

LOCAL MODULARITY, PILLAY'S ONE-BASEDNESS AND CF-PROPERTY IN $\mathcal{o}$ -MINIMAL STRUCTURES

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ABSTRACT. In $\mathcal{o}$ -minimal structures, weak one-basedness, weak local modularity, strong linearity, generic linearity, linearity and CF-property are equivalent. We discuss the implications among modularity, local modularity, Pillay's one-basedness and CF-property.

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1. INDEPENDENCE RELATION IN ROSY THEORIES AND DIMENSION IN $U^p = 1$ THEORIES

Let $\mathcal{M}$ be a sufficiently saturated model of an L -theory T . $e \in \mathcal{M}^{\text{eq}}$ is an imaginary iff $e = \bar{a}/E$, where $E(\bar{x}, \bar{y})$ is an $\emptyset$ -definable equivalence relation with $\text{lh}(\bar{a}) = \text{lh}(\bar{x}) = \text{lh}(\bar{y})$, where $\bar{a} \subset \mathcal{M}$ is a finite tuple. For $e \in \mathcal{M}^{\text{eq}}$ and $A \subset \mathcal{M}^{\text{eq}}$ we write $e \in \text{acl}^{\text{eq}}(A)$ if $|\{\sigma(e) : \sigma \in \text{Aut}(\mathcal{M}^{\text{eq}}/A)\}|$ is finite. $\bar{a}, \bar{b}, \bar{c}, \dots$ denote finite tuples of $\mathcal{M}^{\text{eq}}$ and $A, B, C, D, \dots$ denote small subset of $\mathcal{M}^{\text{eq}}$.

The independence calculus See [A].

A *symmetric* ternary relation $* \downarrow_* *$ on $\mathcal{M}^{\text{eq}}$ has the independence calculus if the following 8 conditions hold:

- (1) Invariance: $A \downarrow_B C$ and $ABC \equiv A'B'C'$ imply $A' \downarrow_{B'} C'$
- (2) Normality: $A \downarrow_B C$ implies $A \downarrow_B BC$.
- (3) Monotonicity: $A \downarrow_B C$ and $A_0 \subseteq A$ imply $A_0 \downarrow_B C$
- (4) *Transitivity* : If $B \subseteq C \subseteq D$, then
 $A \downarrow_B D$ iff $A \downarrow_B C$ and $A \downarrow_C D$
- (5) Extension: There exists $A' \equiv_B A$ such that $A' \downarrow_B C$.

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- (6) Finite character: If $\bar{a} \downarrow_B C$ for any finite tuple $\bar{a} \subseteq A$, then $A \downarrow_B C$.
- (7) *Local character*: For any $B, A \subset \mathcal{M}$, there exist a cardinal $\kappa(B)$ and $A_0 \subseteq A$ such that $|A_0| \leq \kappa(B)$ and $B \downarrow_{A_0} A$.
- (8) Anti-reflexivity: $\bar{a} \downarrow_A \bar{a}$ implies $\bar{a} \in \text{acl}^{\text{eq}}(A)$.

We say $\mathcal{M}^{\text{eq}}$ is rosy if it has the independence calculus.

If $\downarrow$ is the thorn independence, $\text{symmetry} \Leftrightarrow \text{transitivity} \Leftrightarrow \text{local character}$ modulo the above other properties. In any $\text{U}^{\text{p}} = 1$ (rosy of rank one) structure $\mathcal{M}$, (i.e. $a \in \text{acl}^{\text{eq}}(A)$ iff $\text{U}^{\text{p}}(a/A) = 0$, where $a \in \mathcal{M}$. $b \notin \text{acl}^{\text{eq}}(A)$ iff $\text{U}^{\text{p}}(b/A) = 1$, where $b \in \mathcal{M}$.) for any $a, b \in \mathcal{M}$ and $B \subset \mathcal{M}^{\text{eq}}$, $a \in \text{acl}^{\text{eq}}(Bb) \setminus \text{acl}^{\text{eq}}(B)$ implies $b \in \text{acl}^{\text{eq}}(Ba)$, as $a \not\downarrow_B b$ iff $b \not\downarrow_B a$, we can define $\dim(\bar{a}/A)$ and $\dim(\bar{a}/Ae)$. For any $\bar{a} \subset \mathcal{M}$, $e = \bar{a}/E \in \mathcal{M}^{\text{eq}}$ and $A \subset \mathcal{M}^{\text{eq}}$, we define $\dim(e/A) := \dim(\bar{a}/A) - \dim(\bar{a}/Ae)$.

Fact 1.1. Any o-minimal structure is rosy of rank one (i.e. $\text{U}^{\text{p}} = 1$)

2. CF-PROPERTY

For real tuple $\bar{a} \in \mathcal{M}^n$ we write $\bar{a} \in \text{dcl}^{\text{eq}}(A)$ if $\{\sigma(\bar{a}) : \sigma \in \text{Aut}(\mathcal{M}^{\text{eq}}/A)\} = \{\bar{a}\}$.

We say that $(\mathcal{M}, <, \dots)$ is o-minimal structure if any definable set $X \subseteq \mathcal{M}$ is a finite union of intervals and points. $\text{acl}^{\text{eq}}(*) = \text{dcl}^{\text{eq}}(*)$ in the real sort $\mathcal{M}^n$ ($n < \omega$) by lexicographic order in o-minimal structures.

Let $\varphi(\bar{z})$ be an $\emptyset$ -definable set and let $C(x, y, \bar{z})$ be an $\emptyset$ -definable set such that

$$C(x, y, \bar{z}) \rightarrow \varphi(\bar{z}) \wedge y \in \text{dcl}(\bar{z}, x) \wedge \dim(x, y/\bar{z}) \leq 1$$

Then we write $y = f_{\bar{z}}(x)$.

For any $a \in \mathcal{M}$, we define an a -definable equivalent relation on U : For $\bar{c}, \bar{c}' \models \varphi(\bar{z})$, $\bar{c} \sim_a \bar{c}'$ iff there exists $a' > a$ such that $f_{\bar{c}}(x) = f_{\bar{c}'}(x)$ on the interval (a, a') . Put $\hat{\bar{z}}_a = \bar{z}/\sim_a \in \mathcal{M}^{\text{eq}}$.

Definition 2.1. We say that an o-minimal structure $(\mathcal{M}, <, \dots)$ has **Collapse of Families of functions-property** if for any $a \in \mathcal{M}$ and any $\bar{c} \models \varphi(\bar{z})$ we have

$$\dim(\hat{\bar{c}}_a/a) \leq 1.$$

We call $\hat{\bar{c}}_a$ Peterzil's germ of $\bar{c}$ at a .

An example of non CF-property: Multiplication violates CF-property.

In $(\mathbb{R}, +, \cdot, <)$, put $f_{(b_1, b_2)}(x) = b_1x + b_2$. Then $(b_1, b_2) \sim_a (c_1, c_2)$ iff $(b_1, b_2) = (c_1, c_2)$. So $\hat{b}_a = (b_1, b_2)/\sim_a$ is interdefinable with (b_1, b_2) over a . So $\dim(\hat{b}_a/a) = 2$.

3. AN EXAMPLE OF NON-LOCALLY MODULAR o-MINIMAL STRUCTURE WITH CF-PROPERTY

Definition of partial endomorphisms:

Suppose that $(I, +, <)$ is a group-interval (roughly speaking it has a partial definable group operation on the interval I) in an o-minimal structure. We say that an $\emptyset$ -definable partial unary function f is a partial endomorphism if

- (1) $\text{dom}(f) = I$ or $\text{dom}(f) = (-c, c)$ for some $c \in I$.
- (2) If $a, b, a+b \in \text{dom}(f)$, then $f(a+b) = f(a) + f(b)$.

Fact 3.1. [LPe]: Suppose that $(I, +, <)$ is a group-interval in a locally modular o-minimal structure $\mathcal{M}$ and let f be a partial endomorphism in I . Then there exists a total endomorphism with $\text{dom}(g) = \mathcal{M}$ such that $f = g|_{\text{dom}(f)}$.

In $\mathbb{R}$, put $\pi|(-1, 1)(x) = \pi \cdot x$ for each $x \in (-1, 1)$.

We can define $\pi \cdot y = n \cdot \pi|(-1, 1)\left(\frac{y}{n}\right)$ for each $y \in (-n, n)$, where $n \in \mathbb{N}$.

We can not extend $\pi|(-1, 1)$ to non-standard part, so $\text{Th}(\mathbb{R}, +, 0, 1, <, \pi|(-1, 1))$ is not locally modular by Fact 3.1. As $\text{Th}(\mathbb{R}, +, 0, 1, <, \pi|(-1, 1))$ does not interpret any field-interval, it has CF-property by trichotomy theorem [PeS].

4. PILLAY'S ONE-BASEDNESS FOR STRONG TYPES OF REAL TUPLES IN o -MINIMAL STRUCTURES

For any real type p over $C = \text{acl}^{\text{eq}}(C)$, we can find $\bar{a}\bar{b} \models p$ such that $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ and $\bar{b} = f(\bar{a}, \bar{c})$, where f is an $\emptyset$ -definable function and $\bar{c} \subseteq C$. We say that $\bar{a}$ is a generic tuple for p .

The germs of definable functions:

We define an $\bar{a}$ -definable equivalence relation $E_{f, \bar{a}}$ as follows. $E_{f, \bar{a}}(\bar{c}, \bar{c}') \Leftrightarrow$ there exists an open neighborhood U of $\bar{a}$ such that $f(\bar{x}, \bar{c}), f(\bar{x}, \bar{c}')$ are defined on U and $f(\bar{x}, \bar{c})|U = f(\bar{x}, \bar{c}')|U$ OR neither of $f(\bar{x}, \bar{c}), f(\bar{x}, \bar{c}')$ is defined on an open neighborhood of $\bar{a}$.

$\bar{c}_{E_{f, \bar{a}}} := \bar{c}/E_{f, \bar{a}}$ is the germ of definable function f around $\bar{a}$.

We call $\bar{c}_{E_{f, \bar{a}}}$ Pillay's germ of $\bar{c}$ at $\bar{a}$ (with respect to f).

Weak canonical bases for strong type $\text{stp}(\bar{a}\bar{b}/C) := \text{tp}(\bar{a}\bar{b}/\text{acl}^{\text{eq}}(C))$:

An algebraically closed set D is said to be weak canonical base of $\text{stp}(\bar{a}\bar{b}/C)$ if $D = \text{acl}^{\text{eq}}(D)$ is the smallest subset of $\text{acl}^{\text{eq}}(C)$ with $\bar{a}\bar{b} \downarrow_D C$, and we write $\text{wcb}(\text{stp}(\bar{a}\bar{b}/C))$ for the D .

If weak canonical bases exist, the germs of definable functions will be interdefinable with weak canonical bases over generic tuples.

A strong type without weak canonical base:

We work in $\text{Th}(\mathbb{R}, +, 0, 1, <, \pi|(-1, 1))$. Take $a, b, c > \mathbb{R}$ with $|a - b| < 1, |c - \pi \cdot b| < 1$ and a, b, c are independent in a saturated model of $\text{Th}(\mathbb{R}, +, 0, 1, <, \pi(*))$, where $\pi(*)$ is totally defined. Put

$$d := \pi(a - b) + c.$$

Then $\text{wcb}(\text{stp}(a, d/b, c))$ does not exist. (If it existed, then $\pi(a) \in \text{dcl}(a)$ would follow in $\text{Th}(\mathbb{R}, +, 0, 1, <, \pi|(-1, 1))$. Let $f(x, y, z) = \pi \cdot (x - y) + z$.

Claim: $(b, c)_{E_{f, a}} \in \text{dcl}(a, d)$.

Let $\sigma \in \text{Aut}(\mathcal{M}/a, d)$. Then we have

$$f(a, b, c) = \pi(a - b) + c = d = \sigma(d) = \pi(a - \sigma(b)) + \sigma(c) = f(a, \sigma(b), \sigma(c)),$$

$$E_{f, a}((b, c), (\sigma(b), \sigma(c)))$$

Later we will say that $\text{stp}(a, d/b, c)$ is Pillay's one-based.

Fact 4.1. [Pi] Suppose that $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ and $\bar{b} = f(\bar{a}, \bar{c})$, where f is an $\emptyset$ -definable function and $\bar{c} \subseteq C = \text{acl}^{\text{eq}}(C)$.

- (1) If $\text{wcb}(\text{stp}(\bar{a}\bar{b}/C))$ exists, there exists $\bar{d} \subseteq \text{wcb}(\text{stp}(\bar{a}\bar{b}/C))$ such that $\text{dcl}^{\text{eq}}(\bar{d}, \bar{a}) = \text{dcl}^{\text{eq}}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$.
- (2) If there exists $\bar{d} \subseteq C$ such that $\text{acl}^{\text{eq}}(\bar{d}, \bar{a}) = \text{acl}^{\text{eq}}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$, then $\text{acl}^{\text{eq}}(\bar{d}) = \text{wcb}(\text{stp}(\bar{a}\bar{b}/C))$, the weak canonical base of $\text{stp}(\bar{a}\bar{b}/C)$ exists.

Pillay's germ $\bar{c}_{E_{f,\bar{a}}}$ is an almost weak canonical base over a generic tuple $\bar{a}$.

Definition 4.2. Suppose that $\dim(\bar{a}\bar{b}/C) = \dim(\bar{a}/C) = |\bar{a}|$ and $\bar{b} = f(\bar{a}, \bar{c})$, where f is an $\emptyset$ -definable function and $\bar{c} \subseteq C = \text{acl}^{\text{eq}}(C)$.

We say that $\text{stp}(\bar{a}, \bar{b}/C)$ is Pillay's one-based if

$$\bar{c}_{E_{f,\bar{a}}} \in \text{acl}^{\text{eq}}(\bar{a}, f(\bar{a}, \bar{c})) = \text{acl}^{\text{eq}}(\bar{a}, \bar{b}).$$

If weak canonical bases exist, Pillay's one-basedness implies usual one-basedness: We have $\text{wcb}(\bar{a}, \bar{b}/\text{acl}^{\text{eq}}(C)) \subseteq \text{acl}^{\text{eq}}(\bar{a}, \bar{b})$, so $\dim(\bar{a}, \bar{b}/C) = \dim(\bar{a}, \bar{b}/C \cap \text{acl}^{\text{eq}}(\bar{a}, \bar{b})) \geq \dim(\bar{a}, \bar{b}/C)$.

- Question 4.3.** (1) In o -minimal structures, modularity implies Pillay's one-basedness for any strong type of real tuples. Does local modularity imply Pillay's one-basedness for any strong type of real tuples in o -minimal structures? It is known that local modularity implies CF-property [Pe].
- (2) In o -minimal theories, is there non Pillay's one-based strong type without its weak canonical base? There exists Pillay's one-based strong type without its weak canonical base as we mentioned above in this section.
- (3) Local modularity implies one-basedness under assuming the existence of weak canonical bases for any strong type in rosy theories [Y]. Is there locally modular rosy theory with a strong type without its weak canonical base?

5. PILLAY'S ONE-BASEDNESS FOR ANY STRONG TYPE OF REAL TUPLE IN o -MINIMAL CASE IMPLIES CF-PROPERTY

Peterzil's germ and Pillay's one are almost same by the following fact.

Fact 5.1. [LPe]: Assume dense linear ordering. $\hat{c}_a = \hat{c}'_a \Leftrightarrow \bar{c}_{E_{f,a}} = \bar{c}'_{E_{f,a}}$. In particular, we see $\text{dcl}^{\text{eq}}(\hat{c}_a, a) = \text{dcl}^{\text{eq}}(\bar{c}_{E_{f,a}}, a)$ and $\dim(\hat{c}_a/a) = \dim(\bar{c}_{E_{f,a}}/a)$.

Proof. ($\Leftarrow$): Clear. ($\Rightarrow$): $\hat{c}_a = \hat{c}'_a$ iff $f(x, \bar{c}) = f(x, \bar{c}')$ on (a, a') for some $a' > a$. Take $\sigma \in \text{Aut}(\mathcal{M})$ such that $\sigma(a) \models \text{tp}(a/\bar{c}, \bar{c}') \cup \{a < x < a'\}$. (If $a \not\perp \bar{c}, \bar{c}'$, then $f(x, \bar{c}) = f(x, \bar{c}') = d \in \text{dcl}^{\text{eq}}(\bar{c}, \bar{c}')$ is constant around a : Let M_0 be a prime model over $\bar{c}, \bar{c}'$. Let $a_+ \models \{x : a < x < e \text{ if } a < e \in M_0\}$. We have $a_+ \notin M_0$. By $f(a_+, \bar{c}) = f(a_+, \bar{c}')$ and dense linear ordering, there exists a closed interval $J = [e_1, e_2]$ containing a_+ such that $e_i \in M_0$ and $f(x, \bar{c}) = f(x, \bar{c}')$ on J . Note that $e_1 \leq a$. Otherwise $a < e_1$, then $a_+ < e_1$ follows, a contradiction.) So we may assume $a \perp \bar{c}, \bar{c}'$. Take a thorn non-forking extension of $\text{tp}(a/\bar{c}, \bar{c}')$ over $\bar{c}, \bar{c}', a, a'$ satisfying $a < x < a'$. As $f(x, \bar{c}) = f(x, \bar{c}')$ on the interval $I = (a, a')$ containing $\sigma(a)$, we have $\bar{c}_{E_{f,\sigma(a)}} = \bar{c}'_{E_{f,\sigma(a)}}$. By considering σ^{-1} which fixes $\bar{c}$ and $\bar{c}'$, we see $\bar{c}_{E_{f,a}} = \bar{c}'_{E_{f,a}}$. $\square$

Remark 5.2. For any generic tuple $\bar{a}$, we have $f_{\bar{c}}(\bar{a}) \in \text{dcl}(\bar{c}_{E_{f,\bar{a}}}, \bar{a})$.

Suppose that $\sigma \in \text{Aut}(\mathcal{M}/\bar{a})$ such that $\bar{c}_{E_{f,\bar{a}}} = \sigma(c)_{E_{f,\bar{a}}}$. Then we have $f(x, \bar{c}) = f(x, \sigma(\bar{c}))$ on an open neighborhood containing $\bar{a}$. So we see $f_{\bar{c}}(\bar{a}) = f_{\sigma(\bar{c})}(\bar{a}) =$

$\sigma(f_{\bar{c}}(\bar{a}))$ as desired.

Proposition 5.3. *Pillay's one-basedness implies CF-property.*

By the above remark and Pillay's one-basedness, $f_{\bar{c}}(a)$ is interalgebraic with $\bar{c}_{E_{f,a}}$ over a . Fact 5.1 and $\dim(f_{\bar{c}}(a)/a) \leq 1$ imply $\dim(\hat{c}_a/a) = \dim(\bar{c}_{E_{f,a}}/a) = \dim(f_{\bar{c}}(a)/a) \leq 1$. $\square$

6. DOES CF-PROPERTY IMPLY PILLAY'S ONE-BASEDNESS IN o -MINIMAL STRUCTURES?

A consideration for unary generic tuple a :

We have $\dim(\bar{c}_{E_{f,a}}/a) \leq 1$ by CF-property and Fact 5.1 and $f_{\bar{c}}(a) \in \text{dcl}(\bar{c}_{E_{f,a}}, a)$ by Remark 5.2.

Assume that $\dim(f_{\bar{c}}(a)/a) = 1$.

We have $\dim(a, f_{\bar{c}}(a), \bar{c}_{E_{f,a}}) = \dim(f_{\bar{c}}(a)/\bar{c}_{E_{f,a}}, a) + \dim(\bar{c}_{E_{f,a}}/a) + \dim(a) \leq 2$. On the other hand we have $\dim(a, f_{\bar{c}}(a), \bar{c}_{E_{f,a}}) = \dim(\bar{c}_{E_{f,a}}/a, f_{\bar{c}}(a)) + \dim(f_{\bar{c}}(a)/a) + \dim(a) = \dim(\bar{c}_{E_{f,a}}/a, f_{\bar{c}}(a)) + 2$.

Therefore we have $\bar{c}_{E_{f,a}} \in \text{acl}^{\text{eq}}(a, f_{\bar{c}}(a))$.

A consideration for n -ary generic tuple $\bar{a}$:

We have $\dim(\bar{a}, f_{\bar{c}}(\bar{a}), \bar{c}_{E_{f,\bar{a}}}) = \dim(f_{\bar{c}}(\bar{a})/\bar{c}_{E_{f,\bar{a}}}, \bar{a}) + \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) + \dim(\bar{a}) = \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) + \dim(\bar{a})$. On the other hand, we have $\dim(\bar{a}, f_{\bar{c}}(\bar{a}), \bar{c}_{E_{f,\bar{a}}}) = \dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}, f_{\bar{c}}(\bar{a})) + \dim(f_{\bar{c}}(\bar{a})/\bar{a}) + \dim(\bar{a})$.

Remark 6.1. The following are equivalent.

- (1) $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}, f_{\bar{c}}(\bar{a})) = 0$
- (2) $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) = \dim(f_{\bar{c}}(\bar{a})/\bar{a})$.
- (3) $\dim(\bar{c}_{E_{f,\bar{a}}}/\bar{a}) \leq \dim(f_{\bar{c}}(\bar{a})/\bar{a})$.

Proof. Use Remark 5.2 $f_{\bar{c}}(\bar{a}) \in \text{dcl}(\bar{a}, \bar{c}_{E_{f,\bar{a}}})$. $\square$.

Question 6.2. *Is there an o -minimal theory with CF-property with a non Pillay's one-based strong type? CF-property with elimination of imaginaries and the existence of weak canonical bases for any strong type imply modularity in o -minimal theories [Y].*

7. APPENDIX : WEAKLY LOCAL MODULARITY IMPLIES STRONG LINEARITY IN GEOMETRIC STRUCTURES

The following proof appears at Propostion 2.17 in [BV], but it is complicated.

We give a modified proof.

Suppose that $(\mathcal{M}, \perp)$ is weakly local modular and $\hat{\mathcal{C}} := \{\hat{C}(x, y, \hat{a}) : \hat{a} \models \hat{\varphi}(\hat{z})\}$ is 2-dimensional almost normal interpretable family of plane curve.

So $\dim(\hat{\varphi}(\hat{z})) \geq 2$. We seek a contradiction.

Suppose that $\mathcal{M} \models \hat{\varphi}(\hat{a})$ and $\dim(\hat{a}) \geq 2$. Put $k = \dim(\bar{a})$ and $\bar{a} = a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_n$, where $a_{\leq k}$ is acl-independent and $a_j \in \text{acl}(a_{\leq k})$ for $j = k+1, \dots, n$. Take c, d such that $\hat{C}(c, d, \hat{a}), c \perp \bar{a}, \dim(cd) = 2$. By weakly local modularity, there exists B such that $B \perp \bar{a}, cd$ and $\bar{a} \perp_{\text{acl}(\bar{a}, B) \cap \text{acl}(cdB)} cd$. Put $X = \text{acl}(\bar{a}, B)$ and $Y = \text{acl}(cdB)$.

Then we have $X \perp_{X \cap Y} Y$.

Claim 1 : $\dim(X \cap Y/B) = 1$

$\dim(XY/B) = \dim(Y/XB) + \dim(X/B) = \dim(cd/\bar{a}, B) + \dim(X/B)$ (as $cd \downarrow_{\bar{a}} B$) = $\dim(cd/\bar{a}) + \dim(X/B) = 1 + k$. Since $k + 1 = \dim(XY/B) = \dim(X/Y) + \dim(Y/B) = \dim(X/X \cap Y) + 2$, we have $\dim(X/X \cap Y) = k - 1 < k = \dim(X/B)$. So Claim 1 follows.

Take t such that $\text{acl}(tB) = X \cap Y$. We have $\bar{a} \downarrow_{tB} cd$. As $\bar{a} \downarrow_{tB} cd$ and $d \in \text{acl}(c\bar{a}, B)$, $d \in \text{acl}(ctB)$ follows. On the other hand, we have $\dim(\bar{a}/tB) = \dim(X/X \cap Y) < \dim(X/B) = k$, we may assume $a_k \in \text{acl}(a_{\leq k-1}tB)$. Take $\bar{u}, \bar{v} \subset B$ such that $d \in \text{acl}(ct\bar{u})$ and $a_k \in \text{acl}(a_{\leq k-1}t\bar{v})$. Put $\bar{w} := \bar{u}\bar{v} \subset B$. As $\bar{w} \downarrow cd\bar{a}$, we have $a_k \notin \text{acl}(a_{\leq k-1}\bar{w})$ and $d \notin \text{acl}(c, \bar{w})$. So we have $a_k \in \text{acl}(a_{\leq k-1}t\bar{w}) \setminus \text{acl}(a_{\leq k-1}\bar{w})$ and $d \in \text{acl}(ct\bar{w}) \setminus \text{acl}(c\bar{w})$. By exchange property of algebraic closure for geometric structures, $t \in \text{acl}(a_{\leq k}\bar{w})$ and $t \in \text{acl}(cd\bar{w})$.

Take $\bar{b} \models \text{tp}(\bar{a}/\text{acl}(cd\bar{w}))$ such that $\bar{a} \downarrow_{cd\bar{w}} \bar{b}$. Note that $\text{acl}(cd\bar{w}) = \text{acl}(cdt\bar{w}) = \text{acl}(ct\bar{w})$. Take $\bar{b} \models \text{tp}(\bar{a}/\text{acl}(cd\bar{w}))$ such that $\bar{a} \downarrow_{cd\bar{w}} \bar{b}$. Note that $\text{acl}(cd\bar{w}) = \text{acl}(cdt\bar{w}) = \text{acl}(ct\bar{w})$.

Claim 2: $c \notin \text{acl}(\bar{a}, \bar{b}, t\bar{w})$

As $c \downarrow \bar{a}$ and $\bar{a}, cd \downarrow B$, we have $c \downarrow \bar{a}, B$. As $t, \bar{w} \in \text{acl}(\bar{a}, B)$, $c \downarrow \bar{a}, t\bar{w}$ follows. By automorphism fixing $\text{acl}(cd\bar{w})$ which sends $\bar{a}$ to $\bar{b}$, we have $c \downarrow \bar{b}, t\bar{w}$. So $c \downarrow_{t\bar{w}} \bar{b}$ follows. Since $\bar{a} \downarrow_{ct\bar{w}} \bar{b}$, we have $\bar{a}, c \downarrow_{t\bar{w}} \bar{b}$. So $c \downarrow_{\bar{a}, t, \bar{w}} \bar{b}$ and $c \notin \text{acl}(\bar{a}, t\bar{w})$, Claim 2 follows. Take $\bar{b} \models \text{tp}(\bar{a}/\text{acl}(cd\bar{w}))$ such that $\bar{a} \downarrow_{cd\bar{w}} \bar{b}$. Note that $\text{acl}(cd\bar{w}) = \text{acl}(cdt\bar{w}) = \text{acl}(ct\bar{w})$.

Claim 3: $\hat{a} \notin \text{acl}(cd\bar{w})$

Note that $c \downarrow \bar{a}$ and $d \in \text{acl}(c\hat{a})$. As $B \downarrow cd\bar{a}$ and $\bar{w} \subset B$, we have $\bar{w} \downarrow_{cd} \bar{a}$. So $\dim(\hat{a}/cd\bar{w}) = \dim(\hat{a}/cd) = \dim(cd\hat{a}) - \dim(cd) = \dim(c\hat{a}) - 2 = \dim(c) + \dim(\hat{a}) - 2 = \dim(\hat{a}) - 1 \geq 1$, because we assume $\dim(\hat{a}) \geq 2$.

Claim 4: $\hat{a} \notin \text{acl}(\hat{b})$: By Claim 3 and $\hat{a} \downarrow_{cd\bar{w}} \hat{b}$.

By Claim 2, $c \notin \text{acl}(\bar{a}, \bar{b})$. As $\bar{b} \models \text{tp}(\bar{a}/\text{acl}(cd\bar{w}))$, we have $\hat{C}(c, d, \hat{a}) \wedge \hat{C}(c, d, \hat{b})$. $\{\hat{a} \models \varphi(\hat{z}) : \hat{C}(x, y, \hat{a}) \wedge \hat{C}(x, y, \hat{b}) \text{ is infinite}\}$ is infinite by Claim 4. Here, we use elimination of $\exists^\infty$ for geometric structures. $\hat{\mathcal{C}} := \{\hat{C}(x, y, \hat{a}) : \hat{a} \models \varphi(\hat{z})\}$ is not almost normal, a contradiction. $\square$

I want a direct proof of the converse implication i.e. strong linearity implies weakly local modularity. Suppose non-weakly local modularity. Construct a 2-dimensional almost normal interpretable family of plane curve.

REFERENCES

- [A] H.Adler, Explanation of independence, Ph.D thesis, Freiburg, 2005, arXiv:math.LO/0511616 v1, 24 Nov 2005.
- [BV] A.Berenstein and E.Vassiliev, Weakly one-based geometric structures, J.Symbolic Logic 77 (2012), 392-422.
- [LPe] J.Loveys and Y.Peterzil, Linear o -minimal structures, Israel J. Math. 81 (1993), 1-30.
- [Pe] Y.Peterzil, Constructing a group-interval in o -minimal structures, J.Pure Appl.Algebra 94 (1994), 85-100.
- [Pi] A.Pillay, Canonical bases in o -minimal and related structures, (2006) unpublished?
- [PeS] Y.Peterzil and S.Starchenko, A trichotomy theorem for o -minimal structures, Proc. London Math Soc. (3) 77 (1998), 481-523.
- [Y] I.Yoneda, Weak canonical bases and geometrically lower rosiness theory, in preparation, February 2024.

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