

The behaviour of dimension functions on unions of closed subsets I

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1 Introduction

All spaces we shall consider here are separable metrizable spaces.

It is well known that there exist (transfinite) dimension functions d such that $d(X_1 \cup X_2) > \max\{dX_1, dX_2\}$ even if the subspaces X_1 and X_2 are closed in the union $X_1 \cup X_2$.

Let $\mathcal{K}$ be a class of spaces, β, α be ordinals such that $\beta < \alpha$, and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. Define $m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^k X_i, \text{ where } X_i \text{ is closed in } X \text{ and } dX_i \leq \beta\}$, $m_{\mathcal{K}}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$ and $M_{\mathcal{K}}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\}$.

We will say that $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ do not exist if there is no space X from $\mathcal{K}$ with $dX = \alpha$ which is the union of finitely many closed subsets with $d \leq \beta$. It is evident that either $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ satisfy $2 \leq m_{\mathcal{K}}(d, \beta, \alpha) \leq M_{\mathcal{K}}(d, \beta, \alpha) \leq \infty$ or they do not exist.

Two natural questions arise.

Question 1.1 Determine the values of $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$ for given $\mathcal{K}, d, \beta, \alpha$.

Question 1.2 Find a (transfinite) dimension function d having for given pair $2 \leq k \leq l \leq \infty$, $m_{\mathcal{K}}(d, \beta, \alpha) = k$ and $M_{\mathcal{K}}(d, \beta, \alpha) = l$.

Let $\mathcal{C}$ be the class of metrizable compact spaces and $\mathcal{P}$ be the class of separable completely metrizable spaces. By $\text{trind}(\text{trInd})$ we denote Hurewicz's (Smirnov's) transfinite extension of ind (Ind) and Cmp is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let $\alpha = \lambda(\alpha) + n(\alpha)$ be the natural

decomposition of the ordinal $\alpha \geq 0$ into the sum of a limit number $\lambda(\alpha)$ (observe that $\lambda(\text{an integer } \geq 0) = 0$) and a nonnegative integer $n(\alpha)$. Let $\beta < \alpha$ be ordinals, put $p(\beta, \alpha) = \frac{n(\alpha)+1}{n(\beta)+1}$ and $q(\beta, \alpha) =$ the smallest integer $\geq p(\beta, \alpha)$. We have the following theorems. The outline of the proof will be presented in section 2.

Theorem 1.1 1. Let $0 \leq \beta < \alpha$ be finite ordinals. Then we have $m_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = q(\beta, \alpha)$ and $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$.
2. Let $\beta < \alpha$ be infinite ordinals. Then we have

$$m_{\mathcal{C}}(\text{trInd}, \beta, \alpha) = \begin{cases} q(\beta, \alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

$$M_{\mathcal{C}}(\text{trInd}, \beta, \alpha) = \begin{cases} \infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\ \text{does not exist}, & \text{otherwise} \end{cases}$$

Theorem 1.2 1. For every finite $\alpha \geq 1$ there exists a space $X_{\alpha} \in \mathcal{P}$ such that

- (a) $\text{Cmp}X_{\alpha} = \alpha$;
- (b) $X_{\alpha} = \bigcup_{i=1}^{\infty} Y_i$, where each Y_i is closed in X_{α} and $\text{Cmp}Y_i \leq 0$;
- (c) $X_{\alpha} \neq \bigcup_{i=1}^m Z_i$, where each Z_i is closed in X_{α} and $\text{Cmp}Z_i \leq \alpha - 1$ and m is any integer ≥ 1 .

2. For every infinite α with $n(\alpha) \geq 1$ there exists a space $X_{\alpha} \in \mathcal{C}$ such that

- (a) $\text{trInd}X_{\alpha} = \alpha$;
- (b) $X_{\alpha} = \bigcup_{i=1}^{\infty} Y_i$, where each Y_i is closed in X_{α} and finite-dimensional;
- (c) $X_{\alpha} \neq \bigcup_{i=1}^m Z_i$, where each Z_i is closed in X_{α} and $\text{trInd}Z_i \leq \alpha - 1$ and m is any integer ≥ 1 .

2 Evaluations of $m_{\mathcal{K}}(d, \beta, \alpha)$ and $M_{\mathcal{K}}(d, \beta, \alpha)$

The notation $X \sim Y$ means that the spaces X and Y are homeomorphic. At first we consider the following construction.

Step 1. Let X be a space without isolated points and P a countable dense subset of X . Consider Alexandroff's duplicate $D = X \cup X^1$ of X , where each point of X^1 is clopen in D . Remove from D those points of X^1 which do not correspond to any point from P . Denote the obtained space by $L(X, P)$. Observe that $L(X, P)$ is the disjoint union of X with the countable dense subset P^1 of $L(X, P)$ consisting of points from X^1 corresponding to the points from P . The space $L(X, P)$ is separable and metrizable. It will be compact if X is compact. Put $L_1(X, P) = L(X, P)$. Assume that X is a completely metrizable space (recall that the increment $bX \setminus X$ in any compactification bX of X is an F_{σ} -set in bX). Observe that $L(bX, P)$ is a compactification of $L(X, P)$ and the increment

$L(bX, P) \setminus L(X, P) (\sim bX \setminus X)$ is an F_σ -set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

Step 2. Let X be a space with a countable subset R consisting of isolated points. Let Y be a space. Substitute each point of R in X by a copy of Y . The obtained set W has the natural projection $pr : W \rightarrow X$. Define the topology on W as the smallest topology such that the projection pr is continuous and each copy of Y has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if X and Y are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \setminus R$ of X (which we will call *basic* for the space $L(X, R, Y)$) and countably many clopen copies of Y .

Step 3. Let X be a space without isolated points and P be a countable dense subset of X . Define $L_n(X, P) = L(L_1(X, P), P^1, L_{n-1}(X, P))$, $n \geq 2$. Observe that for any open subset O of $L_n(X, P)$ meeting the basic subset X of $L_n(X, P)$ there is a copy of $L_{n-1}(X, P)$ contained in O . Put $L_*(X, P) = \{*\} \cup \bigoplus_{n=1}^{\infty} L_n(X, P)$. (Here by $\{*\} \cup \bigoplus_{i=1}^{\infty} X_i$ we mean the one-point extension of the free union $\bigoplus_{i=1}^{\infty} X_i$ such that a neighborhood base at the point $*$ consists of the sets $\{*\} \cup \bigoplus_{i=k}^{\infty} X_i$, $k = 1, 2, \dots$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each q . $L_*(X, P)$ will be compact (completely metrizable) if X is the same.

All our dimension functions d are assumed to be monotone with respect to closed subsets and $d(\text{a point}) \leq 0$.

Lemma 2.1 *Let d be a dimension function and X be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where α is an ordinal. Let also P be a countable dense subset of X . Then*

- (a) *for every q we have $L_q(X, P) \neq \bigcup_{i=1}^k X_i$, where each X_i is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;*
- (b) *$L_*(X, P) \neq \bigcup_{i=1}^m X_i$, where each X_i is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and m is any integer ≥ 1 .*

All our classes $\mathcal{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(, ,)$.

Lemma 2.2 *Let $\mathcal{K}$ be a class of topological spaces, α be an ordinal ≥ 0 and d be a dimension function such that $dL(L(S, P), P^1, T) \leq \alpha$ for any S, T from $\mathcal{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any P . Let $X \in \mathcal{K}$ such that $X = \bigcup_{i=1}^k X_i$, where each X_i is closed in X , without isolated points and $dX_i \leq \alpha$. Let also P_i be a countable dense subset of X_i for each i . Then for each q the space $L_q(X, \bigcup_{i=1}^k P_i)$ exists and is the union of k^q closed subsets with $d \leq \alpha$.*

We will say that a dimension function d satisfies *the sum theorem of type A* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \leq \alpha_i$, where each α_i is finite and ≥ 0 , we have $dX \leq \alpha_1 + \alpha_2 + 1$. A space X is *completely decomposable in the sense of the dimension function d* if $dX = \alpha$, where α is an integer ≥ 1 , and $X = \bigcup_{i=1}^{\alpha+1} X_i$, where each X_i is closed in X and $dX_i = 0$. Observe that if this space X belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq \alpha + 1$ for each β with $0 \leq \beta < \alpha$.

We will say that a transfinite dimension function d satisfies *the sum theorem of type A_{tr}* if for any X being the union of two closed subspaces X_1 and X_2 with $dX_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $dX \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $dX \leq \alpha_2 + n(\alpha_1) + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$. A space X is *completely decomposable in the sense of the transfinite dimension function d* if $dX = \alpha$, where α is an infinite ordinal with $n(\alpha) \geq 1$, and $X = \bigcup_{i=1}^{n(\alpha)+1} X_i$, where each X_i is closed in X and $dX_i = \lambda(\alpha)$. Observe that if this space X belongs to a class $\mathcal{K}$ of topological spaces then $m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq n(\alpha) + 1$ for each β with $\lambda(\alpha) \leq \beta < \alpha$.

To every space X one assigns *the large inductive compactness degree* Cmp as follows.

- (i) $\text{Cmp } X = -1$ iff X is compact;
- (ii) $\text{Cmp } X = 0$ iff there is a base $\mathcal{B}$ for the open sets of X such that the boundary $\text{Bd } U$ is compact for each U in $\mathcal{B}$;
- (iii) $\text{Cmp } X \leq \alpha$, where α is an integer ≥ 1 , if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that $\text{Cmp } C \leq \alpha - 1$;
- (iv) $\text{Cmp } X = \alpha$ if $\text{Cmp } X \leq \alpha$ and $\text{Cmp } X > \alpha - 1$;
- (v) $\text{Cmp } X = \infty$ if $\text{Cmp } X > \alpha$ for every positive integer α .

Recall also the definitions of *the transfinite inductive dimensions* trind and trInd .

- (i) $\text{trInd } X = -1$ iff $X = \emptyset$;
- (ii) $\text{trInd } X \leq \alpha$, where α is an ordinal ≥ 0 , if for each pair of disjoint closed subsets A and B of X there exists a partition C between A and B in X such that $\text{trInd } C < \alpha$;
- (iii) $\text{trInd } X = \alpha$ if $\text{trInd } X \leq \alpha$ and $\text{trInd } X \leq \beta$ holds for no $\beta < \alpha$;
- (iv) $\text{trInd } X = \infty$ if $\text{trInd } X \leq \alpha$ holds for no ordinal α .

The definition of trind is obtained by replacing the set A in (ii) with a point of X .

Remark 2.1 (i) Note that Cmp satisfies the sum theorem of type A ([ChH, Theorem 2.2]) and for each integer $\alpha \geq 1$ there exists a separable completely metrizable space C_α with $\text{Cmp } C_\alpha = \alpha$ which is completely decomposable in the sense of Cmp ([ChH, Theorem 3.1]). For the convenience of the reader, we recall that $C_\alpha = \{0\} \times ([0, 1]^\alpha \setminus (0, 1)^\alpha) \cup \bigcup_{i=1}^\infty \{x_i\} \times [0, 1]^\alpha \subset I^{\alpha+1}$, where $\{x_i\}_{i=1}^\infty$ is a sequence of real numbers such that $0 < x_{i+1} < x_i \leq 1$ for all i and $\lim_{i \rightarrow \infty} x_i = 0$. Note that the closed subsets in the decomposition of C_α can be assumed without isolated points.

(ii) Note also that trInd satisfies the sum theorem of type A_{tr} ([E, Theorem 7.2.7]) and for each infinite ordinal α with $n(\alpha) \geq 1$ there exists a metrizable compact space S^α (Smirnov's

compactum) with $\text{trInd} S^\alpha = \alpha$ which is completely decomposable in the sense of trInd ([Ch, Lemma 3.5]). Recall that Smirnov's compacta $S^0, S^1, \dots, S^\alpha, \dots, \alpha < \omega_1$, are defined by transfinite induction: S^0 is the one-point space, $S^\alpha = S^\beta \times [0, 1]$ for $\alpha = \beta + 1$, and if α is a limit ordinal, then $S^\alpha = \{*_\alpha\} \cup \bigcup_{\beta < \alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined S^β 's, where $*_\alpha$ is the compactifying point. Note that the closed subsets in the decomposition of S^α can be assumed without isolated points.

(iii) Observe that trind satisfies another sum theorem. Namely, for any X being the union of two closed subspaces X_1 and X_2 with $\text{trind} X_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $\text{trind} X \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $\text{trind} X \leq \alpha_2 + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$ [Ch, Theorem 3.9].

Proposition 2.1 (i) Let $\mathcal{K}$ be a class of topological spaces, d be a dimension function satisfying the sum theorem of type A, α be an integer ≥ 1 and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d . Then for any integer $0 \leq \beta < \alpha$ we have $m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$.

(ii) Let $\mathcal{K}$ be a class of topological spaces, d be a transfinite dimension function satisfying the sum theorem of type A_{tr} , α be an infinite ordinal with $n(\alpha) \geq 1$ and X be a space from $\mathcal{K}$ with $dX = \alpha$ which is completely decomposable in the sense of d . Then for any infinite ordinal $\beta < \alpha$ we have $m_{\mathcal{K}}(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$ if $\lambda(\beta) = \lambda(\alpha)$ and $m_{\mathcal{K}}(d, \beta, \alpha)$ does not exist otherwise.

The deficiency def is defined in the following way: For a space X ,

$$\text{def } X = \min\{\dim(Y \setminus X) : Y \text{ is a metrizable compactification of } X\}.$$

Recall that $\text{Cmp } X \leq \text{def } X$ and $\text{def } X = 0$ iff $\text{Cmp } X = 0$.

Lemma 2.3 (i) $\text{def } L(L(X, P), P^1, Y) = \max\{\text{def } X, \text{def } Y\}$ for any X, P, Y . In particular, we have $\text{Cmp } L(L(X, P), P^1, Y) \leq 0$ if $\text{Cmp } X \leq 0$ and $\text{Cmp } Y \leq 0$.

(ii) $\text{trInd } L(L(X, P), P^1, Y) = \max\{\text{trInd } X, \text{trInd } Y\}$ for any compacta X, Y and any P .

Proof. (i) Let bX and bY be metrizable compactifications of X and Y respectively such that $\dim(bX \setminus X) = \text{def } X$ and $\dim(bY \setminus Y) = \text{def } Y$. Observe that the space $L(L(bX, P), P^1, bY)$ is a compactification of $L(L(X, P), P^1, Y)$ and the increment $Z = L(L(bX, P), P^1, bY) \setminus L(L(X, P), P^1, Y)$ is the union of countably many closed subsets, one of which is homeomorphic to $bX \setminus X$ and the others are homeomorphic to $bY \setminus Y$. So by the countable sum theorem for $\dim$ we get that $\dim Z = \max\{\dim(bX \setminus X), \dim(bY \setminus Y)\} = \max\{\text{def } X, \text{def } Y\}$. Hence $\text{def } L(L(X, P), P^1, Y) \leq \max\{\text{def } X, \text{def } Y\}$, thereby $\text{def } L(L(X, P), P^1, Y) = \max\{\text{def } X, \text{def } Y\}$.

(ii) At first let us prove the statement when Y is a singleton. Observe that in this case $L(L(X, P), P^1, Y) = L(X, P)$. Consider two disjoint closed subsets A and B of $L(X, P)$.

Recall that $L(X, P)$ contains a copy of X . Choose a partition C between $A \cap X$ and $B \cap X$ in X . Extend the partition to a partition C_1 between A and B in $L(X, P)$. Consider another partition C_2 between A and B in $L(X, P)$ which is "thin" (i.e. $\text{Int}_{L(X, P)} C_2 = \emptyset$) and is in C_1 . Observe that $C_2 \subset C$. Hence $\text{trInd} L(X, P) = \text{trInd} X$.

Now let us consider the general case. Assume that A and B are disjoint closed subsets in $L(L(X, P), P^1, Y)$. Recall that there is the natural continuous projection $pr : L(L(X, P), P^1, Y) \rightarrow L(X, P)$. Consider the closed subsets prA and prB of $L(X, P)$. If they are disjoint, choose a partition C_2 between prA and prB in $L(X, P)$ like in the previous part. Observe that $pr^{-1}C_2$ is a partition between A and B in $L(L(X, P), P^1, Y)$ such that $pr^{-1}C_2$ is homeomorphic to a closed subset of C . Assume now that $prA \cap prB \neq \emptyset$. Note that $Q^1 = prA \cap prB$ is finite and $L(L(X, P), P^1, Y)$ is the free union of $L(L(X, (P \setminus Q)), P^1 \setminus Q^1, Y)$, where Q is the finite subset of P corresponding to Q^1 and finitely many copies of Y . Choose a partition between A and B in X and a partition between A and B in each of the copies of Y corresponding to points of Q . It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X, P), P^1, Y)$ between A and B . We conclude that $\text{trInd} L(L(X, P), P^1, Y) = \max\{\text{trInd} X, \text{trInd} Y\}$. $\square$

Proof of Theorem 1.1.

(i) Because of Remark 2.1 and Proposition 2.1, we need only establish that $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$. Consider the space $C_\alpha = \bigcup_{i=1}^{\alpha+1} X_i$, where each X_i is closed in X , without isolated points and $\text{Cmp } X_i = 0$, from Remark 2.1. Let P_i be a countable dense subset of X_i . Put $P = \bigcup_{i=1}^{\alpha+1} P_i$. Recall that $\text{def } C_\alpha = \alpha$ ([ChH, Theorem 3.1]). So by Lemma 2.3 for any integer q we have $\text{def } L_q(C_\alpha, P) = \alpha$ and hence $\text{Cmp } L_q(C_\alpha, P) = \alpha$. Observe that by Lemmas 2.2 and 2.3, we get that the completely metrizable space $L_q(C_\alpha, P)$ is the union of $(\alpha+1)^q$ many closed subspaces with $\text{Cmp} \leq 0$. Hence $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \leq (\alpha+1)^q$. Since Cmp satisfies the sum theorem of type A, C_α cannot be represented as α -many closed subsets with $\text{Cmp} \leq 0$. By Lemma 2.1, we have $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \geq q\alpha \geq q$. Since $\lim_{q \rightarrow \infty} q = \infty$ we get $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_{\mathcal{C}}(\text{trInd}, \beta, \alpha) = \infty$, if $\lambda(\beta) = \lambda(\alpha)$; and does not exist otherwise. $\square$

Proof of Theorem 1.2.

(i) Put $X_\alpha = \{*\} \cup \bigoplus_{i=1}^{\infty} L_i(C_\alpha, P)$. Observe that X_α is completely metrizable and is the union of countably many closed subspaces with $\text{Cmp} \leq 0$. Since $\text{def } X_\alpha = \alpha$, we have $\text{Cmp } X_\alpha = \alpha$. Now observe that $\lim_{i \rightarrow \infty} m(L_i(C_\alpha, P), \text{Cmp}, \alpha-1, \alpha) = \infty$. Hence X_α cannot be written as the finite union of closed subsets with $\text{Cmp} \leq \alpha-1$.

(ii) Put $X_\alpha = \{*\} \cup \bigoplus_{i=1}^{\infty} L_i(S^\alpha, P)$. Observe that X_α is compact and is the union of countably many finite-dimensional closed subspaces (recall that S^α and therefore $L_i(S^\alpha, P)$ have the same property). Since for each i , $\text{trInd } L_i(S^\alpha, P) = \alpha$, we have $\text{trInd } X_\alpha = \alpha$. Now

observe that $\lim_{i \rightarrow \infty} m(L_i(S^\alpha, P), \text{trInd}, \alpha - 1, \alpha) = \infty$. Hence X_α cannot be written as the finite union of closed subsets with $\text{trInd} \leq \alpha - 1$. $\square$

Remark 2.2 Let Q be the set of rational numbers of the closed interval $[0, 1]$. Recall that for the spaces $X = Q \times [0, 1]^n$ and $Y = ([0, 1] \setminus Q) \times I^n$ we have $\text{Cmp } X = \text{def } X = \text{Cmp } Y = \text{def } Y = n$ ([AN, p. 18 and 56]). It is easy to observe that X satisfies points (a)-(c) of Theorem 1.2 (i). However, X is not completely metrizable. Note that Y is completely metrizable and satisfies points (a) and (c) of Theorem 1.2 (i) but not (b). Observe that Smirnov's compactum S^α with $n(\alpha) \geq 1$ satisfies points (a) and (b) of Theorem 1.2 (ii) but not (c). Note also that any Cantor manifold Z with $\text{trInd } Z = \alpha$, where α is infinite ordinal with $n(\alpha) \geq 1$, (see for such spaces for example in [O]) satisfies points (a) and (c) of Theorem 1.2 (ii) but not (b).

Let d be a (transfinite) dimension function. A space X with $dX \neq \infty$ is said to have property $(*)_d$ if for every open nonempty subset O of the space X there exists a closed in X subset $F \subset O$ with $dF = dX$.

Observe that the spaces X, Y from Remark 2.2 have property $(*)_{\text{Cmp}}$ and Z has property $(*)_{\text{trInd}}$.

Proposition 2.2 Let X be a completely metrizable space with $dX \neq \infty$. Then $X \neq \bigcup_{i=1}^\infty X_i$, where each X_i is closed in X and $dX_i < dX$ iff there exists a closed subspace Y of X such that

- (i) $dY = dX$ and
- (ii) Y has the property $(*)_d$.

Remark 2.3 This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij's example ([E, p. 140]), Chatyrko, Kozlov and Pasynkov [ChKP, Remark 3.15 (b)] presented for each $n = 3, 4, \dots$ a compact Hausdorff space X_n such that $\text{ind } X_n = 2$ and $m(X_n, \text{ind}, 1, 2) = n$. Hence it is clear that $m_{\mathcal{N}}(\text{ind}, 1, 2) = 2$ and $M_{\mathcal{N}}(\text{ind}, 1, 2) = \infty$, where $\mathcal{N}$ is the class of compact Hausdorff spaces. In [K] Kotkin constructed a compact Hausdorff space X with $\text{ind } X = 3$ which is the union of three one-dimensional in the sense of ind closed subspaces. Hence, $m_{\mathcal{N}}(\text{ind}, 1, 3) = 3$ and $m_{\mathcal{N}}(\text{ind}, 2, 3) = 2$. Filippov in [F] presented for every n a compact Hausdorff space F_n with $\text{ind } F_n = n$, which is the union of finitely many one-dimensional in the sense of ind closed subspaces, thereby $m_{\mathcal{N}}(\text{ind}, k, n) < \infty$ for each $1 \leq k < n$. By the sum theorem from Remark 2.1 (iii) for ind which is valid in fact for all regular spaces, one can get that $m_{\mathcal{N}}(\text{ind}, 1, n) \geq 2^{n-2} + 1$ for each n .

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