

THE NAVIER-STOKES FLOW FOR GLOBALLY LIPSCHITZ CONTINUOUS INITIAL DATA

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ABSTRACT. Consider the Cauchy problem of the incompressible Navier-Stokes equations with initial velocity U_0 of the form $U_0(x) := u_0(x) - f(x)$, where f is a Lipschitz function and $u_0 \in L^p_\sigma(\mathbb{R}^n)$. It is shown that under these assumptions the equations of Navier-Stokes admit a unique local in time mild solution.

1. INTRODUCTION

We consider the flow of an incompressible, viscous fluid in the whole space $\mathbb{R}^n$, $n \geq 2$ described by the Cauchy problem for the system of the Navier-Stokes equations, i.e.,

$$(1.1) \quad \begin{cases} U_t - \Delta U + (U, \nabla)U + \nabla P &= F, & \text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot U &= 0 & \text{in } \mathbb{R}^n \times (0, T), \\ U|_{t=0} &= U_0 \quad (\text{with } \nabla \cdot U_0 = 0) & \text{in } \mathbb{R}^n. \end{cases}$$

Here, $U = (U^1, \dots, U^n)$ and P represent the unknown velocity and the unknown pressure of the fluid; U_0 is the given initial velocity, and F is a given external force term.

There is a vast literature on existence of solutions of (1.1) in $\mathbb{R}^n$, see e.g. [1, 7, 9, 12, 16, 19, 22]. All these results assume that the initial data decay as $|x| \rightarrow \infty$. In particular, when $F = 0$, it is well known that there exists a locally-in-time smooth solution to (1.1) provided the initial velocity U_0 belongs to $L^p_\sigma(\mathbb{R}^n)$ and $p \geq n$ (see e.g. [15, 19]).

On the other hand, there is strong interest in equation (1.1) for initial data which do not decay at infinity. For results in this direction, we refer to [5, 6, 13] and [8]. Also, H. Okamoto [24] showed that for certain concrete flow problems there exist exact solutions to (1.1) which have the property that u grows linearly as $|x| \rightarrow \infty$.

In this paper, we consider initial data of the form

$$(1.2) \quad U_0(x) = u_0(x) - f(x), \quad x \in \mathbb{R}^n,$$

where $u_0 \in L^p(\mathbb{R}^n)^n$ satisfies $\nabla \cdot u_0 = 0$ and f fulfills the following three conditions:

- (H1) $\nabla \cdot f = 0$,
- (H2) $\Delta f \in L^p_\sigma$,
- (H3) $\exists \Pi$: scalar function s.t. $(f, \nabla)f + \nabla \Pi \in L^p_\sigma$.

The particular case where $f(x) = Mx$ was considered in [17]. Here M denotes a real $n \times n$ matrix having $\text{tr } M = 0$. It was shown that this case there exists a unique, local solution to (1.1). It was also shown that this solution is analytic in the spatial variables provided M is skew-symmetric. In this paper, we generalize the result of [17] to the case of Lipschitz continuous functions f satisfying (H1), (H2) and (H3).

For the time being consider again the case where $f(x) = Mx$. Then it is known that (1.1) admits many exact solutions, which are studied e.g. in [10, 21, 25]. In fact, let f be of the form

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$f(x) = Mx + V$, where $M = (m_{ij})_{i,j}$ is an $n \times n$ real-valued constant matrix satisfying that $\text{tr } M = 0$ and such that M^2 is symmetric. Moreover, let V be a vector. Then $\Delta f = 0$ and

$$\text{tr } M = 0 \Leftrightarrow (\text{H1}) \quad \text{and} \quad M^2 \text{ is symmetric} \Leftrightarrow (\text{H3}).$$

In fact, take $\Pi = \frac{1}{2}(M^2x, x) + (V, M^Tx)$. Then (U, P) given by $U = -f$ and $P = -\Pi$ solves (1.1) with $F \equiv 0$ provided $\Delta f = 0$ and (H3) holds.

The particular case, where $M = R$ describes pure rotation, was investigated by Hishida and by Babin, Mahalov and Nicolaenko. Indeed, Hishida constructed in [18] a local solution to the equation (1.3) written below in the L^2 context and provided u_0 belongs to a certain fractional power space. Babin, Mahalov and Nicolaenko [2, 3] proved the existence of a local solution and even a global solution to (1.1)-(1.2) provided the speed of rotation is fast enough. Further, the case $f(x) = (ax_1, ax_2, -2ax_3)$ with some constant $a \in \mathbb{R}$, was investigated by Giga and Kambe [14]. They studied the axisymmetric irrotational flow and the stability of the vortex.

In [26], the third author proved the existence of a local solution of (1.1)-(1.2), still for $M = R$ provided u_0 belongs to the homogeneous Besov space $\dot{B}_{\infty,1}^0$. Although $\dot{B}_{\infty,1}^0$ is strictly smaller than L^∞ , this space still contains the nondecaying function $f(x) = \sin x$. He also showed the uniqueness of the solution for general matrices M ; see [27].

In the following consider the the substitutions $u := U + f$ and $\tilde{P} := P + \Pi$. Then the pair (U, P) satisfies (1.1) in the classical sense, if and only if $(u, \tilde{P})$ satisfies

$$(1.3) \quad \begin{cases} u_t - \Delta u + (u, \nabla)u - (f, \nabla)u - (u, \nabla)f + \nabla \tilde{P} &= \tilde{F} \quad \text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\ u(0) &= u_0 \quad \text{with } \nabla \cdot u_0 = 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

Here $\tilde{F} := F + \Delta f - (f, \nabla)f - \nabla \Pi$. Of course, if (f, Π) is a stationary solution to (1.1) with $F = F(x)$, then $\tilde{F} \equiv 0$. Our approach to equation (1.1) is based on equation (1.3).

2. MAIN RESULTS

Let $u_0 \in L_\sigma^p(\mathbb{R}^n)$ for some p satisfying $1 < p < \infty$. Moreover, let f be a vector-valued globally Lipschitz continuous function satisfying hypothesis (H1), (H2), (H3).

We then rewrite the first equation of (1.3) as the abstract equation

$$(2.1) \quad u' + Au + (u, \nabla)u - 2(u, \nabla)f + \nabla \tilde{P} = \tilde{F}.$$

with A being an operator in $L_\sigma^p(\mathbb{R}^n)$ defined by

$$(2.2) \quad Au := -\Delta u - (f, \nabla)u + (u, \nabla)f.$$

Equipped with the domain $D(A) := \{u \in W^{2,p}(\mathbb{R}^n) \cap L_\sigma^p(\mathbb{R}^n); (f, \nabla)u \in L^p(\mathbb{R}^n)\}$, $-A$ generates a C_0 -semigroup $\{e^{-tA}\}_{t \geq 0}$ on L_σ^p for $1 < p < \infty$. This follows from the results in [20] and standard perturbation theory.

Applying the Helmholtz projection $\mathbb{P}$ to (2.1), we may rewrite (1.3) as

$$(2.3) \quad \begin{cases} u' + Au + \mathbb{P}(u, \nabla)u - 2\mathbb{P}(u, \nabla)f &= \tilde{F} \\ u(0) &= u_0. \end{cases}$$

Note that in our case the Helmholtz projection $\mathbb{P}$ can be expressed explicitly by $\mathbb{P} := (\delta_{ij} + R_i R_j)_{i,j}$, where δ_{ij} stands for Kronecker's delta, and R_i is the Riesz transform defined by $R_i := \partial_i (-\Delta)^{-1/2}$ for $i = 1, \dots, n$. Observe that A and $\mathbb{P}$ commute in our case, since $\nabla \cdot Au = 0$ if $\nabla \cdot u = 0$. Since u , F and f are divergence-free, $\mathbb{P}u = u$ as well as $\mathbb{P}\tilde{F} = \tilde{F}$.

For $T > 0$ we call a function $u \in C([0, T]; L_\sigma^p(\mathbb{R}^n))$ a *mild solution* of (2.3) if u satisfies the integral equation

$$(2.4) \quad \begin{aligned} u(t) = & e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}(u(s), \nabla)u(s)ds \\ & + 2 \int_0^t e^{-(t-s)A}\mathbb{P}(u(s), \nabla)f(s)ds + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds \end{aligned}$$

for $t \in (0, T)$, and $u(0) = u_0$.

We now state the our existence and uniqueness results for mild solutions of (2.3) in L^p spaces.

2.1. Theorem. *Let $n \geq 2$, $T > 0$, $p \in [n, \infty)$ and $q \in [p, \infty)$. Let f be a vector-valued globally Lipschitz continuous function satisfying (H1), (H2) and (H3). Assume that $u_0 \in L_\sigma^p(\mathbb{R}^n)$, and that $F \in C(0, T; L_\sigma^p(\mathbb{R}^n))$. Then there exist $T_0 \in (0, T)$ and a unique mild solution u of (2.3) such that*

$$(2.5) \quad [t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}u(t)] \in C([0, T_0]; L_\sigma^q(\mathbb{R}^n))$$

$$(2.6) \quad [t \mapsto t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\nabla u(t)] \in C([0, T_0]; L^q(\mathbb{R}^n)).$$

2.2. Remark. (i) The semigroup $\{e^{-tA}\}_{t \geq 0}$ is not analytic.

(ii) Consider the case $p = \infty$ and $u_0 \in L_\sigma^\infty(\mathbb{R}^n)$ or $u_0 \in BUC_\sigma$, i.e., u_0 do not decay at space infinity. In this case, one might expect to obtain the existence result for the mild solutions $u \in C([0, T_0]; \dot{B}_{\infty,1}^0)$ satisfying (2.3) provided that $u_0 \in \dot{B}_{\infty,1}^0(\mathbb{R}^n)$ and $\nabla \cdot u_0 = 0$. In [27], this is discussed for the case $f(x) = Mx$.

The proof of Theorem 2.1 is based on Kato's iteration procedure. The key is to derive appropriate smoothing estimates for the semigroup and its gradient; see Proposition 3.3. Uniqueness follows the by Gronwall's inequality.

3. ESTIMATES FOR THE SEMIGROUP

In this section we prepare the linear estimates needed for the iteration scheme. Let f be a vector-valued globally Lipschitz continuous function satisfying (H1), (H2) and (H3).

We the define the realization of the operator

$$(3.1) \quad \mathcal{L}u := -\Delta u - (f, \nabla)u, \quad x \in \mathbb{R}^n,$$

in $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ as follows. Set

$$\begin{aligned} Lu &:= \mathcal{L}u \\ D(L) &:= \{u \in W^{2,p}(\mathbb{R}^n); (f, \nabla)u \in L^p(\mathbb{R}^n)\}. \end{aligned}$$

Then the following result was proved by Lunardi and Metafune [20].

3.1. Proposition. *Let $1 < p < \infty$. Then the operator $-L$ generates a C_0 -semigroup $\{e^{-tL}\}_{t \geq 0}$ on $L^p(\mathbb{R}^n)$.*

3.2. Remark. (i) The semigroup $\{e^{-tL}\}_{t \geq 0}$ is not analytic; see [20].

(ii) The family $\{e^{-tL}\}_{t \geq 0}$ is also a semigroup on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$, which in the latter case is not strongly continuous.

(iii) If $f(x) = Mx$ where M is a constant matrix, the semigroup $\{e^{-tL}\}_{t \geq 0}$ has an explicit representation given by

$$e^{-tL}\varphi(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} \varphi(e^{tM}x - y)e^{-\frac{1}{4}(Q_t^{-1}y, y)}dy, \quad x \in \mathbb{R}^n, \quad t > 0,$$

where Q_t for $t > 0$ is given by $Q_t := \int_0^t e^{sM}e^{sM^T}ds$.

For the iteration scheme described in the next section it is essential that the associated semigroup maps an L^p -function u with $\nabla \cdot u = 0$ into the space of L^p -functions which are divergence free. We therefore introduce the operator $\mathcal{A}$ by

$$\mathcal{A}u := \mathcal{L}u + (u, \nabla)f,$$

where $u = (u^1, \dots, u^n)$. Thus $\mathcal{A}$ is an $n \times n$ operator matrix given by

$$\mathcal{A} = \mathcal{L} \text{Id} + (\nabla f)$$

where Id denotes the identity matrix. Observe that

$$\nabla \cdot \{(f, \nabla)u - (u, \nabla)f\} = 0, \quad \text{provided } \nabla \cdot u = 0 \text{ and } \nabla \cdot f = 0.$$

Hence, we define the realization A of $\mathcal{A}$ in $L_\sigma^p(\mathbb{R}^n)$ as

$$(3.2) \quad \begin{aligned} Au &:= \mathcal{A}u, \\ D(A) &:= D(\mathcal{L})^n \cap L_\sigma^p(\mathbb{R}^n). \end{aligned}$$

By standard perturbation theory, $-A$ generates a C_0 -semigroup $\{e^{-tA}\}_{t \geq 0}$ on L_σ^p for all $p \in (1, \infty)$.

In the case where $f(x) = Mx$, the semigroup $\{e^{-tA}\}_{t \geq 0}$ is given by

$$(3.3) \quad (e^{-tA}u)(x) := \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} e^{-tM} \int_{\mathbb{R}^n} u(e^{tM}x - y) e^{-\frac{1}{4}(Q_t^{-1}y, y)} dy.$$

We cannot expect to have such a formula for the semigroup $\{e^{-tA}\}_{t \geq 0}$, in general.

We are now state $L^p - L^q$ smoothing properties for the semigroup e^{-tA} as well as gradient estimates for e^{-tA} . Note that due to the non-analyticity of $\{e^{-tA}\}_{t \geq 0}$, gradient estimates for e^{-tA} do not follow from the general theory of semigroups (like the Stokes semigroup). Notice also that in the special case where $f(x) = x$, $L^p - L^q$ smoothing estimates as well as gradient estimates for e^{-tA} were obtain by Gallay and Wayne [11]. For $f(x) = Mx$, these estimates were obtained in [17]. For the general case, we rely on the recent results of Lunardi and Metafuno [20] and Bertholdi and Lorenzi [4].

3.3. Proposition. [[20], Prop. 5.4], [[4], Thm. 4.7, Cor. 4.8]. *Let $n \geq 2$, $1 < p < \infty$ and $p \leq q \leq \infty$.*

a) *Then there exist constants $C > 0$ and $\omega \in \mathbb{R}$ such that*

$$(3.4) \quad \|e^{-tA}\varphi\|_q \leq C e^{\omega t} t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_p, \quad t \geq 0, \varphi \in L^p(\mathbb{R}^n),$$

$$(3.5) \quad \|\nabla e^{-tA}\varphi\|_p \leq C e^{\omega t} t^{-\frac{1}{2}} \|\varphi\|_p, \quad t \geq 0, \varphi \in L^p(\mathbb{R}^n).$$

b) *There exist constants $C' > 0$ and $\omega' \in \mathbb{R}$ such that*

$$(3.6) \quad \|\nabla^2 e^{-tA}\varphi\|_p \leq C' e^{\omega' t} t^{-1} \|\varphi\|_p, \quad t \geq 0, \varphi \in L^p(\mathbb{R}^n).$$

c) *Moreover, let $1 < p < q \leq \infty$ and $\varphi \in L^p(\mathbb{R}^n)$. Then*

$$(3.7) \quad t^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \|e^{-tA}\varphi\|_q \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$(3.8) \quad t^{\frac{1}{2}} \|\nabla e^{-tA}\varphi\|_p \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$(3.9) \quad t \|\nabla^2 e^{-tA}\varphi\|_p \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

4. PROOF OF THE MAIN RESULT

For a given globally Lipschitz continuous function f satisfying (H1), (H2), (H3), consider the substitution $u(x, t) := U(x, t) + f(x)$ and $\tilde{P}(x, t) := P(x, t) + \Pi(x)$. Then (U, P) is a solution of (1.1) in the classical sense if and only if $(u, \tilde{P})$ satisfies (1.3). We thus consider in the following (1.3) and its abstract formulation in (2.3), or (2.4). We only show the proof for the case $p = n$; the case $p > n$ is similar.

Proof of Theorem 2.1. Let $n \geq 2$ and $u_0 \in L_\sigma^n(\mathbb{R}^n)$. Assume that $F \in C(0, \infty; L_\sigma^n(\mathbb{R}^n))$. Recall that $\tilde{F} = F + \Delta f - (f, \nabla)f - \nabla \Pi$ and $\nabla \cdot \tilde{F} = 0$. For $j \geq 1$ and $t > 0$ we define functions u_j successively by

$$(4.1) \quad u_1(t) := e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\tilde{F}(s)ds,$$

$$(4.2) \quad u_{j+1}(t) := u_1(t) - \int_0^t e^{-(t-s)A}\mathbb{P}(u_j(s), \nabla)u_j(s)ds + 2 \int_0^t e^{-(t-s)A}\mathbb{P}(u_j(s), \nabla)fds.$$

Since $\{e^{-tA}\}_{t \geq 0}$ acts on $L_\sigma^p(\mathbb{R}^n)$ for $p \in (1, \infty)$, it follows from the definition of the Helmholtz projection that the functions u_j are divergence-free for all $t > 0$ and all j .

For $T \in (0, 1]$ and $\delta \in (0, 1)$ we define

$$K_0 := \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \|e^{-tA}u_0\|_{n/\delta} \quad \text{and} \quad K'_0 := \sup_{0 < t \leq T} t^{1/2} \|\nabla e^{-tA}u_0\|_n.$$

By (3.7) and (3.8) in Proposition 3.3-(c), $K_0 \rightarrow 0$ and $K'_0 \rightarrow 0$ as $T \rightarrow 0$. Similarly, we define $K_j := K_j(T)$ and $K'_j := K'_j(T)$ for $j \geq 1$ by

$$K_j(T) := \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \|u_j(t)\|_{n/\delta} \quad \text{and} \quad K'_j(T) := \sup_{0 < t \leq T} t^{1/2} \|\nabla u_j(t)\|_n.$$

Let us estimate K_1 and K'_1 : by definition and the $L^p - L^q$ smoothing property (3.4), we have

$$\begin{aligned} K_1 &= \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \|u_1(t)\|_{n/\delta} \\ &\leq K_0 + C \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \int_0^t \|e^{-(t-s)A}\tilde{F}(s)\|_{n/\delta} ds \\ &\leq K_0 + C \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \int_0^t (t-s)^{-\frac{1-\delta}{2}} \|\tilde{F}(s)\|_n ds \\ &\leq K_0 + CT(\|-\Delta f + (f, \nabla)f + \Pi\|_n + \|F\|_{L^\infty(0, T; L^n(\mathbb{R}^n))}). \end{aligned}$$

Similarly,

$$K'_1 \leq K'_0 + CT(\|-\Delta f + (f, \nabla)f + \Pi\|_n + \|F\|_{L^\infty(0, T; L^n(\mathbb{R}^n))}).$$

We thus have

$$(4.3) \quad K_1, K'_1 \rightarrow 0 \quad \text{as} \quad T \rightarrow 0.$$

Next, it follows from (4.2), the $L^p - L^q$ smoothing of the semigroup and from the boundedness of $\mathbb{P}$ from $L^p(\mathbb{R}^n)$ into $L_\sigma^p(\mathbb{R}^n)$ that

$$\begin{aligned} &\|u_{j+1}(t)\|_{n/\delta} \\ &\leq \|u_1\|_{n/\delta} + \int_0^t \|e^{-(t-s)A}\mathbb{P}(u_j(s), \nabla)u_j(s)\|_{n/\delta} ds + 2 \int_0^t \|e^{-(t-s)A}\mathbb{P}(u_j(s), \nabla)f\|_{n/\delta} ds \\ &\leq t^{-\frac{1-\delta}{2}} K_1 + C \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r} - \frac{\delta}{n})} \|(u_j(s), \nabla)u_j(s)\|_r ds + C \int_0^t \|u_j(s)\|_{n/\delta} ds, \end{aligned}$$

where $r = \frac{n}{1+\delta}$. In order to estimate the second term on the right hand side of last inequality, we apply Hölder's inequality to conclude that

$$\|(u_j(s), \nabla)u_j(s)\|_r \leq \|u_j(s)\|_{n/\delta} \|\nabla u_j(s)\|_n \leq K_j K'_j s^{-\frac{1-\delta}{2} - \frac{1}{2}}.$$

This implies

$$\|u_{j+1}(t)\|_{n/\delta} \leq t^{-\frac{1-\delta}{2}} K_1 + CK_j K'_j \int_0^t (t-s)^{-\frac{1}{2}} s^{-1+\frac{\delta}{2}} ds + CK_j \int_0^t s^{-\frac{1-\delta}{2}} ds.$$

Multiplying with $t^{\frac{1-\delta}{2}}$ and taking $\sup_{0 < t \leq T}$ on both sides, we obtain

$$K_{j+1} \leq K_1 + C_1 K_j K'_j + C_2 T K_j$$

with some constants C_1, C_2 , independent of j and T .

Similarly, applying ∇ to (4.2) and estimating it with respect to the L^n -norm, it follows from (3.4) and (3.5) that

$$K'_{j+1} \leq K'_1 + C_3 K_j K'_j + C_4 T K_j$$

for some constants C_3 and C_4 . By (4.3), for any $\lambda > 0$ there exists $\tilde{T}_0 > 0$ such that $K_1, K'_1 \leq \lambda$ for all $T \leq \tilde{T}_0$. So, we fix $\tilde{T}_0 \leq \min(1, \frac{1}{3C_2}, \frac{1}{3C_4})$ provided $\lambda \leq \min(\frac{1}{9C_1}, \frac{1}{9C_3})$. We thus obtain bounds for $K_j(T)$ and $K'_j(T)$ for any $T \leq \tilde{T}_0$ uniformly in j provided that $\tilde{T}_0$ is small enough. Indeed, $\sup_j K_j, K'_j \leq 3\lambda$ for $T \leq \tilde{T}_0$.

The uniform bounds of K_j and K'_j imply that $t^{\frac{1}{2}-\frac{n}{2q}} \|u_j(t)\|_q$ as well as $t^{1-\frac{n}{2q}} \|\nabla u_j(t)\|_q$ are bounded for $q \in [n, \infty)$, $t \leq \tilde{T}_0$ and all $j \in \mathbb{N}$. The continuity of the above functions follows from similar calculations and (3.7).

We finally derive estimates for the differences $u_{j+1} - u_j$. Indeed, for all $j \geq 1$ put

$$L_j(T) := \sup_{0 < t \leq T} t^{\frac{1-\delta}{2}} \|u_{j+1}(t) - u_j(t)\|_{n/\delta} \quad \text{and} \quad L'_j(T) := \sup_{0 < t \leq T} t^{1/2} \|\nabla u_{j+1}(t) - \nabla u_j(t)\|_n.$$

Similarly as before, we have for all $j \geq 1$

$$L_j(T) \leq C_5 \lambda (L_{j-1} + L'_{j-1}) + C_6 T L_{j-1},$$

$$L'_j(T) \leq C_7 \lambda (L_{j-1} + L'_{j-1}) + C_8 T L_{j-1}$$

with some positive constants C_5, C_6, C_7 and C_8 . We now choose $T_0 \leq \tilde{T}_0$ small enough so that $T_0 \leq \min(\frac{1}{8C_6}, \frac{1}{8C_8})$ provided $4(C_5 + C_7)\lambda \leq 1$. Hence we have $(L_{j+1} + L'_{j+1})/(L_j + L'_j) \leq 1/2$ for all $j \geq 1$ and $T \leq T_0$. This implies that L_j and L'_j tend to zero as $j \rightarrow \infty$. It thus follows that the above sequences are Cauchy sequences and we conclude that there are unique limit functions

$$[t \mapsto t^{\frac{1}{2}-\frac{n}{2q}} u(t)] \in C([0, T_0]; L^q_\sigma), \quad [t \mapsto t^{1-\frac{n}{2q}} v(t)] \in C([0, T_0]; L^q)$$

of the sequences $\{t^{\frac{1}{2}-\frac{n}{2q}} u_j(t)\}_{j \geq 1}$ (if necessary, we shall take its subsequence) and $\{t^{1-\frac{n}{2q}} \nabla u_j(t)\}_{j \geq 1}$. Finally, note that $v(t) = t^{1/2} \nabla u(t)$ and that u is a mild solution of (2.3) on $[0, T_0]$.

Uniqueness of mild solutions follows from standard Gronwall's inequality. This completes the proof of the first assertion of Theorem 2.1. $\square$

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