluthge transformation and its generalization, known as the generalized Aluthge transformation. By the generalized Aluthge transformation of $T \in B(\mathcal{H})$, we mean the bounded operator $T(s, t)$ on $\mathcal{H}$ for which

$$T(s, t) = |T|^s U |T|^t, \quad \text{where } s \geq 0 \text{ and } t \geq 0,$$

Especially, $T(1, 0) = |T|^1 U |T|^0 = |T| U U^* U = |T| U$ and $T(0, 1) = |T|^0 U |T|^1 = U^* U U |T|$. In recent years, one can find number of articles in which various relations among T , $\tilde{T}$ and $T(s, t)$ are obtained. It is obvious that $\|\tilde{T}\| \leq \|T\|$. Okubo [8] gave a non-obvious extension of this inequality by deriving $\|f(\tilde{T})\| \leq \|f(T)\|$ for any polynomial $f(t)$ by proving more general result. As corollary to this inequality, he showed that $W(f(\tilde{T})) \subseteq \overline{W(f(T))}$,

extending some results known to be true [6] or in case either $f(t) = t$ [7, 9, 11]. Our main object of the present report is to compare the numerical range of T with that of $T(s, t)$ for some restricted values of s and t .

In section 2, some results are given that will be of use in the succeeding sections. Section 3 is devoted to establishing inclusion relations among the numerical ranges of rational functions of operators $T(0, 1)$, $T(1, 0)$ and T . The inequality that says $\|f(\tilde{T})\| \leq \|f(T)\|$ for every polynomial f is extended further in section 4 by proving $\|f(T(s, t))\| \leq \|f(T)\|$ with $s + t = 1$ for every rational function f for which $f(T)$ exists. Finally, in section 5, we introduce a numerical range value function on $[0, 1]$ and obtain an improvement over a characterization of convexoid matrices due to Ando [2].

In what follows, we assume, unless it is stated otherwise, that f will be a rational function with poles off $\sigma(T)$.

2. FUNDAMENTAL PROPERTIES

Lemma 2.1. *Let $T = U|T|$ be the polar decomposition of T . Then $\dim \ker T \leq \dim \ker T^*$ if and only if there exists an isometry V such that $V|T| = U|T|$.*

Although our first lemma is well known [5, p. 75], [10, p. 4], we would like to present it with a proof.

Proof. Let $\mathcal{H} = \overline{R(|T|)} \oplus R(|T|)^\perp = \overline{R(T)} \oplus R(T)^\perp$. Then U is an isometry from $\overline{R(|T|)}$ to $\overline{R(T)}$. On the other hand, there exists an isometry $U_1 : R(|T|)^\perp \rightarrow R(T)^\perp$ if and only if $\dim(R(|T|)^\perp) \leq \dim(R(T)^\perp)$. By $R(|T|)^\perp = \ker |T| = \ker T$ and $R(T)^\perp = \ker T^*$, it is equivalent to $\dim \ker T \leq \dim \ker T^*$. So the underlying kernel condition ensures the existence of an isometry $U_1 : R(|T|)^\perp \rightarrow R(T)^\perp$. Let

$$V = UU^*U + U_1(I - U^*U) = U + U_1(I - U^*U).$$

The facts that $I - U^*U$ is the projection onto $\ker U = \ker T = \ker |T| = R(|T|)^\perp = (\ker U_1)^\perp$, $U_1^*U_1x = x$ on $R(|T|)^\perp$ and $R(U_1) \subseteq R(T)^\perp = \ker T^* = \ker U^*$ will give

$$\begin{aligned} V^*V &= \{U + U_1(I - U^*U)\}^* \{U + U_1(I - U^*U)\} \\ &= U^*U + (I - U^*U)U_1^*U + U^*U_1(I - U^*U) + (I - U^*U)U_1^*U_1(I - U^*U) \\ &= U^*U + \{U^*U_1(I - U^*U)\}^* + U^*U_1(I - U^*U) + I - U^*U \\ &= U^*U + I - U^*U \\ &= I. \end{aligned}$$

Thus V is an isometry. Moreover,

$$V|T| = \{U + U_1(I - U^*U)\}|T| = U|T|.$$

□

Lemma 2.2. *Let $A \in B(\mathcal{H})$. Then the following assertions hold:*

(i) *If P is a projection with $PAP = AP$, then*

$$f(AP) = Pf(A)P + f(0)(I - P).$$

(ii) *If V is an isometry, then*

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

Proof. (i). Let $\mathcal{H} = (\ker P)^\perp \oplus \ker P$. Then by the assumption $PAP = AP$, A can be expressed as follows:

$$A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \quad \text{on } \mathcal{H} = (\ker P)^\perp \oplus \ker P.$$

Hence

$$f(AP) = f\left(\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} f(X) & 0 \\ 0 & f(0)I \end{pmatrix}.$$

On the other hand,

$$f(A) = f\left(\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}\right) = \begin{pmatrix} f(X) & Y' \\ 0 & f(Z) \end{pmatrix}.$$

Hence we have

$$Pf(A)P + f(0)(I - P) = \begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & f(0)I \end{pmatrix} = f(AP).$$

(ii). For an isometry V , note that $\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix}$ is unitary. Then we have

$$\begin{aligned} \begin{pmatrix} f(VAV^*) & 0 \\ 0 & f(0)I \end{pmatrix} &= f\left(\begin{pmatrix} VAV^* & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix}\right) \\ &= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} f\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\ &= \begin{pmatrix} V & I - VV^* \\ 0 & V^* \end{pmatrix} \begin{pmatrix} f(A) & 0 \\ 0 & f(0)I \end{pmatrix} \begin{pmatrix} V^* & 0 \\ I - VV^* & V \end{pmatrix} \\ &= \begin{pmatrix} Vf(A)V^* + f(0)(I - VV^*) & 0 \\ 0 & f(0)I \end{pmatrix}. \end{aligned}$$

Hence

$$f(VAV^*) = Vf(A)V^* + f(0)(I - VV^*).$$

□

The following result is a modification of [6, Proposition 4.5].

Proposition 2.3. *Let $A, B \in B(\mathcal{H})$. Then the following assertions are mutually equivalent:*

- (i) $\overline{W(f(A))} \subseteq \overline{W(f(B))}$ for all f .
- (ii) $w(f(A)) \leq w(f(B))$ for all f .
- (iii) $\|f(A)\| \leq \|f(B)\|$ for all f .

The proof is almost identical to the one given for Proposition 4.5 of [6].

3. NUMERICAL RANGES OF $T(0, 1)$ AND $T(1, 0)$

The primary object of the present section is to establish the connection among the numerical ranges of T , $T(0, 1)$ and $T(1, 0)$.

Theorem 3.1. *Let $T \in B(\mathcal{H})$. Then the following assertions hold:*

- (i) $W(f(T(0, 1))) \subseteq W(f(T))$.
- (ii) $W(f(T(1, 0))) \subseteq W(f(T))$.

Proof. (i). Let $T = U|T|$ be the polar decomposition of T , and $\mathcal{H} = (\ker T)^\perp \oplus \ker T$. Then

$$T(0, 1) = U^*UU|T| = U^*UT = U^*UTU^*U.$$

Since U^*U is a projection, (i) of Lemma 2.2 yields

$$(3.1) \quad f(T(0, 1)) = U^*Uf(T)U^*U + f(0)(I - U^*U).$$

In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.1), $f(T(0, 1)) = f(T)$, and hence

$$W(f(T(0, 1))) = W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.1), we obtain

$$W(f(T(0, 1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(0, 1))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

(ii). *Step 1.* We shall show the following equality:

$$(3.2) \quad f(T(1, 0)) = U^*f(T)U + f(0)(I - U^*U).$$

We shall establish this equality separately for each of the cases when $\dim \ker T \leq \dim \ker T^*$ and $\dim \ker T \geq \dim \ker T^*$.

(a) The case $\dim \ker T \leq \dim \ker T^*$. By the Lemma 2.1, there is an isometry V satisfying $U|T| = V|T|$. Note that in the proof of Lemma 2.1,

$$(3.3) \quad V = U + U_1(I - U^*U), \text{ where } U_1 \text{ is isometry with } R(U_1) \subseteq \ker T^*.$$

Then by (3.3), we have

$$UU^*TUU^* = TUU^* = U|T|UU^* = V|T|UV^*.$$

Hence by (ii) Lemma 2.2, we obtain

$$f(TUU^*) = f(V|T|UV^*) = Vf(|T|U)V^* + f(0)(I - VV^*).$$

Moreover since V is isometry, we have

$$f(|T|U) = V^*f(TUU^*)V.$$

Therefore

$$\begin{aligned} f(|T|U) &= V^*f(TUU^*)V \\ &= V^*\{UU^*f(T)UU^* + f(0)(I - UU^*)\}V \quad \text{by (i) of Lemma 2.2} \\ &= U^*f(T)U + f(0)(I - U^*U) \quad \text{by (3.3).} \end{aligned}$$

On the other hand, by (3.3),

$$U|T^*| = UU^*UU|T|U^* = VU^*UTV^*.$$

Then by Lemma 2.2, we obtain

$$\begin{aligned} f(U|T^*|) &= f(VU^*UTV^*) \\ &= Vf(U^*UT)V^* + f(0)(I - VV^*) \quad \text{by (ii) of Lemma 2.2} \\ (3.4) \quad &= V\{U^*Uf(T)U^*U + f(0)(I - U^*U)\}V^* + f(0)(I - VV^*) \\ &\quad \text{by (i) of Lemma 2.2} \\ &= Uf(T)U^* + f(0)(I - UU^*) \quad \text{by (3.3).} \end{aligned}$$

(b) The case $\dim \ker T \geq \dim \ker T^*$. Replacing T by T^* in (3.4), we have

$$\begin{aligned} f(U^*|T|) &= U^*f(T^*)U + f(0)(I - U^*U) \\ &\iff f(|T|U) = U^*f(T)U + f(0)(I - U^*U). \end{aligned}$$

Step 2. In case $\ker T = \{0\}$. In this case U must be isometry. Then by (3.2), $f(T(1, 0)) = f(|T|U) = U^*f(T)U$, and hence

$$W(f(T(1, 0))) \subseteq W(f(T)).$$

In case $\ker T \neq \{0\}$. By (3.2), we obtain

$$W(f(T(1, 0))) = W(f(|T|U)) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\}.$$

Here by $\ker T \neq \{0\}$, we have $f(0) \in W(f(T))$, and

$$W(f(T(1, 0))) \subseteq \text{conv} \{W(f(T)) \cup \{f(0)\}\} = W(f(T)).$$

Hence the proof is complete. □

Corollary 3.2. *Let $T = U|T|$. Then*

- (i) $W(T(1, 0)) = W(T)$ if $\ker T^* \subseteq \ker T$.
- (ii) $W(T(0, 1)) = W(T)$ if $\ker T \subseteq \ker T^*$.
- (iii) $W(T(0, 1)) \subseteq W(T(1, 0))$ if $\ker T^* \subseteq \ker T$.
- (iv) $W(T(1, 0)) \subseteq W(T(0, 1))$ if $\ker T \subseteq \ker T^*$.

Proof. (i). In view of Theorem 3.1, only we have to prove $W(T) \subseteq W(T(1, 0))$.

If $\ker T^* \subseteq \ker T$, then $T = U|T| = U|T|UU^* = UT(1, 0)U^*$, and we have

$$W(T) \subseteq \text{conv}\{W(T(1, 0)) \cup \{0\}\}.$$

If $\ker T \neq \{0\}$, then $0 \in W(T(1, 0))$ and we have

$$W(T) \subseteq \text{conv}\{W(T(1, 0)) \cup \{0\}\} = W(T(1, 0)).$$

If $\ker T = \{0\}$, then $\{0\} = \ker T \supset \ker T^*$, and U^* must be an isometry. Hence we have $W(T) \subseteq W(T(1, 0))$.

(ii). If $\ker T \subseteq \ker T^*$, then $U^*UU = U$ holds. Hence $T(0, 1) = U^*UU|T| = U|T| = T$, and $W(T(0, 1)) = W(T)$.

(iii). If $\ker T^* \subseteq \ker T$, then by (i) and Theorem 3.1, we have

$$W(T(0, 1)) \subseteq W(T) = W(T(1, 0)).$$

(iv). If $\ker T \subseteq \ker T^*$, then by (ii) and Theorem 3.1, we have

$$W(T(1, 0)) \subseteq W(T) = W(T(0, 1)).$$

□

Remark 3.3. *If we drop the kernel condition from the statements of Corollary 3.2, then we may not get the same conclusions as following indicate.*

Example 3.4. Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $|T| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $U = T$. Clearly $W(T(1, 0)) = \{0\} \neq W(T)$.

Example 3.5. For $\alpha > 0$, let

$$T = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(0, 1) = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T(1, 0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Also $W(T(0,1)) = \{z \in \mathbb{C} : |z| \leq \frac{\alpha}{2}\}$ and $W(T(1,0)) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Then

- (i) for $\alpha \in (0,1)$, $W(T(0,1)) \subsetneq W(T(1,0))$,
- (ii) for $\alpha > 1$, $W(T(1,0)) \subsetneq W(T(0,1))$.

4. NORM INEQUALITY INVOLVING A RATIONAL FUNCTION OF T AND $T(s,t)$.

Theorem 4.1. *Let $T \in B(\mathcal{H})$. Then*

$$\|f(T(s,t))\| \leq \|f(T)\|$$

holds for $s, t \geq 0$ with $s + t = 1$.

Proof. Let $T = U|T|$ be the polar decomposition of T . Let $|T_\varepsilon| = |T| + \varepsilon I > 0$. Note that

$$\lim_{\varepsilon \rightarrow +0} |T_\varepsilon|^{-1}|T| = \lim_{\varepsilon \rightarrow +0} (|T| + \varepsilon I)^{-1}|T| = U^*U.$$

We prepare the important inequality due to [4]. For $X \in B(\mathcal{H})$ and positive operators A and B ,

$$(4.1) \quad \|A^s X B^s\| \leq \|AXB\|^s \|X\|^{1-s}$$

holds for $s \in [0,1]$. Then we have

$$\begin{aligned} \|f(T(s,t))\| &= \|f(|T|^s U |T|^t)\| \\ &= \|f(|T_\varepsilon|^s |T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s |T_\varepsilon|^{-s})\| \\ &= \| |T_\varepsilon|^s f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s) |T_\varepsilon|^{-s} \| \\ &\leq \| |T_\varepsilon| f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s) |T_\varepsilon|^{-1} \|^s \|f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s)\|^t \quad \text{by (4.1)} \\ &= \|f(|T_\varepsilon|^{1-s} |T|^s U |T|^t |T_\varepsilon|^{s-1})\|^s \|f(|T_\varepsilon|^{-s} |T|^s U |T|^t |T_\varepsilon|^s)\|^t \\ &\longrightarrow \|f(|T| U U^* U)\|^s \|f(U^* U U |T|)\|^t \quad \text{as } \varepsilon \rightarrow +0 \\ &= \|f(T(1,0))\|^s \|f(T(0,1))\|^t \\ &\leq \|f(T)\| \quad \text{by Theorem 3.1 and Proposition 2.3.} \end{aligned}$$

Hence the proof is complete. $\square$

Remark 4.2. *Above theorem is not true if $s + t \neq 1$ as can be illustrated with the following Example 4.3.*

Example 4.3. *Let $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$. Then $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $|T| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Also $T(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It easy to find that $\overline{W(T(2,1))} = [0,2]$. Moreover $\overline{W(T)}$ is a closed elliptic disc with foci at 0 and 1, and the major axis $\sqrt{2}$ and the minor axis 1. This fact shows that $\overline{W(T)}$ excludes 2 and therefore $\overline{W(T(2,1))}$ is not a*

subset of $\overline{W(T)}$. If the theorem were true for $s+t \neq 1$, then we would have in particular, $\|T(s, t) - zI\| \leq \|T - zI\|$ for all z . Then $\overline{W(T(s, t))} \subseteq \overline{W(T)}$, which is not correct.

As simple consequence of Proposition 2.3 and Theorem 4.1, we obtain the following corollary.

Corollary 4.4. *Let $T \in B(\mathcal{H})$. Then*

$$\overline{W(f(T(s, t)))} \subseteq \overline{W(f(T))}$$

holds for $s, t \geq 0$ with $s + t = 1$.

$$5. \text{ WHEN } F(x) = \overline{W(f(T(x, 1-x)))}.$$

Theorem 5.1. *For an operator T and $x \in [0, 1]$, let*

$$F(x) = \overline{W(f(T(x, 1-x)))}.$$

Then

$$(5.1) \quad F(\alpha x + (1-\alpha)y) \subseteq \alpha F(x) + (1-\alpha)F(y)$$

holds for all $x, y \in [0, 1]$ and $\alpha \in [0, 1]$.

As a consequence of Theorem 5.1, the function $\Phi(x) = w(T(x, 1-x))$ turns out to be a convex function on $[0, 1]$.

Proof. Let $T = U|T|$ be the polar decomposition. Firstly, we shall prove

$$(5.2) \quad F\left(\frac{x+y}{2}\right) \subseteq \frac{1}{2}\{F(x) + F(y)\}.$$

Note that for a positive invertible operator S and $A \in B(\mathcal{H})$,

$$\|A\| \leq \frac{1}{2}\|SAS^{-1} + S^{-1}AS\|.$$

in [3]. Let $\varepsilon > 0$ and $|T_\varepsilon| = |T| + \varepsilon I > 0$. By the above inequality, we obtain

$$\begin{aligned} & \|f\left(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)\right)\| \\ &= \|f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}})\| \\ &\leq \frac{1}{2} \| |T_\varepsilon|^{\frac{x-y}{2}} f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}}) |T_\varepsilon|^{\frac{y-x}{2}} + |T_\varepsilon|^{\frac{y-x}{2}} f(|T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}}) |T_\varepsilon|^{\frac{x-y}{2}} \| \\ &= \frac{1}{2} \|f(|T_\varepsilon|^{\frac{x-y}{2}} |T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}} |T_\varepsilon|^{\frac{y-x}{2}}) + f(|T_\varepsilon|^{\frac{y-x}{2}} |T|^{\frac{x+y}{2}} U |T|^{1-\frac{x+y}{2}} |T_\varepsilon|^{\frac{x-y}{2}})\| \\ &\longrightarrow \frac{1}{2} \|f(|T|^x U |T|^{1-x}) + f(|T|^y U |T|^{1-y})\| \quad \text{as } \varepsilon \rightarrow +0 \\ &= \frac{1}{2} \|f(T(x, 1-x)) + f(T(y, 1-y))\|. \end{aligned}$$

Hence for any complex number λ ,

$$\|f(T(\frac{x+y}{2}, 1 - \frac{x+y}{2})) - \lambda I\| \leq \|\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2} - \lambda I\|.$$

Since

$$\overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \{z \in \mathbb{C} : |z - \lambda| \leq \|T - \lambda I\|\},$$

we have

$$\begin{aligned} F\left(\frac{x+y}{2}\right) &= \overline{W\left(f\left(T\left(\frac{x+y}{2}, 1 - \frac{x+y}{2}\right)\right)\right)} \\ &\subseteq \overline{W\left(\frac{f(T(x, 1-x)) + f(T(y, 1-y))}{2}\right)} \\ &\subseteq \frac{1}{2}\{\overline{W(f(T(x, 1-x)))} + \overline{W(f(T(y, 1-y)))}\} \\ &= \frac{1}{2}\{F(x) + F(y)\}. \end{aligned}$$

Next, we will extend (5.2) to (5.1). From (5.2), one can easily derive

$$F\left(\frac{x_1 + x_2 + \cdots + x_{2^n}}{2^n}\right) \subseteq \frac{1}{2^n}\{F(x_1) + F(x_2) + \cdots + F(x_{2^n})\}$$

for all $x_i \in [0, 1]$ ($i = 1, 2, \dots$). Hence for any rational number $\alpha \in [0, 1]$, we have (5.1).

Since F is continuous, we have (5.1) for any real number $\alpha \in [0, 1]$.

This completes the proof. $\square$

Remark 5.2. The conclusion of Theorem 5.1 cannot be strengthened further to

$$F(\alpha x + (1 - \alpha)y) = \alpha F(x) + (1 - \alpha)F(y)$$

as Example 5.3 will show. However, whether the range of F is convex remains as an open problem.

Example 5.3. For $\alpha > 0$, let

$$T = \begin{pmatrix} 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$T(s, t) = \begin{pmatrix} 0 & 16^t & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $F(x) = W(T(x, 1-x)) = \{z : |z| \leq \frac{16^{1-x}}{2}\}$. Let $x = \frac{1}{4}$, $y = \frac{3}{4}$ and $\alpha = \frac{1}{2}$. Then

$$(i) \quad F\left(\frac{1}{2}\right) = F(\alpha x + (1 - \alpha)y) = \{z : |z| \leq 2\}.$$

$$(ii) F(x) = \{z : |z| \leq 4\}.$$

$$(iii) F(y) = \{z : |z| \leq 1\}.$$

Hence

$$\begin{aligned} F\left(\frac{1}{2}\right) &= F(\alpha x + (1 - \alpha)y) = \{z : |z| \leq 2\} \\ &\subseteq \{z : |z| \leq \frac{5}{2}\} = \alpha F(x) + (1 - \alpha)F(y). \end{aligned}$$

Corollary 5.4. *Let T be an operator. Then*

$$\begin{aligned} \overline{W(f(\tilde{T}))} &= F\left(\frac{1}{2}\right) \subseteq \frac{1}{2}\{F(s) + F(1 - s)\} \\ &\subseteq \frac{1}{2}\{F(t) + F(1 - t)\} \subseteq \overline{W(f(T))} \end{aligned}$$

holds for all $\frac{1}{2} \leq s \leq t \leq 1$.

Proof. Since $\frac{1}{2} = \frac{s+1-s}{2}$ and $F(x) = \overline{W(f(T(x, 1-x)))} \subseteq \overline{W(f(T))}$ for $x \in [0, 1]$, we have

$$\begin{aligned} \overline{W(f(\tilde{T}))} &= F\left(\frac{1}{2}\right) \subseteq \frac{1}{2}\{F(s) + F(1 - s)\} \quad \text{by Theorem 5.1} \\ &\subseteq \overline{W(f(T))} \quad \text{by Corollary 4.4.} \end{aligned}$$

Next, let $\frac{1}{2} \leq s \leq t \leq 1$. Then we have $[1 - s, s] \subseteq [1 - t, t]$. Then there exist $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$s = \alpha_1 t + (1 - \alpha_1)(1 - t) \quad \text{and} \quad 1 - s = \alpha_2 t + (1 - \alpha_2)(1 - t).$$

By an easy calculation, we have $\alpha_1 + \alpha_2 = 1$, and by Theorem 5.1, we have

$$\begin{aligned} \frac{1}{2}\{F(s) + F(1 - s)\} &\subseteq \frac{1}{2}\{\alpha_1 F(t) + (1 - \alpha_1)F(1 - t) + \alpha_2 F(t) + (1 - \alpha_2)F(1 - t)\} \\ &= \frac{1}{2}\{F(t) + F(1 - t)\}. \end{aligned}$$

□

As a simple consequence of Corollary 4.4, one can see that if T is convexoid then so is $T(s, t)$ with $\overline{W(T(s, t))} = \overline{W(T)}$. The converse is obvious. However, if we do not assume $\overline{W(T(s, t))} = \overline{W(T)}$, then mere convexoidity of $T(s, t)$ does not guarantee that T is convexoid even if $\mathcal{H}$ is finite dimensional. To see this, we refer to Example 4.3. That convexoidity of $\tilde{T} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is clear from the fact that it is selfadjoint. On the other hand as $\text{conv } \sigma(T) = [0, 1] \neq \overline{W(T)}$, T is not convexoid. However, if $\mathcal{H}$ is finite-dimensional, then our next result will show that the condition $\overline{W(T(s, t))} = \overline{W(T)}$ is just equivalent to the convexoidity of T . In case $\mathcal{H}$ is infinite dimensional, we do not know the validity of this result.

Corollary 5.5. *For a $n \times n$ matrix T , the following assertions are mutually equivalent:*

- (i) T is convexoid.
- (ii) $W(\tilde{T}) = W(T)$.
- (iii) $W(T(s_0, 1 - s_0)) = W(T)$ for a fixed $s_0 \in (0, 1)$.
- (iv) $W(T(s, 1 - s)) = W(T)$ for all $s \in [0, 1]$.

In order to prove Corollary 5.5, we shall need the following theorem, a remarkable result due to Ando [2].

Theorem A ([2]). *Let T be a $n \times n$ matrix. Then T is convexoid if and only if $W(\tilde{T}) = W(T)$.*

Proof. (i) $\iff$ (ii) has been shown in Theorem A. (iv) $\implies$ (ii), (iii) are obvious. So only we have to show (ii) $\implies$ (iv) and (iii) $\implies$ (ii).

Proof of (ii) $\implies$ (iv). Since $W(T(s, 1 - s)) \subseteq W(T)$ and $W(T(1 - s, s)) \subseteq W(T)$ for all $s \in [0, 1]$ hold and Corollary 5.4, we have

$$\begin{aligned} W(T) = W(\tilde{T}) &\subseteq \frac{1}{2} \{W(T(s, 1 - s)) + W(T(1 - s, s))\} \\ &\subseteq \frac{1}{2} \{W(T(s, 1 - s)) + W(T)\} \subseteq W(T). \end{aligned}$$

Then we have

$$(5.3) \quad \frac{1}{2} \{W(T(s, 1 - s)) + W(T)\} = W(T).$$

For any $\theta \in [0, 2\pi)$, let λ be an extreme point of $\operatorname{Re} e^{i\theta} W(T)$. Then by (5.3), there exist $\lambda_1 \in \operatorname{Re} e^{i\theta} W(T)$ and $\mu_1 \in \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$ such that

$$\lambda = \frac{\lambda_1 + \mu_1}{2}.$$

Since $\operatorname{Re} e^{i\theta} W(T)$ is a line segment, and λ is a extreme point of $\operatorname{Re} e^{i\theta} W(T)$, it must be $\lambda = \lambda_1 = \mu_1 \in \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$, i.e., $\operatorname{Re} e^{i\theta} W(T) \subseteq \operatorname{Re} e^{i\theta} W(T(s, 1 - s))$ for any $\theta \in [0, 2\pi)$. Since $W(T)$ is convex, and $W(T(s, 1 - s)) \subseteq W(T)$ always holds, we have $W(T) = W(T(s, 1 - s))$ for all $s \in [0, 1]$.

Proof of (iii) $\implies$ (ii). We may assume $s_0 > \frac{1}{2}$. For each $s_0 \in (\frac{1}{2}, 1)$, there exists $\alpha \in (0, 1)$ such that

$$s_0 = \alpha \frac{1}{2} + (1 - \alpha) \cdot 1.$$

Then by Theorem 5.1,

$$W(T) = W(T(s_0, 1 - s_0)) \subseteq \alpha W(\tilde{T}) + (1 - \alpha) W(T(1, 0)) \subseteq W(T).$$

By the same argument of the above one, we have $W(\tilde{T}) = W(T)$. □

Remark 5.6. In (iii) of Corollary 5.5, s_0 must not be 0 or 1, because if T is invertible, then U is unitary and $W(T) = W(T(0, 1)) = W(T(1, 0))$. But in general, $W(T) \neq W(\tilde{T})$.

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