

A Criterion for Finding Existence and Nonexistence Domains of Solutions of Nonlinear Equations

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Let X and Y be Banach spaces, f and g be operators on $D \subseteq X$ with values in Y , where f is Fréchet differentiable in an open convex set $D_0 \subset D$, while the differentiability of g is not assumed. Let $B(x^0, r)$ be the open ball with center x^0 and radius r in X , $\bar{B}(x^0, r)$ denote its closure and $\bar{B}(x^0, R) \subset D_0$.

To find a solution x^* of the equation

$$f(x) + g(x) = 0, \quad (1)$$

several authors[6,7,9-11] have considered the iteration

$$x_0 = x^0, \quad x_{n+1} = x_n - f'(x_n)^{-1}(f(x_n) + g(x_n)), \quad n \geq 0.$$

In this paper, we consider the iteration

$$x_0 \in \bar{B}(x^0, R), \quad x_{n+1} = x_n - A(x_n)^{-1}(f(x_n) + g(x_n)), \quad n \geq 0, \quad (2)$$

where $A(x)$ is an approximation for $f'(x)$. Assume that $A(x^0)^{-1}$ exists and that for any $x, y \in \bar{B}(x^0, r) \subset \bar{B}(x^0, R)$,

the following hold:

$$\|A(x^0)^{-1}(A(x)-A(x^0))\| \leq w_0(\|x-x^0\|)+b,$$

$$\begin{aligned} & \|A(x^0)^{-1}(f'(x+t(y-x))-A(x))\| \\ & \leq w(\|x-x^0\|+t\|y-x\|)-w_0(\|x-x^0\|)+c, \quad t \in [0,1], \end{aligned}$$

$$\|A(x^0)^{-1}(g(x)-g(y))\| \leq e(r)\|x-y\|,$$

where $w(r+\tau)-w_0(r), \tau \geq 0$ and $e(r)$ are nondecreasing functions with $w(0)=w_0(0)=e(0)=0$, $w_0(r)$ is differentiable $w_0'(r) > 0$ at every point of $[0, R]$, and the constants b, c satisfy $b \geq 0, c \geq 0$ and $b+c < 1$. Put

$$\eta = \|A(x^0)^{-1}(f(x^0)+g(x^0))\| > 0, \quad \phi(r) = \eta - r + \int_0^r w(t)dt,$$

$$\psi(r) = \int_0^r e(t)dt, \quad x(r) = \phi(r) + \psi(r) + (b+c)r,$$

and denote by x^* and r^* the minimal value and the minimal point of $x(r)$ in $[0, R]$. Furthermore, let

$$\omega(r) = 1 - w_0(r) - b, \quad r_x = \|x - x^0\|,$$

$$\eta_x = \|A(x)^{-1}(f(x)+g(x))\|, \quad v_x(r) = \omega(r_x)\eta_x - x(r_x) + x(r).$$

Theorem 1. For some $x_0 \in \bar{B}(x^0, R)$, the iteration (2) is well defined for all $n \geq 0$, and $\{x_n\}$ converges to a solution $x^* \in B(x^0, r^*)$ of (1) if and only if there exists an $x \in B(x^0, r^*)$ such that $v_x(r^*) < 0$.

Theorem 2. Let $x \in B(x^0, r^*)$.

(i) If $v_x(r^*) > 0$, then there is no solution of (1) in $B(x, \delta) \cap \bar{B}(x^0, R)$, where δ is a unique positive root of the scalar equation

$$q(t) = w(r_x)^{-1} (x(t+r_x) - x(r_x)) + 2t - \eta_x = 0.$$

We have $\delta \geq \min\{r^* - r_x, a_x/2\}$ and $a_x > (x(r_x) - x(r^*)) / w(r_x)$.

(ii) If $v_x(r^*) \leq 0$, then there exists a unique solution x^* of (1) in $\bar{B}(x, r^* - r_x) \subseteq \bar{B}(x^0, r^*)$, which can be obtained by (2) with $x_0 = x$.

For the case of Newton's method applied to the equation

$$f(x) = 0, \quad (3)$$

we assume that $f'(x^0)^{-1}$ exists and $f'(x^0)^{-1}f'(x)$ satisfies a Lipschitz condition in D_0 with a Lipschitz constant $K > 0$ and $1/K \leq R$. Then we can take $A(x) = f'(x)$, $g(x) = 0$, $w_0(r) = w(r) = Kr$, $b = c = e(r) = 0$, so that we have $x(r) = \eta - r + Kr^2/2$. Hence Theorems 1 and 2 reduce to the following:

Corollary 1. For some $x_0 \in \bar{B}(x^0, R)$, Newton's method is well defined for all $n \geq 0$ and $\{x_n\}$ converges to a solution $x^* \in B(x^0, 1/K)$ of (3), if and only if there is an $x \in B(x^0, 1/K)$ such that $h_x = Ka_x / (1 - Kr_x) < 1/2$.

Corollary 2. Let $x \in B(x^0, 1/K)$.

(i) If $h > 1/2$, then there is no solution of (3) in $B(x, \delta) \cap \bar{B}(x^0, R)$, where $\delta = 2a_x / (1 + \sqrt{1 + 2h_x}) \geq \min\{r^* - r_x, a_x/2\}$, and $a_x > (1 - Kr_x)/(2K)$.

(ii) If $h_x \leq 1/2$, then there exists a unique solution of (3) in $B(x, 1/K - r_x) \subseteq \bar{B}(x^0, 1/K)$, which can be obtained by Newton's method starting from x .

Theorems 1 and 2 generalize and deepen Rheinboldt-Dennis's results for Newton-like methods. Furthermore, Theorem 2 gives a foundation for constructing an algorithm which finds all the solutions of the equation (1) in a domain.

References

- 1). X. Chen and T. Yamamoto, Convergence domains of certain iterative methods for solving nonlinear equations, preprint;
- 2). J. E. Dennis, On the convergence of Newton-like methods, in Numerical Methods for Nonlinear Algebraic Equations, ed. P. Rabinowitz, Gordon and Breach, New York (1970), 163-181;
- 3). W. B. Gragg and R. A. Tapia, Optimal error bounds for the Newton-Kantorovich theorem, SIAM J. Numer. Anal. 11(1974), 10-13;

- 4). L.V.Kantorovich, On Newton's method for functional equations, Dokl.Akad.Nauk SSSR59(1948),1237-1240;
- 5). J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York(1970);
- 6). W.C. Rheinboldt, A unified convergence theory for a class of iterative process, SIAM J. Numer. Anal. 5(1968),42-63;
- 7). T. Yamamoto, Error bounds for Newton-like methods under Kantorovich type assumptions, Japan J. Appl. Math. 3(1986),295-313;
- 8). T. Yamamoto, A convergence theorem for Newton-like methods in Banach spaces, Numer.Math.51(1987),545-557;
- 9). T. Yamamoto, A note on a posteriori error bound of Zabrejko and Nguen for Zincenko's iteration, Numer. Funct. Anal. and Optimiz.9(1987),987-994;
- 10). P. P. Zabrejko and D. F. Nguen, The majorant method in the theory of Newton-Kantorovich approximations and the Pták error estimates, Numer. Funct. Anal. and Optimiz. 9(1987),671-684;
- 11). A. I. Zincenko, Some approximate methods of solving equations with nondifferentiable operators Dopovidi Akad. Nauk Ukrain. RSR(1963),156-161.