

ON THE ORDER OF STRONGLY CLOSE-TO-CONVEXITY OF STRONGLY CONVEX FUNCTIONS

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ABSTRACT. In this work the order of strongly close-to-convexity of strongly convex functions is discussed. The sufficient conditions for function to be Bazilevič function are also considered.

1. INTRODUCTION

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote by

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + \dots\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + \dots\},$$

so $\mathcal{A} = \mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$.

The class $\mathcal{S}_\alpha^*$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}_\alpha^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{U} \right\}.$$

The class $\mathcal{S}_\alpha^*$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\begin{aligned} \mathcal{K}_\alpha &:= \left\{ f \in \mathcal{A} : \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathbb{U} \right\} \\ &= \{f \in \mathcal{A} : zf' \in \mathcal{S}_\alpha^*\} \end{aligned}$$

introduced Robertson in [13]. If $\alpha \in [0; 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}_0^* = \mathcal{S}^*$, $\mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively.

Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order β , $0 < \beta \leq 1$,

$$\mathcal{SS}^*(\beta) := \left\{ f \in \mathcal{A} : \left| \operatorname{Arg} \frac{zf'(z)}{f(z)} \right| < \frac{\beta\pi}{2}, z \in \mathbb{U} \right\},$$

which was introduced in [14] and [1]. Furthermore, $\mathcal{SK}(\beta) = \{f \in \mathcal{A} : zf' \in \mathcal{SS}^*(\beta)\}$ denote the class of strongly convex functions of order β . The class $\mathcal{S}^*[A, B]$

$$\mathcal{S}^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, z \in \mathbb{U} \right\}$$

was investigated in [2] for $-1 \leq B < A \leq 1$. Recall, that we write $f \prec g$ and say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathbb{U}$, if and only if there exists an analytic

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function $w \in \mathcal{H}$ such that $|w(z)| < |z|$ and $f(z) = g[w(z)]$ for $z \in \mathbb{U}$. Therefore, $f \prec g$ in $\mathbb{U}$ implies $f(\mathbb{U}) \subset g(\mathbb{U})$. In particular if g is univalent in $\mathbb{U}$, then

$$f \prec g \quad \Leftrightarrow \quad [f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U})].$$

2. PRELIMINARIES

To prove the main results, we need the following Nunokawa's Lemma.

Lemma 2.1. [8], [9] *Let p be analytic function in $|z| < 1$ with $p(0) = 1$, $p(z) \neq 0$. If there exists a point z_0 , $|z_0| < 1$, such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\alpha}{2},$$

where

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \quad \text{and} \quad a > 0.$$

We need also the following four authors lemma [10].

Lemma 2.2. [10] *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic function in $|z| < 1$. If there exists a point z_0 , $|z_0| < 1$, such that*

$$\Re p(z) > c \quad \text{for} \quad |z| < |z_0|$$

and

$$\Re p(z_0) = c, \quad p(z_0) \neq c$$

for some $0 < c < 1$, then we have

$$\Re \left\{ \frac{z_0 p'(z_0)}{p(z_0)} \right\} \leq \gamma(c) = \begin{cases} \frac{-c}{2(1-c)} & \text{when } c \in (0, \frac{1}{2}), \\ \frac{c-1}{2c} & \text{when } c \in (\frac{1}{2}, 1). \end{cases}$$

3. MAIN RESULT

Theorem 3.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that in $|z| < 1$

$$(3.1) \quad \left| \arg \left(1 + \frac{z f''(z)}{f'(z)} \right) \right| < \tan^{-1} \frac{\beta}{1-\alpha},$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. Then we have

$$(3.2) \quad \left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi\beta}{2} \quad \text{in } |z| < 1,$$

for some $g \in \mathcal{K}_{1-\alpha}$.

Proof. Let us put $g'(z) = (f'(z))^\alpha$. By (3.1) $\Re \{1 + f''(z)/f'(z)\} > 0$ so

$$\begin{aligned} & \Re \left\{ 1 + \frac{z g''(z)}{g'(z)} \right\} \\ &= \Re \left\{ 1 - \alpha + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} > 1 - \alpha > 0, \end{aligned}$$

hence

$$(3.3) \quad g \in \mathcal{K}_{1-\alpha}.$$

Next, let us put

$$p(z) = f'(z), \quad p(0) = 1.$$

Then it follows that

$$1 + \frac{z f''(z)}{f'(z)} = 1 + \frac{z p'(z)}{p(z)}.$$

If there exists a point z_0 , $|z_0| < 1$, such that

$$|\arg p(z)| < \frac{\pi\beta}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\beta}{2},$$

then by Nunokawa's Lemma 2.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\beta k,$$

where

$$k \geq 1 \quad \text{when} \quad \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -1 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi\beta}{2}.$$

For the case $\arg p(z_0) = \pi\beta/2$, we have

$$\begin{aligned} & \arg \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} = \arg \left\{ 1 + \frac{i\beta k}{1-\alpha} \right\} \\ & \geq \arg \left\{ 1 + \frac{i\beta}{1-\alpha} \right\} = \tan^{-1} \frac{\beta}{1-\alpha}. \end{aligned}$$

This contradicts hypothesis of the Theorem 3.1 and for the case $\arg p(z_0) = -\pi\beta/2$, applying the same method as the above, we have

$$\arg \left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} \leq -\tan^{-1} \frac{\beta}{1-\alpha}.$$

This is also the contradiction and therefore, it completes the proof. $\square$

Recall that $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}_\alpha(\beta)$, [3], of close-to-convex functions of order β , $0 \leq \beta < 1$, if and only if there exist $g \in \mathcal{K}_\alpha$, $\varphi \in \mathbb{R}$, such that

$$(3.4) \quad \Re \left\{ e^{i\varphi} \frac{f'(z)}{g'(z)} \right\} > \beta, \quad z \in \mathbb{U}.$$

Reade [12] introduced the class of strongly close-to-convex functions of order $\beta < 1$ defined by $|\arg \{e^{i\varphi} f'(z)/g'(z)\}| < \pi\beta/2$ instead of (3.4). Therefore, the conditions (3.2) and (3.3) mean that f is strongly close-to-convex functions of order β with respect convex functions of order $1-\alpha$. Functions defined by (3.4) with $\varphi = 0$ were considered earlier by Ozaki [11], see also Umezawa [16, 17]. Moreover, Lewandowski [4, 5] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{U})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski's class is identical with the Kaplan's class $\mathcal{C}_0(0)$, see [3]. If we put $g'(z) = (f'(z))^\alpha$ in Theorem 3.1 and if we denote $\lambda = \beta/(1-\alpha)$, $\lambda \in (0, \infty)$, then we obtain the following corollary.

Corollary 3.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that*

$$\left| \arg \left(1 + \frac{z f''(z)}{f'(z)} \right) \right| < \tan^{-1} \lambda \quad \text{in } |z| < 1,$$

where $0 < \lambda < \infty$. Then we have

$$|\arg \{f'(z)\}| < \frac{\pi\lambda}{2} \quad \text{in } |z| < 1,$$

Remark 3.3. *For the case $0 < \beta < 1$, it is trivial that there exists α , $0 < \alpha < 1$, which satisfies*

$$\begin{aligned} & \frac{\beta}{1-\alpha} > \tan \left(\frac{\pi}{2} \gamma(\beta) \right) \\ &= \tan \left\{ \frac{\pi\beta}{2} + \tan^{-1} \frac{\beta \varrho(\beta) \sin \left(\frac{\pi(1-\beta)}{2} \right)}{\rho(\beta) + \beta \varrho(\beta) \cos \left(\frac{\pi(1-\beta)}{2} \right)} \right\}, \end{aligned}$$

where

$$\rho(\beta) = (1+\beta)^{(1+\beta)/2} \quad \text{and} \quad \varrho(\beta) = (1-\beta)^{(\beta-1)/2}$$

and

$$\frac{\beta}{1-\alpha} > \tan \frac{\pi\beta}{2} + \frac{\beta \left(\frac{1-\beta}{1+\beta} \right)^{(1+\beta)/2}}{(1-\beta) \cos(\pi\beta/2)}.$$

The right hand sides of the above estimate are Nunokawa's and Mocanu's estimate of the order of strongly starlikeness in the class of strongly convex functions $SK(\beta)$, for details see [9] and [7] or [6, p. 266].

Theorem 3.4. Assume that $1/2 \leq \alpha < 1$, $\beta \geq 1$ and $0 < c < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that

$$(3.5) \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad \text{for } |z| < 1.$$

Furthermore, let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $|z| < 1$ such that

$$(3.6) \quad \Re \left\{ \frac{zg'(z)}{g(z)} \right\} \leq \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta} \quad \text{for } |z| < 1,$$

where $\gamma(c)$ is given in Lemma 2.2, and where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha \neq \frac{1}{2}, \\ \frac{1}{2 \log 2} & \text{for } \alpha = \frac{1}{2}. \end{cases}$$

Then we have

$$\Re \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)} > c \quad \text{for } |z| < 1.$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f^{1-\beta}(z)g^{\beta}(z)}, \quad p(0) = 1.$$

Then it follows that

$$(3.7) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + (1-\beta) \frac{zf'(z)}{f(z)} + \beta \frac{zg'(z)}{g(z)}.$$

If there exists a point z_0 , $|z_0| < 1$, such that

$$\Re p(z) > c \quad \text{for } |z| < |z_0|$$

and

$$\Re p(z_0) = c, \quad p(z_0) \neq c,$$

then by Lemma 2.2, we have

$$(3.8) \quad \begin{aligned} \Re \left\{ \frac{z_0 p'(z_0)}{p(z_0)} \right\} &\leq \gamma(c) \\ &= \begin{cases} -\frac{c}{2(1-c)} & \text{when } c \in (0, 1/2], \\ -\frac{1-c}{2c} & \text{when } c \in (1/2, 1). \end{cases} \end{aligned}$$

Furthermore, by (3.5) $f \in \mathcal{K}_{\alpha}$ thus $f \in \mathcal{S}_{\delta(\alpha)}^*$, see [18]. Because $\beta \geq 1$, then in $|z| < 1$

$$(3.9) \quad \Re \left\{ (1-\beta) \frac{zf'(z)}{f(z)} \right\} \leq (1-\beta)\delta(\alpha).$$

Substituting (3.6), (3.8) and (3.9) in (3.7) we get

$$\begin{aligned}
 & 1 + \Re \frac{z_0 f''(z_0)}{f'(z_0)} \\
 &= \Re \left\{ \frac{z_0 p'(z_0)}{p(z_0)} + (1 - \beta) \frac{z_0 f'(z_0)}{f(z_0)} + \beta \frac{z_0 g'(z_0)}{g(z_0)} \right\} \\
 &\leq \gamma(c) + (1 - \beta)\delta(\alpha) + \beta \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta} \\
 &= \alpha.
 \end{aligned}$$

This contradicts hypothesis of the Theorem 3.5 and therefore, it completes the proof. $\square$

Remark 3.5. For the case $1 < \beta$, if α, β and f satisfy the conditions of Theorem 3.4, then f is a Bazilevič function of order c , $0 < c < 1$, see [15, p. 353].

Applying the same method as in the proof of Theorem 3.4, we have the following theorem.

Theorem 3.6. Assume that $1/2 \leq \alpha < 1$, $\beta > 1$ and $0 < c < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that

$$\Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad \text{for } |z| < 1.$$

Furthermore, let $g \in \mathcal{S}^*[A, B]$ and let

$$\frac{1-A}{1-B} \leq \frac{\alpha - \gamma(c) + (\beta - 1)\delta(\alpha)}{\beta} \quad \text{for } |z| < 1,$$

where $\gamma(c)$ is given in Lemma 2.2, and where

$$\delta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha \neq \frac{1}{2}, \\ \frac{1}{2 \log 2} & \text{for } \alpha = \frac{1}{2}. \end{cases}$$

Then we have

$$\Re \frac{z f'(z)}{f^{1-\beta}(z) g^{\beta}(z)} > c \quad \text{for } |z| < 1.$$

Remark 3.7. If f satisfies the conditions of Theorem 3.6, then f is a Bazilevič function.

For $\beta = 1$ Theorem 3.6 gives the following corollary.

Corollary 3.8. Assume that $1/2 \leq \alpha < 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that

$$\Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad \text{for } |z| < 1.$$

Furthermore, let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be analytic in $|z| < 1$ such that

$$\Re \left\{ \frac{z g'(z)}{g(z)} \right\} \leq \alpha - \gamma(c) \quad \text{for } |z| < 1,$$

where $c \in (0, 1)$ is such that $\alpha - \gamma(c) > 1$. Then we have

$$\Re \frac{z f'(z)}{g(z)} > c \quad \text{for } |z| < 1.$$

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