

SUM OF THE EDGE LENGTHS OF A GEODESIC GRAPH

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ABSTRACT

Consider an embedding of a complete graph of order four into the 2-sphere such that each edge becomes a shortest geodesic connecting its endpoints. Then we show that the sum of the edge lengths is at most 4π , and is bigger than 3π if the graph is not contained in any hemisphere.

はじめに

2002 年 12 月に京都大学数理解析研究所で行われた研究集会「双曲空間に関連する研究とその展望」では、「Area of a cellular complex in a hyperbolic manifold」という題目で、双曲多様体に測地的に埋め込まれた 2 次元胞体複体の面積に関して発表しました。本稿では、その主定理の証明において鍵となった、Guddum の定理 [1] の拡張を与えます。尚、研究集会で発表した結果に関しては、プレプリント [3] を参照下さい。

1. RESULTS

In this article, we consider a finite graph geodesically embedded into a surface with constant curvature metric, and estimate the sum of the edge lengths. As usual, we regard a finite graph as a 1-dimensional cellular complex by setting a vertex as a 0-cell and an edge as a closed 1-cell. Given an embedding f of a finite graph into a surface, its image G is obviously identified with the original graph. Thus we say that the image of a vertex

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and an edge under f a *vertex* and an *edge* of G . By S^2 , we mean the two dimensional sphere endowed with the Riemannian metric of constant curvature $+1$. Then our result is the following.

Theorem 1. *Let G be the image of an embedding of the complete graph K_4 of order 4 into S^2 . Suppose that*

- (1) *each edge of G is a shortest geodesic arc on S^2 connecting its endpoints, and*
- (2) *G is not contained in any hemisphere of S^2 .*

Let E be the sum of the length of the edges of G . Then $3\pi < E \leq 4\pi$ holds.

This almost follows from the result of Guddum [1]. In fact, his theorem in [1] implies the inequality $3\pi \leq E \leq 4\pi$. In the next section, we will prove the theorem above using purely elementary spherical geometry.

A generalization of this estimate to the case of any graph embedded in n -sphere S^n will appear in [4]. Our estimate depends upon the combinatorics of the graph only.

Here we append an easy observation for more general cases. Let F_g be a closed, orientable surface of genus $g \geq 2$ with a fixed Riemannian metric of constant curvature -1 . For convenience, let F_0 denote S^2 .

Proposition 1. *Let G be the image of an embedding of a graph into F_g where $g \neq 1$. Suppose that*

- (1) *each edge e of G is a shortest geodesic arc on F_g connecting its endpoints, and*
- (2) *the closure of each component of $F_g - G$ is a convex polygon on F_g .*

Then the sum of the length of the edges of G is greater than $\pi|2 - 2g|$.

Proof. Let σ be a complementary face of G , i.e., the closure of a component of $F_g - G$. By $Area(\sigma)$ and $Length(\partial\sigma)$, we denote the area of σ and the length of the boundary $\partial\sigma$ of σ respectively. We consider the ratio $Area(\sigma)/Length(\partial\sigma)$. This ratio is strictly less than the corresponding ratio for the disk on F_g which has equilong boundary as σ . See [2] for a survey. By elementary calculations, the ratio for such a disk is shown

to be less than 1 for any $g \neq 1$. This implies $\text{Length}(\partial\sigma) > \text{Area}(\sigma)$ holds. By summing the inequalities up over all complementary faces, we have $\sum \text{Length}(\partial\sigma) > \sum \text{Area}(\sigma) = 2\pi|2 - 2g|$, where the last equality follows from the Gauss-Bonne's theorem. Then the sum of the length of the edges of G , which is equal to the half of $\sum \text{Length}(\partial\sigma)$, is greater than $\pi|2 - 2g|$. $\square$

2. PROOF

Let us start with recalling fundamentals of spherical geometry. Let u_1, u_2, u_3 be points on S^2 such that no two of them are antipodal and no great circle includes all the three points. Let Λ_i be the closed hemisphere whose boundary contains the other two points than u_i and whose interior contains u_i for $i = 1, 2, 3$. The *spherical triangle* Δ with the vertices u_1, u_2, u_3 is defined as the intersection $\Lambda_1 \cap \Lambda_2 \cap \Lambda_3$. Then we have the following:

- Δ is convex, i.e., any pair of points in Δ is connected by a geodesic arc in Δ . Moreover the arc is shortest among the arcs connecting the points, and the length is equal to the spherical distance between the points which is strictly less than π .
- The length of an edge of Δ is less than the sum of the length of the other two edges (*the triangle inequality*).

Proof of Theorem 1. Let v_1, v_2, v_3, v_4 be the vertices of G . Let e_{ij} denote the edge of G connecting v_i and v_j for $1 \leq i, j \leq 4$. Note that the assumption (1) implies that the length of e_{ij} is equal to the spherical distance d_{ij} on S^2 between v_i and v_j for $1 \leq i, j \leq 4$. Thus it suffice to show that

$$3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi .$$

In the following, the antipodal point of v_i is denoted by v_{i+4} for $1 \leq i \leq 4$. Also d_{ij} denotes the spherical distance between v_i and v_j for $1 \leq i, j \leq 8$.

First we consider the case that a couple of the vertices, say v_1 and v_2 , are antipodal, equivalently, $d_{12} = \pi$. This implies that $d_{13} + d_{32} = d_{14} + d_{42} = \pi$ holds. Together with $0 < d_{34} \leq \pi$, we have $3\pi < \sum_{1 \leq i < j \leq 4} d_{ij} \leq 4\pi$.

Next consider the case that all the four vertices are contained in a great circle. Suppose for example that v_1, v_2, v_3, v_4 lies in a great circle Γ in this

order. Since G is the image of an embedding, the edge e_{13} is not contained in Γ . This implies that e_{13} is a half of a great circle and $d_{13} = \pi$. Also we see that $d_{24} = \pi$ and so we obtain $\sum_{1 \leq i < j \leq 4} d_{ij} = 4\pi$.

Thus, in the following, we assume that $d_{ij} \neq \pi$ for $1 \leq i, j \leq 4$ and at most three vertices of G lie on a great circle.

Next consider the case that three vertices are contained in a great circle. Suppose for example that v_1, v_2 and v_3 lie on a great circle. Then, by the triangle inequality, we have $d_{41} + d_{42} > d_{12}$, $d_{42} + d_{43} > d_{23}$ and $d_{43} + d_{41} > d_{31}$. These are added to obtain

$$2(d_{41} + d_{42} + d_{43}) > d_{12} + d_{23} + d_{31} = 2\pi .$$

Thus

$$\sum_{1 \leq i < j \leq 4} d_{ij} = (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} > \pi + 2\pi = 3\pi .$$

In the same way as above, we have $d_{45} + d_{46} + d_{47} > \pi$. Since $d_{4j} = \pi - d_{4(j+4)}$ for $j = 1, 2, 3$,

$$\begin{aligned} \sum_{1 \leq i < j \leq 4} d_{ij} &= (d_{41} + d_{42} + d_{43}) + d_{12} + d_{23} + d_{31} \\ &= 3\pi - (d_{45} + d_{46} + d_{47}) + d_{12} + d_{23} + d_{31} \\ &< 3\pi - \pi + 2\pi = 4\pi \end{aligned}$$

holds.

Finally we consider the case that the four vertices are in a general position: We assume that $d_{ij} \neq \pi$ for $1 \leq i, j \leq 4$ and at most two vertices of G lie on a great circle. This means that for any three of the points there is a triangular face which includes the three points as vertices.

Then, by the triangle inequality, we have $d_{53} + d_{63} > d_{56}$, $d_{54} + d_{64} > d_{56}$, $d_{53} + d_{54} > d_{34}$ and $d_{63} + d_{64} > d_{34}$. Add these to obtain

$$d_{53} + d_{63} + d_{54} + d_{64} > d_{34} + d_{56} .$$

Here note that $d_{ij} = \pi - d_{(i-4)j}$ for $i = 5, 6$, $j = 1, 2, 3$, and $d_{56} = d_{12}$. These imply that

$$4\pi - (d_{13} + d_{23} + d_{14} + d_{24}) > d_{34} + d_{12} .$$

Consequently we have

$$4\pi > \sum_{1 \leq i < j \leq 4} d_{ij} .$$

In the following, let Δ be the spherical triangle bounded by e_{12} , e_{23} and e_{31} .

Claim 1. *The antipodal point v_8 of v_4 is included in the interior of Δ .*

Proof. Let Γ_i be the great circle including an edge of Δ but not including v_i for $i = 1, 2, 3$. By the assumption above, v_4 and hence v_8 never lie on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Note that $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ decomposes S^2 into eight spherical triangles.

Assume for a contradiction that v_8 is not included in the interior of Δ . Then v_4 is included in the interior of one of the seven spherical triangles other than the antipodal image of Δ . This implies all the four points v_1 , v_2 , v_3 and v_4 are included in the closed hemisphere bounded by one of Γ_1 , Γ_2 or Γ_3 . Since the four vertices are assumed in a general position, there is a hemisphere which contains whole G . This contradicts the assumption (2) of the theorem. $\square$

Claim 2. *The inequality $d_{12} + d_{13} > d_{82} + d_{83}$ holds.*

Proof. Since the length of each edge is less than π , the edge e_{13} intersects the great circle including v_2 and v_8 at just one point v_9 . Let d_{i9} or d_{9i} denote the distance between v_i and v_9 for $1 \leq i \leq 9$. The distance d_{19} is realized by a geodesic arc included in e_{13} and also is d_{93} . Thus $d_{13} = d_{19} + d_{93}$ holds.

The distance d_{29} is realized by a geodesic arc e_{29} in Δ since Δ is convex. In particular, the arc e_{29} contains v_8 and so $d_{29} = d_{28} + d_{89}$ holds.

Together with the triangle inequality $d_{12} + d_{19} > d_{29}$ and $d_{98} + d_{93} > d_{83}$, we conclude

$$d_{12} + d_{13} = d_{12} + d_{19} + d_{93} > d_{29} + d_{93} = d_{28} + d_{89} + d_{93} > d_{28} + d_{83} .$$

$\square$

In the same way, we have $d_{21} + d_{23} > d_{81} + d_{83}$ and $d_{31} + d_{32} > d_{81} + d_{82}$. By adding these inequalities, we obtain

$$d_{12} + d_{23} + d_{31} > d_{81} + d_{82} + d_{83} .$$

Together with the equations $d_{8j} = \pi - d_{4j}$ for $i = 1, 2, 3$, we conclude that

$$\sum_{1 \leq i < j \leq 3} d_{ij} > 3\pi - \sum_{1 \leq k \leq 3} d_{k4}.$$

This completes the proof. □

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