

The Complements of Projective Plane Curves

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In this note, we shall study rational plane curves. Our subject is the logarithmic Kodaira dimension of a rational plane curve, introduced by Iitaka[1].

First of all, we recall the logarithmic Kodaira dimension. Let X be a nonsingular surface defined over the complex number field $\mathbb{C}$. We find a smooth completion $\bar{X}$ of X such that $D := \bar{X} - X$ is a divisor with simple normal crossings. Let K be a canonical divisor of $\bar{X}$. Then $\bar{P}_m(X) := \dim H^0(\bar{X}, m(K + D))$ is called the logarithmic m -genus of X . Then the logarithmic Kodaira dimension $\bar{\kappa}(X)$ of X is defined by $\bar{\kappa}(X) = \max \{ m \mid \bar{P}_m(X) > 0 \}$. It is easy to check that $\bar{P}_m(X)$ and $\bar{\kappa}(X)$ do not depend on the choice of $\bar{X}$ and D .

Let C be an irreducible curve in $\mathbb{P}^2$. We use the following notations:

$g(C)$: the genus of the normalization of C

$s(C)$: the number of singular points of C

$r(C)$: the number of cuspidal singular points of C

Assume $g(C) = 0$. Then there exists a canonical inclusion $\text{Reg}(C) \longrightarrow \mathbb{P}^1$, where $\text{Reg}(C)$ is the regular locus of C .

$t(C)$: the number of $\mathbb{P}^1 - \text{Reg}(C)$.

We summarize known results by Wakabayashi[4].

- (1) If $g(C) > 0$ and C is not a nonsingular elliptic curve, then $\bar{\kappa}(P^2 - C) = 2$.
- (2) If C is a nonsingular elliptic curve, then $\bar{\kappa}(P^2 - C) = 0$.
- (3) If $g(C) = 0$ and $s(C) \geq 3$, then $\bar{\kappa}(P^2 - C) = 2$.
- (4) If $g(C) = 0$ and $s(C) = r(C) = 2$, then $\bar{\kappa}(P^2 - C) \geq 0$.
- (5) If $g(C) = 0$, $s(C) = 1$ and $t(C) \geq 3$, then $\bar{\kappa}(P^2 - C) > 0$.
- (6) If $g(C) = 0$, $s(C) = 1$ and $t(C) = 2$, then $\bar{\kappa}(P^2 - C) \geq 0$.
- (7) If $g(C) = s(C) = 0$, then $\bar{\kappa}(P^2 - C) = -\infty$.

Our results on the remaining cases are stated as follows.

Proposition 1. If $g(C) = 0$ and $r(C) = s(C) = 2$, then $\bar{\kappa}(P^2 - C) > 0$.

Proposition 2. If $g(C) = 0$ and $s(C) = r(C) = 1$, then $\bar{\kappa}(P^2 - C) \neq 0$.

Let a be an integer greater than 2, let $\delta, \gamma_1, \dots, \gamma_a$ be complex numbers and let ε, γ_0 be nonzero complex numbers. Then the P_{ki} 's ($0 \leq i \leq a-1$, $0 \leq k \leq a$) are defined by the equations as follows:

$$\begin{aligned} & \binom{a}{i} (\delta u + \varepsilon)^{a-i} (\gamma_0 + \gamma_1 u + \dots + \gamma_a u^a) \\ &= P_{ia} + P_{ia-1} u + \dots + P_{i0} u^a + (\text{higher terms}), \end{aligned}$$

where u is an indeterminate. Let (x, y, z) be a system of homogeneous coordinates in P^2 . Let $C_{a, \gamma_0, \dots, \gamma_a, \delta, \epsilon}$ be the curve defined by the equation

$$\begin{aligned} & (y^{a-1}z - (\gamma_0 x^a + \gamma_1 x^{a-1}y + \dots + \gamma_a y^a))^a z \\ & + \sum_{i=0}^{a-1} \sum_{k=0}^a p_{ik} x^k y^{a^2 - ai + 1 - k} (y^{a-1}z - (\gamma_0 x^a + \gamma_1 x^{a-1}y + \dots + \gamma_a y^a))^i \\ & = 0. \end{aligned}$$

Proposition 3. The curve $C_{a, \gamma_0, \dots, \gamma_a, \delta, \epsilon}$ has the following properties:

- (1) $C_{a, \gamma_0, \dots, \gamma_a, \delta, \epsilon} - \{p\} \cong A^1$, for some point p of C ,
- (2) $\bar{\kappa}(P^2 - C_{a, \gamma_0, \dots, \gamma_a, \delta, \epsilon}) = 1$.

Theorem 4. Let C be a projective plane curve satisfying the conditions:

- (1) $C - \{p\} \cong A^1$, for some point p of C ,
- (2) $\bar{\kappa}(P^2 - C) = 1$.

Then C is isomorphic to $C_{a, \gamma_0, \dots, \gamma_a, \delta, \epsilon}$, up to projective equivalence for some $(a, \gamma_0, \dots, \gamma_a, \delta, \epsilon)$.

Let C be a projective plane curve such that $C - \{p\} \cong \mathbb{A}^1$,
for some point p of C . This curve was studied by Yoshihara[5].
His results are as follows:

- (1) If $\deg C \geq 3 \operatorname{mult}_p(C)$, then $\bar{\kappa}(P^2 - C) = 2$,
- (2) there exist no curves such that $\deg C = 6$ and $\operatorname{mult}_p(C) = 2$.

Now, we have the following:

Proposition 5. Let C be as above. Then,

$$\deg C \leq 3 \operatorname{mult}_p(C) + 2.$$

Furthermore, if $\deg C > 192$, then

$$\deg C < 3 \operatorname{mult}_p(C).$$

By the above result, we naturally have the following

Conjecture: Under the above notations,

$$\deg C < 3 \operatorname{mult}_p(C).$$

Finally, we explain the outline of the proofs of Theorem 4
and the first part of Proposition 5.

The proof of Theorem 4: Let $\mu : \bar{X} \longrightarrow \mathbb{P}^2$ be a composite of blowing-ups such that $D := \mu^{-1}(C)$ has only simple normal crossings. Assume that μ is the shortest among such birational morphisms. We set $X := \mathbb{P}^2 - C = \bar{X} - D$. We denote by $\Pi : X \longrightarrow \Delta$ a rational map associated with $|n(K(\bar{X}) + D)|$ for sufficiently large n . Since $\bar{\kappa}(\mathbb{P}^2 - C) = 1$, we can apply Kawamata's results[2]. Then, Π is a morphism and a general fiber of $\Pi|_X$ is G_m or an elliptic curve. Since X is affine, a general fiber of $\Pi|_X$ is G_m , whence a general fiber of Π is $\mathbb{P}^1$. Hence, there exist a Hirzebruch surface $\bar{Y}$ and a birational morphism $\rho : \bar{X} \longrightarrow \bar{Y}$ such that $\Pi \cdot \rho^{-1}$ is a morphism. We put $\psi = \Pi \cdot \rho^{-1}$ and denote by ℓ a general fiber of ψ .

By taking a suitable $\bar{Y}$, we may assume that $\Gamma = \rho_*(D)$ is either

- (i) a sum of a 2 - section and at most three fibers, or
- (ii) a sum of two sections for the fibration ψ and at most three fibers.

Note that

(1) each irreducible component of D has a negative self-intersection number and

(2) the exceptional curve in D is unique.

It follows from (1) and (2) that Γ is a sum of two sections and three fibers. Using these facts, we conclude that (X, X, D) is a resolution of $C_{a, \gamma_0, \dots, \gamma_a, \delta, \varepsilon}$ for some $(a, \gamma_0, \dots, \gamma_a, \delta, \varepsilon)$.

Q.E.D.

The proof of Proposition 5: We shall only prove the first part of Proposition 5. By a rather easy argument, we can obtain that $\deg C \leq 3 \operatorname{mult}_p(C) + 2$. Put $n := \deg C$ and $e := \operatorname{mult}_p(C)$. First, consider the shortest succession of blowing-ups

$$\bar{X}_0 \xleftarrow{f_1} \bar{X}_1 \xleftarrow{\quad} \cdots \xleftarrow{f_s} \bar{X}_s := \bar{X}$$

such that (1) $X_0 = P^2$, (2) the center p_i of the blowing-up f_i lie over $p = \operatorname{Sing}(C)$ and (3) $D = f^{-1}(C)$ has simple normal crossings, where $f = f_1 \dots f_s$. We denote by e_i the multiplicity of C at the center of f_i . Note that $e_1 = e$. By the Plücker formula, we have

$$(n-1)(n-2) = \sum_{i=1}^r e_i(e_i-1). \quad (1)$$

By Yoshihara's result, we have $\bar{K}(P^2 - C) = 2$. Hence, we can use the following fact: Under the above situations,

$$4 e(P^2 - C) \geq (K(\bar{X}) + D)^2,$$

where $e(P^2 - C)$ is the Euler number of $P^2 - C$ (cf. Sakai[3]).

In the present case, since C is a rational plane curve which has only one cuspidal singular point, we have $e(P^2 - C) = 1$. Furthermore,

$$\begin{aligned} (K(\bar{X}) + D)^2 &= (K(\bar{X}), K(\bar{X}) + D) + (D, K(\bar{X}) + D) \\ &= (K(\bar{X}), K(\bar{X}) + D) - 2, \end{aligned}$$

where $(D, K(\bar{X}) + D) = -2$, because D is connected and the dual

graph of D is a tree. Hence, we have

$$6 \geq (K(\bar{X}) + D, K(\bar{X})). \quad (2)$$

We shall next compute $(K(\bar{X}), K(\bar{X}) + C')$, where C' is the proper transform of C . Since $(C')^2 = n^2 - \sum e_i^2$, we have $(K(\bar{X}), C') = -2 - n^2 + \sum e_i^2$. By (1), we have

$$(K(\bar{X}), K(\bar{X}) + C') = 9 - s + \sum e_i - 3n. \quad (3)$$

Note that

$$(K(\bar{X}), D - C') \geq 0. \quad (4)$$

In fact, since f is a resolution of a cuspidal singular point, we see that one of the two irreducible components (except C') meeting the exceptional curve of the first kind in D has self-intersection number ≤ -3 . From (1), (2), (3) and (4), we conclude that $n \leq 3e + 2$. Q.E.D.

References

- [1] S. Iitaka, On logarithmic Kodaira dimension of algebraic varieties, J. Fac. Sci. Univ. Tokyo, 23 (1976), 525 - 544.
- [2] Y. Kawamata, Classification of non-complete algebraic surfaces, Lecture Notes in Math. 732, Springer-Verlag, Berlin Heidelberg New York, 1978, 215-232.
- [3] F. Sakai, Semi-stable curves on algebraic surfaces and logarithmic pluricanonical maps, Math. Ann., 254 (1980), 89-120..

- [4] I. Wakabayashi, On the logarithmic Kodaira dimension of the complement of a curve in $\mathbf{P}^2$, Proc. Japan Acad., 54 (1978), 157 - 162.
- [5] H. Yoshihara, On plane rational curves, Proc. Japan Acad., 55 (1979), 152 - 155.