

Exponential Stability of Dual Flexible Arms for Grasping and Orientation Control

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Abstract: This paper studies the stability of two one-link flexible arms for grasping and orientation control of an object. Flexible arms are modeled by Euler-Bernoulli beam model, and the overall system is represented by a hybrid PDE-ODE model. Our primary concern is the stability analysis of this hybrid PDE-ODE model. In particular, we use the frequency domain method and prove the exponential stability of this system under our previously proposed boundary controller. In addition, we discuss the robustness of the closed-loop system with respect to the several disturbances including disturbances distributed over the arms and disturbances acting at boundaries. Finally, simulation results are presented to investigate the theoretical results.

1 Introduction

This paper deals with the stability analysis of dual flexible arms for grasping and orientation control of an object, where the flexible arms consist of flexible links modeled by Euler-Bernoulli beam model. Since the flexible arm has a lightweight link, it has the benefits of low energy use, faster movement, safer operation due to reduced inertia, and others [1–3]. Thus, grasping an object by flexible arms keeps the above benefits and is safer for the grasped object than the rigid arm, because the flexible arm has intrinsic large compliance compared to the rigid arm [4]. These advantages lead to many applications including a surgical instrument like flexible beams for grasping task in minimally invasive surgery [5], a flexible dual-arm space robot for capturing space debris [6], and a grasping task in the food manufacturing industry [7] and the aerial manipulation field [8].

The dynamics of flexible links are presented by partial differential equations (PDEs), and the dynamics of actuators and tip load are expressed by ordinary differential equations (ODEs), and thus the system is a hybrid PDE-ODE model. Therefore, this paper focuses on grasping and orientation control of an object by dual flexible arms based on infinite dimensional settings.

There are many studies about the control of the flexible structures based on the hybrid PDE-ODE model [9–19]. However, these studies have been treated only the vibration suppression problem. On the other hand, several studies have investigated the grasping by flexible arms based on the hybrid PDE-ODE model [20–23]. These studies discussed the cooperative control by two one-link flexible arms and realized stable grasping of an object. However, they assumed that the contacts between the arm-tips and the grasped object were fixed. Thus, the positions of the contact points cannot change, and two one-link flexible arms could grasp an object but could not control its orientation.

On the other hand, by taking rolling contacts between the arm-tips and grasped object, two one-link flexible arms have the potential to grasp and manipulate an object at the same time. In particular, in the research field of grasping and manipulation by rigid robotic fingers, many studies using the rolling contact between the robotic fingertip and the grasped object have been researched for a long time, and it is known that the position of the contact points can be changed through rolling the fingertips on the object [24–26]. From these points of view, by taking the rolling constraints between the arm-tips and the grasped object, we realized the stable grasping and orientation control of an object by two one-link flexible arms based on a hybrid PDE-ODE model [27, 28]. In particular, the flexible

arms were modeled by Euler-Bernoulli beams in [27], and the flexible arms were modeled by Timoshenko beams in [28]. Although the derived controllers in [27, 28] could asymptotically stabilize the closed-loop system, the controllers cannot exponentially stabilize the system because of the result of the compact perturbation [29, 30], and it is better to construct an exponentially stabilizing controller with regard to its control performance.

From these points of view, this paper utilizes our previously proposed boundary controller [22] and proves the exponential stability of the closed-loop system; that is, the controller realizes stable grasping and orientation control of an object by two one-link flexible arms modeled by Euler-Bernoulli beam model, and the closed-loop system becomes exponentially stable. Here, note that our previous study [22] proposed the boundary controller for grasping an object by dual flexible arms, but the controller could not realize the orientation control of the object because the rolling constraints were not introduced. Therefore, our objective in this paper is to establish well-posedness and exponential stability of two one-link Euler-Bernoulli arms for grasping and orientation control of an object under our previous controller in [22]. In addition, we discuss the robustness of the closed-loop system with respect to the disturbances, because the influences of the disturbances are an unavoidable issue in real situations. Then, we show our system is robust with respect to the several disturbances including disturbances distributed over the flexible arms and disturbances acting at boundaries. The paper is organized as follows: Section 2 formulates the control problem. The semi-group setting of the closed-loop system and well-posedness of the system are presented in Section 3. Section 4 gives exponential stability and the disturbance response. The numerical simulation results, described in Section 5, demonstrate the validity of the theoretical results. Finally, Section 6 presents our conclusions.

2 Problem Description

2.1 Mathematical Model of a Controlled System

In a controlled system, two Euler-Bernoulli arms grasp a rectangular rigid object as shown in Fig. 1. One end of the arm i , $i = 1, 2$, is clamped to the rotational motor i , and the other end has a tip mass in the shape of a wheel called arm-tip i . Fig. 1 (b) shows the side view of the arm-tip, and the arm is connected to the tip's center of mass. The arm-tip makes contact with the surface of the object and rotates on the surface. Using the motors, the arms rotate in the $O - XY$ plane, and thus are not affected by the acceleration of gravity. Here, $O - XY$ is an absolute coordinate system.

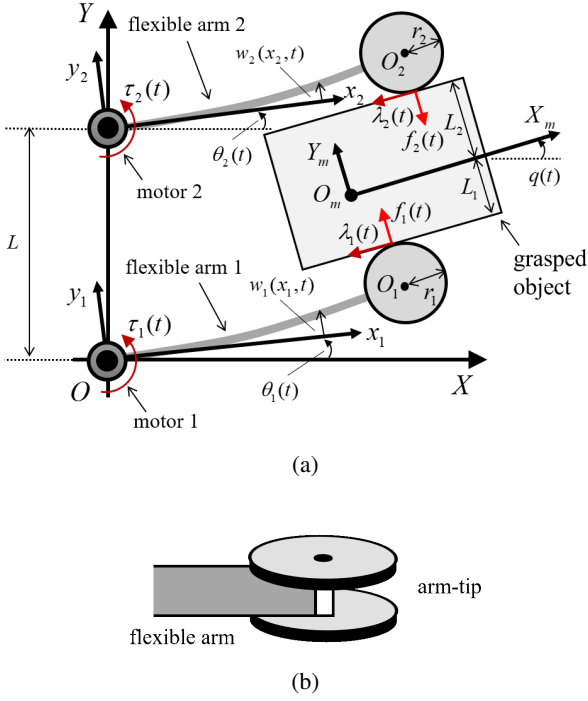


Fig. 1: Controlled system: (a) two Euler-Bernoulli arms grasp an object; (b) side view of the arm-tip.

Let $\theta_i(t)$ and $\tau_i(t)$ be the angle and torque, respectively, of motor i , and let $w_i(x_i, t)$ be the transverse displacement of arm i , where x_i is the spatial point on the local coordinate system $x_i y_i$, that rotates with the rotor of motor i , and t is a time. For the grasped object, let $[x_m(t), y_m(t)]$ be the position vector of the center of mass O_m in $O - XY$, and let $q(t)$ be the orientation of the object, where $O_m - X_m Y_m$ is a local coordinate system that rotates with the grasped object. The physical parameters are summarized in Table 1.

Assumption 1. We assume that all values of the movement are small. That is, $w_i(x_i, t)$ is small, and $\theta_i(t)$, $x_m(t)$, $y_m(t)$, and $q(t)$ are expressed as

$$\begin{cases} \theta_i(t) = \bar{\theta}_i + \theta_{ie}(t), & x_m(t) = \bar{x}_m + x_{me}(t), \\ y_m(t) = \bar{y}_m + y_{me}(t), & q(t) = \bar{q} + q_e(t), \end{cases}$$

where $\bar{\theta}_i$, $\bar{x}_m$, $\bar{y}_m$, and $\bar{q}$ are constant stationary angle/position, and $\theta_{ie}(t)$, $x_{me}(t)$, $y_{me}(t)$, and $q_e(t)$ are small.

Here, note that $\bar{\theta}_i$, $\bar{x}_m$, $\bar{y}_m$, and $\bar{q}$ are corresponding to the initial values, and these values need to satisfy the constraints (65) and (66) (see Appendix A) at $t = 0$.

Table 1 Physical Parameters

Symbol	Description
J_i	Moment of inertia of rotor of motor i
l_i	Length of arm i
ρ_i	Linear density of arm i
EI_i	Flexural rigidity of arm i
M	Mass of the grasped object
I	Moment of inertia of the grasped object
L_i	Length between the object's surface on the side of arm i and center of mass of the object
m_i	Mass of arm-tip i
r_i	Radius of arm-tip i
I_i	Moment of inertia of arm-tip i
L	Length between two motors

The equations of motion of the system has been derived in [27] as follows (see Appendix A): For $(x_i, t) \in (0, l_i) \times \mathbb{R}^+$ and $i = 1, 2$,

$$\begin{cases} \rho_i \ddot{y}_i(x_i, t) + \mu_i \dot{y}_i(x_i, t) + EI_i y_i''''(x_i, t) = 0, \\ y_i(0, t) = 0, \\ J_i \ddot{y}_i'(0, t) = \tau_i(t) + EI_i y_i''(0, t) + EI_i (w_i^d)''(0), \\ \ddot{\zeta}(t) = \mathbf{G} [EI_1 y_1'''(l_1, t), EI_2 y_2'''(l_2, t), \\ -EI_1 y_1''(l_1, t), -EI_2 y_2''(l_2, t)]^T, \\ \dot{\zeta}(t) = [\dot{y}_1(l_1, t), \dot{y}_2(l_2, t), \dot{y}_1'(l_1, t), \dot{y}_2'(l_2, t)]^T, \end{cases} \quad (1)$$

where

$$y_i(x_i, t) = w_i(x_i, t) - w_i^d(x_i) + x_i [\theta_{ie}(t) - \theta_{ie}^d], \quad (2)$$

$w_i^d(x_i)$ is the desired static deformation of arm i , θ_{ie}^d is the desired angle of motor i , μ_i is the coefficient of viscous damping of arm i , a dot denotes the time derivative, and a prime denotes the partial derivative with respect to the corresponding spatial variable. In addition,

$$\begin{cases} \mathbf{G} = \mathbf{B}_o \mathbf{M}_o^{-1} \mathbf{B}_o^T \in \mathbb{R}^{4 \times 4}, \\ \mathbf{B}_o = -\mathbf{A}_2^{-1} \mathbf{A}_1 \in \mathbb{R}^{4 \times 3}, \quad \mathbf{M}_o = \mathbf{L}^T \hat{\mathbf{M}}_o \mathbf{L} \in \mathbb{R}^{3 \times 3}, \\ \mathbf{L} = \begin{bmatrix} \mathbf{I}_{3 \times 3} \\ \mathbf{B}_o \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} -\sin \bar{q} & \cos \bar{q} & A_{1,1} \\ \sin \bar{q} & -\cos \bar{q} & A_{1,2} \\ -\cos \bar{q} & -\sin \bar{q} & A_{2,1} \\ -\cos \bar{q} & -\sin \bar{q} & A_{2,2} \end{bmatrix}, \\ \mathbf{A}_2 = \begin{bmatrix} -\cos(\bar{\theta}_1 - \bar{q}) & 0 & 0 & 0 \\ 0 & \cos(\bar{\theta}_2 - \bar{q}) & 0 & 0 \\ -\sin(\bar{\theta}_1 - \bar{q}) & 0 & -r_1 & 0 \\ 0 & -\sin(\bar{\theta}_2 - \bar{q}) & 0 & r_2 \end{bmatrix}, \\ A_{1,i} = (-1)^{i+1} [l_i \cos(\bar{\theta}_i - \bar{q}) - \bar{x}_m \cos \bar{q} + g_i \sin \bar{q}], \\ A_{2,i} = l_i \sin(\bar{\theta}_i - \bar{q}) + \bar{x}_m \sin \bar{q} \\ + g_i \cos \bar{q} + (-1)^{i+1} r_i, \\ g_i = L(i-1) - \bar{y}_m, \\ \hat{\mathbf{M}}_o = \text{diag}(M, M, I, m_1, m_2, I_1, I_2), \\ \zeta(t) = \mathbf{B}_o [x_{me}(t) - x_{me}^d, y_{me}(t) - y_{me}^d, q_e(t) - q_e^d]^T, \end{cases} \quad (3)$$

where $[x_{me}^d, y_{me}^d]$ is the desired position vector of the center of the object. Here, $w_i^d(x_i)$ is uniquely determined for any given desired grasping force f^d . In addition, θ_{ie}^d , x_{me}^d , and y_{me}^d are uniquely determined for any given desired orientation of the grasped object q_e^d (see Appendix B). All desired values, f^d , q_e^d , θ_{ie}^d , x_{me}^d , and y_{me}^d , are constants.

The mathematical model in [27] did not include the viscous damping, but we included the viscous damping in the model of this paper. Then, we proved the exponential stability of the closed-loop system in Section 4 by utilizing this damping as well as the controller in [22], which can avoid the following compact perturbation problem [29, 30]: The controller, which is compact operator in the functional state space, cannot exponentially stabilize the system, but the controller, which is noncompact operator in the functional state space has the possibility to exponentially stabilizes the system. In addition, we carried out numerical simulations in Section 5, and we confirmed that our controller has higher control performance than the controller, which is a compact operator.

2.2 Boundary Controller

The forces applied to the object by the arm i , are $f_i(t)$ and $\lambda_i(t)$ as shown in Fig. 1 (a). $f_i(t)$ is the contact force of arm i , which arises in the opposite direction along the normal vector of the surface of

the object, and $\lambda_i(t)$ is the constraint force of arm i , which arises in the tangential direction of the surface of the object. In the steady state, $f_i(t)$ and $\lambda_i(t)$ become constant as follows: f_i^s and λ_i^s . They have the following relations (see Appendix B):

$$f^d := f_1^s = f_2^s, \quad \lambda^d := \lambda_1^s = -\lambda_2^s, \quad (4)$$

$$\lambda^d = \frac{-l_1 \cos(\bar{\theta}_1 - \bar{q}) + l_2 \cos(\bar{\theta}_2 - \bar{q}) + L \sin \bar{q}}{l_1 \sin(\bar{\theta}_1 - \bar{q}) - l_2 \sin(\bar{\theta}_2 - \bar{q}) - L \cos \bar{q} + r_1 + r_2} f^d. \quad (5)$$

From (4), two arms push the object at the same force, and we define the force f_i^s as the grasping force f^d . In addition, constraint forces λ_1^s and λ_2^s are competing forces, and we define $\lambda^d = \lambda_1^s = -\lambda_2^s$. The relation (5) is derived from the balance condition of the moment, and we find that f^d and λ^d cannot be set independently. In addition, in the steady state, if $y_i(x_i, t) = 0$ and $y_i'(0, t) = 0$ hold, that is, if $w_i(x_i, t) = w_i^d(x_i)$ and $\theta_{ie}(t) = \theta_{ie}^d$ hold, then we obtain $f_i(t) = f^d$, $q_e(t) = q_e^d$, $x_{me}(t) = x_{me}^d$, $y_{me}(t) = y_{me}^d$, and $\lambda_i(t) = (-1)^{i+1} \lambda^d$. Thus, the control objective is to realize the following:

$$\begin{cases} y_i(x_i, t) \rightarrow 0, \quad \dot{y}_i(x_i, t) \rightarrow 0, \\ y_i'(0, t) \rightarrow 0, \quad \dot{y}_i'(0, t) \rightarrow 0, \quad \dot{\zeta}(t) \rightarrow 0, \end{cases} \quad \text{as } t \rightarrow \infty. \quad (6)$$

For this, we use the following boundary controller proposed in [22]:

$$\begin{aligned} \tau_i(t) = & -k_{pi}y_i'(0, t) - k_{di}\dot{y}_i'(0, t) + (k_{pi}^s - 1)EI_i y_i''(0, t) \\ & + k_{di}^s EI_i \dot{y}_i''(0, t) - EI_i (w_i^d)''(0), \end{aligned} \quad (7)$$

where k_{pi} , k_{di} , k_{pi}^s , and k_{di}^s are positive constants. In [22], this controller realized stable grasping of an object by two flexible arms. However, the contacts between the arm-tips and the grasped object were fixed and did not allow rolling contact. Therefore, the flexible arms could grasp an object, but could not manipulate it. This paper considers the system taking the rolling constraint and shows well-posedness and exponential stability of the system under the controller (7).

In (7), the first and second terms are PD control of the rotational angle of the motor, and the third and fourth terms are PD control of the bending moment at the root of the arm. The fifth term is the compensation term of the reaction torque in the dynamics of the motor. The first and second terms are expected to realize $y_i'(0, t) \rightarrow 0$ and $\dot{y}_i'(0, t) \rightarrow 0$. The third term is the term for $y_i(x_i, t) \rightarrow 0$. In addition, the fourth term is expected to realize $\dot{y}_i(x_i, t) \rightarrow 0$ and $\dot{\zeta}(t) \rightarrow 0$.

3 Closed-Loop System

3.1 Semigroup Setting

Let us introduce the following new variable:

$$\eta_i(t) = J_i \dot{y}_i'(0, t) + \frac{k_{di}^s k_{pi}}{k_{pi}^s} y_i'(0, t) - k_{di}^s EI_i y_i''(0, t). \quad (8)$$

Here, we introduced $\eta_i(t)$ so that the closed-loop system becomes dissipative as shown in (16). Then, the closed-loop system (1), (7), and (8) can be written as follows: For $(x_i, t) \in (0, l_i) \times \mathbb{R}^+$ and

$i = 1, 2$,

$$\begin{cases} \rho_i \ddot{y}_i(x_i, t) + \mu_i \dot{y}_i(x_i, t) + EI_i y_i''''(x_i, t) = 0, \\ y_i(0, t) = 0, \\ \dot{\eta}_i(t) = -k_{pi} y_i'(0, t) - D_i \dot{y}_i'(0, t) + k_{pi}^s EI_i y_i''(0, t), \\ \eta_i(t) = J_i \dot{y}_i'(0, t) + \frac{k_{di}^s k_{pi}}{k_{pi}^s} y_i'(0, t) - k_{di}^s EI_i y_i''(0, t), \\ \ddot{\zeta}(t) = \mathbf{G} [EI_1 y_1'''(l_1, t), EI_2 y_2'''(l_2, t), \\ -EI_1 y_1''(l_1, t), -EI_2 y_2''(l_2, t)]^T, \\ \dot{\zeta}(t) = [\dot{y}_1(l_1, t), \dot{y}_2(l_2, t), \dot{y}_1'(l_1, t), \dot{y}_2'(l_2, t)]^T, \end{cases} \quad (9)$$

where $D_i = k_{di} - k_{di}^s k_{pi} / k_{pi}^s$.

Assumption 2. Feedback gains satisfy

$$k_{pi}^s k_{di} > k_{di}^s k_{pi},$$

that is, $D_i > 0$.

Through this paper, we assume that feedback gains satisfy this assumption. This assumption is provided because of the followings: a coefficient in the inner product (12) becomes positive; the operator $\mathbb{A}$ defined by (13) becomes dissipative as shown in (16).

As the state space of the closed-loop system (9), we define the following functional space $\mathbb{H}$: For $z = [u_1, v_1, u_2, v_2, \eta_1, \eta_2, \xi^T]^T$ and $\xi = [\xi_1, \xi_2, \xi_3, \xi_4]^T$,

$$\begin{aligned} \mathbb{H} = & \left\{ z \in H^2(0, l_1) \times L^2(0, l_1) \times H^2(0, l_2) \times L^2(0, l_2) \right. \\ & \left. \times \mathbb{C}^2 \times H_b : u_i(0) = 0, P_B[u_1, u_2]^T \in H_b \right\}, \end{aligned} \quad (10)$$

where the state variable z is corresponding to $[y_1(\cdot, t), \dot{y}_1(\cdot, t), y_2(\cdot, t), \dot{y}_2(\cdot, t), \eta_1(t), \eta_2(t), \dot{\zeta}^T]^T$, $H^m(0, l_i)$ is the Sobolev space of order m , $L^2(0, l_i)$ is the square integrable functional space, $\mathbb{C}$ is the set of complex numbers, and H_b is following space with the corresponding inner product:

$$\begin{cases} H_b = [\ker(\mathbf{B}_o^T)]^\perp = \text{range}(\mathbf{B}_o) \subset \mathbb{C}^4, \\ \langle \xi, \tilde{\xi} \rangle_{H_b} = \xi^T \mathbf{G}^+ \tilde{\xi}, \quad \text{for } \xi, \tilde{\xi} \in H_b, \end{cases} \quad (11)$$

where $\mathbf{G}^+$ denotes the Moore-Penrose generalized inverse of matrix $\mathbf{G}$. The rank of $\mathbf{G} \in \mathbb{R}^{4 \times 4}$ is 3, and thus $\mathbf{G}^{-1}$ does not exist. Therefore, we introduce the space (11) as the state space of $\dot{\zeta}$ based on the procedure in [11]. Here, note that $\ker(\mathbf{G}^+) = \text{range}(\mathbf{G})^\perp = \text{range}(\mathbf{B}_o)^\perp = H_b^\perp$. Thus, $\mathbf{G}$ restricted to H_b is positive definite, and $\mathbf{G}^+$ restricted to H_b is also positive definite. Further, following the procedure in [11], the boundary projection operator P_B is defined as follows:

$$\begin{cases} P_B[u_1, u_2]^T = [u_1(l_1), u_2(l_2), u_1'(l_1), u_2'(l_2)]^T, \\ D(P_B) = H^2(0, l_1) \times H^2(0, l_2), \end{cases}$$

where the space $\{(u_1, u_2) \in H^2(0, l_1) \times H^2(0, l_2) : u_i(0) = 0\}$ is contained in $D(P_B)$, and thus the operator P_B is well defined in $\mathbb{H}$. In $\mathbb{H}$, we define the following inner product: For $z = [u_1, v_1, u_2, v_2, \eta_1, \eta_2, \xi^T]^T$, and $\tilde{z} = [\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\xi}^T]^T$,

$$\begin{aligned} \langle z, \tilde{z} \rangle_{\mathbb{H}} = & \sum_{i=1}^2 \left[\int_0^{l_i} (EI_i u_i'' \overline{\tilde{u}_i''} + \rho_i v_i \overline{\tilde{v}_i}) dx_i + C_i \eta_i \overline{\tilde{\eta}_i} \right. \\ & \left. + \frac{k_{pi}}{k_{pi}^s} u_i'(0) \overline{\tilde{u}_i'(0)} \right] + \xi^T \mathbf{G}^+ \tilde{\xi}, \end{aligned} \quad (12)$$

where $C_i = 1/(J_i k_{pi}^s + k_{di}^s D_i) > 0$. It is easy to see that $\mathbb{H}$ with the inner product (12) becomes a Hilbert space. In addition, we

define the operator $\mathbb{A} : D(\mathbb{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ as follows:

$$\mathbb{A}z = \begin{bmatrix} v_1 \\ -(EI_1 u_1'''' + \mu_1 v_1)/\rho_1 \\ v_2 \\ -(EI_2 u_2'''' + \mu_2 v_2)/\rho_2 \\ -k_{p1} u_1'(0) - D_1 v_1'(0) + k_{p1}^s EI_1 u_1''(0) \\ -k_{p2} u_2'(0) - D_2 v_2'(0) + k_{p2}^s EI_2 u_2''(0) \\ \mathbf{GF} \end{bmatrix}, \quad (13)$$

with domain

$$\begin{aligned} D(\mathbb{A}) = & \left\{ z \in H^4(0, l_1) \times H^2(0, l_1) \times H^4(0, l_2) \right. \\ & \times H^2(0, l_2) \times \mathbb{C}^2 \times H_b : u_i(0) = v_i(0) = 0, \\ & \eta_i = J_i v_i'(0) + \frac{k_{di}^s k_{pi}}{k_{pi}^s} u_i'(0) - k_{di}^s EI_i u_i''(0), \\ & \left. \xi = P_B[v_1, v_2]^T \in H_b, P_B[u_1, u_2]^T \in H_b \right\}, \end{aligned} \quad (14)$$

where

$$\mathbf{F} = [EI_1 u_1'''(l_1), EI_2 u_2'''(l_2), -EI_1 u_1''(l_1), -EI_2 u_2''(l_2)]^T.$$

Then, the closed-loop system (9) can be written as the following first order evolution equation on $\mathbb{H}$:

$$\dot{z}(t) = \mathbb{A}z(t), \quad z(0) = z_0, \quad (15)$$

where $z(t) = [y_1(\cdot, t), \dot{y}_1(\cdot, t), y_2(\cdot, t), \dot{y}_2(\cdot, t), \eta_1(t), \eta_2(t), \xi^T]^T$ is the state, and z_0 is the initial value.

3.2 Property of the Closed-Loop System

The well-posedness (there is a solution, the solution is unique, and the solution depends continuously on the initial data) of the system represented by PDE is not trivial. If the operator $\mathbb{A}$ generates a C_0 -semigroup, the system becomes well-posed, and thus we investigate whether the operator $\mathbb{A}$ generates a C_0 -semigroup in Lemma 1.

Lemma 1. Let $\mathbb{H}$ and $\mathbb{A}$ be defined by (10) and (13), respectively. The operator $\mathbb{A}$ generates a C_0 -semigroup of contractions on $\mathbb{H}$.

Proof: We divide the proof into three steps.

(1st step) First, we show that $\mathbb{A}$ is dissipative. For any $z = [u_1, v_1, u_2, v_2, \eta_1, \eta_2, \xi^T]^T \in D(\mathbb{A})$, it follows that

$$\begin{aligned} \langle \mathbb{A}z, z \rangle_{\mathbb{H}} &= \sum_{i=1}^2 \left[\int_0^{l_i} EI_i u_i'' \overline{u_i''} dx_i + \frac{k_{pi}^s}{k_{pi}^s} v_i'(0) \overline{u_i'(0)} \right. \\ &\quad - EI_i \int_0^{l_i} u_i'' \overline{v_i''} dx_i - EI_i u_i''(0) \overline{v_i'(0)} - \mu_i \|v_i\|_{L^2}^2 \\ &\quad \left. + C_i (-k_{pi} u_i'(0) - D_i v_i'(0) + k_{pi}^s EI_i u_i''(0)) \overline{\eta_i} \right] \\ &\quad - \mathbf{F}^T \bar{\xi} + \mathbf{F}^T \mathbf{G}^T \mathbf{G}^+ \bar{\xi} \\ &= \sum_{i=1}^2 \left[\int_0^{l_i} EI_i v_i'' \overline{u_i''} dx_i + \frac{k_{pi}^s}{k_{pi}^s} v_i'(0) \overline{u_i'(0)} \right. \\ &\quad - EI_i \int_0^{l_i} u_i'' \overline{v_i''} dx_i - EI_i u_i''(0) \overline{v_i'(0)} - \mu_i \|v_i\|_{L^2}^2 \\ &\quad \left. + C_i (-k_{pi} u_i'(0) - D_i v_i'(0) + k_{pi}^s EI_i u_i''(0)) \overline{\eta_i} \right], \end{aligned}$$

where we used the integration by parts, the boundary conditions in (14), and the facts $\mathbf{G}$ is symmetric and $\mathbf{G}\mathbf{G}^+$ is the orthogonal projection of $\mathbb{C}$ onto H_b . In addition, $\|\cdot\|_{L^2}^2 = \int_0^{l_i} |\cdot|^2 dx_i$. Similarly, we obtain the following:

$$\begin{aligned} \langle z, \mathbb{A}z \rangle_{\mathbb{H}} &= \sum_{i=1}^2 \left[\int_0^{l_i} EI_i u_i'' \overline{v_i''} dx_i + \frac{k_{pi}^s}{k_{pi}^s} u_i'(0) \overline{v_i'(0)} \right. \\ &\quad - EI_i \int_0^{l_i} v_i'' \overline{u_i''} dx_i - EI_i v_i'(0) \overline{u_i''(0)} - \mu_i \|v_i\|_{L^2}^2 \\ &\quad \left. + C_i \eta_i \overline{(-k_{pi} u_i'(0) - D_i v_i'(0) + k_{pi}^s EI_i u_i''(0))} \right], \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} 2\text{Re}\langle \mathbb{A}z, z \rangle_{\mathbb{H}} &= \langle \mathbb{A}z, z \rangle_{\mathbb{H}} + \langle z, \mathbb{A}z \rangle_{\mathbb{H}} \\ &= -2 \sum_{i=1}^2 C_i \left[\frac{k_{di}^s}{k_{pi}^s} |k_{pi} u_i'(0) - k_{pi}^s EI_i u_i''(0)|^2 \right. \\ &\quad \left. + J_i D_i |v_i'(0)|^2 \right] - 2 \sum_{i=1}^2 \mu_i \|v_i\|_{L^2}^2 \leq 0. \end{aligned} \quad (16)$$

Hence, $\mathbb{A}$ is dissipative.

(2nd step) Next, we show $0 \in \rho(\mathbb{A})$, where $\rho(\mathbb{A})$ is the resolvent set of $\mathbb{A}$. For any given $\hat{z} = [\hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\xi}^T]^T \in \mathbb{H}$, we find a solution $z = [u_1, v_1, u_2, v_2, \eta_1, \eta_2, \xi^T]^T \in D(\mathbb{A})$ of the equation $\mathbb{A}z = \hat{z}$. Here, $\mathbb{A}z = \hat{z}$ means the followings:

$$v_i = \hat{u}_i, \quad u_i'''' = -\frac{1}{EI_i} (\rho_i \hat{v}_i + \mu_i \hat{u}_i), \quad (17)$$

$$-k_{pi} u_i'(0) - D_i v_i'(0) + k_{pi}^s EI_i u_i''(0) = \hat{\eta}_i, \quad (18)$$

$$\mathbf{GF} = \hat{\xi}, \quad (19)$$

$$\begin{cases} u_i(0) = v_i(0) = 0, \\ \eta_i = J_i v_i'(0) + \frac{k_{di}^s k_{pi}}{k_{pi}^s} u_i'(0) - k_{di}^s EI_i u_i''(0), \\ \xi = [v_1(l_1), v_2(l_2), v_1'(l_1), v_2'(l_2)]^T \in H_b, \\ [u_1(l_1), u_2(l_2), u_1'(l_1), u_2'(l_2)] \in H_b. \end{cases} \quad (20)$$

The solution of the second equation in (17) is given by

$$\begin{aligned} u_i(x_i) &= c_{1i} x_i + c_{2i} x_i^2 + c_{3i} x_i^3 \\ &\quad - \frac{1}{6EI_i} \int_0^{x_i} (x_i - s)^3 [\rho_i \hat{v}_i(s) + \mu_i \hat{u}_i(s)] ds, \end{aligned} \quad (21)$$

where $c_{j,i}$, $i = 1, 2$, $j = 1, 2, 3$ are constants, determined by the boundary conditions, and we used $u_i(0) = 0$. Here, multiplying both sides of (19) by $\mathbf{M}_o(\mathbf{B}_o^T \mathbf{B}_o)^{-1} \mathbf{B}_o^T$ from the left side gives

$$\mathbf{B}_o^T \mathbf{F} = \mathbf{f}, \quad (22)$$

where $\mathbf{f} = \mathbf{M}_o(\mathbf{B}_o^T \mathbf{B}_o)^{-1} \mathbf{B}_o^T \hat{\xi} \in \mathbb{C}^{3 \times 1}$. Now, we consider the transformation of the final equation in (20) using the procedure in [11]. Let $\mathbf{b}_i$, $i = 1, 2, 3$, be the column vector of $\mathbf{B}_o$, i.e., $\mathbf{B}_o = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$. Since $[\ker(\mathbf{B}_o^T)]^\perp = \text{span}\{\mathbf{b}_i, i = 1, 2, 3\}$, there exist three constants a_i , $i = 1, 2, 3$, such that

$$[u_1(l_1), u_2(l_2), u_1'(l_1), u_2'(l_2)]^T = a_1 \mathbf{b}_1 + a_2 \mathbf{b}_2 + a_3 \mathbf{b}_3. \quad (23)$$

Substituting (21) into (18), (22), and (23), we obtain the matrix form relation: $\mathbf{M}_p[c_{11}, c_{12}, c_{21}, c_{22}, c_{31}, c_{32}, a_1, a_2, a_3]^T = \mathbf{F}_p$,

where $M_p \in \mathbb{R}^{9 \times 9}$ is a matrix, and $F_p \in \mathbb{C}^{9 \times 1}$ is a vector. Here, $\text{rank}(M_p) = 9$, and thus the coefficients $c_{j,i}$ and a_i can be uniquely determined. The remaining unknowns, v_i , η_i , and ξ can be easily obtained. Thus, $\mathbb{A}^{-1}$ exists.

For the coefficients $c_{j,i}$, we obtain the following:

$$c_{ji} \leq \kappa_1 \sum_{i=1}^2 \left\{ |\hat{u}'_i(0)|^2 + |\hat{\eta}_i|^2 + \|\hat{u}_i\|_{L^2}^2 + \|\hat{v}_i\|_{L^2}^2 \right\} + \mathbf{f}^T \mathbf{f}, \quad (24)$$

where κ_1 is a positive constant. In the derivation of (24), we used the Cauchy-Schwartz inequality and the following

$$|a + b|^2 \leq (|a| + |b|)^2 \leq 2(|a|^2 + |b|^2), \quad \text{for } a, b \in \mathbb{C}. \quad (25)$$

Here, note that there exists $\hat{\mathbf{a}} \in \mathbb{C}^{3 \times 1}$ such that $\hat{\xi} = B_o \hat{\mathbf{a}}$ because of $\hat{\xi} \in H_b$. Then, $\mathbf{f}^T \mathbf{f}$ in (24) can be estimated as $\mathbf{f}^T \mathbf{f} = \hat{\xi}^T (B_o^T G^+)^T B_o^T G^+ \hat{\xi} = \hat{\mathbf{a}}^T M_o^2 \hat{\mathbf{a}} \leq \kappa_2 \hat{\mathbf{a}}^T M_o \hat{\mathbf{a}} = \kappa_2 \langle \hat{\xi}, \hat{\xi} \rangle_{H_b}$, where we used $B_o^T G^+ B_o = M_o$ and κ_2 is a positive constant, and thus (24) can be rewritten as

$$c_{ji} \leq \kappa_3 \left[\sum_{i=1}^2 \left\{ |\hat{u}'_i(0)|^2 + |\hat{\eta}_i|^2 + \|\hat{u}_i\|_{L^2}^2 + \|\hat{v}_i\|_{L^2}^2 \right\} + \|\hat{\xi}\|_{H_b}^2 \right], \quad (26)$$

where κ_3 is a positive constant. By using the Cauchy-Schwartz inequality, (25), and (26), it is easy to see that there exists a positive constant κ_4 such that $\|z\|_{\mathbb{H}} = \|\mathbb{A}^{-1} \hat{z}\|_{\mathbb{H}} \leq \kappa_4 \|\hat{z}\|_{\mathbb{H}}$. Therefore, $\mathbb{A}^{-1}$ is bounded, and thus we obtain $0 \in \rho(\mathbb{A})$.

(3rd step) From the 1st and 2nd steps, $\mathbb{A}$ is dissipative and $0 \in \rho(\mathbb{A})$. Therefore, $\mathbb{A}$ generates a C_0 -semigroup of contractions by the Lumer-Phillips theorem and the contraction mapping theorem [31]. $\square$

Let $\mathbb{T}(t)$ be a C_0 -semigroup of contractions generated by $\mathbb{A}$. Then, Lemma 1 means that (15) has a unique solution $z(t) = \mathbb{T}(t)z_0$ for every $z_0 \in D(\mathbb{A})$, that is, the closed-loop system is well-posed. If the operator $\mathbb{A}$ generates a C_0 -semigroup of contractions, the closed-loop system is well-posed, and the system becomes Lyapunov stable. However, to show the exponential stability of the closed-loop system, we need to investigate some properties. In Lemma 2 and 3, we show that the system satisfies the conditions for exponential stability. In addition, we summarize in Theorem 1 that the system is exponentially stable.

4 Stability and Robustness of the Closed-Loop System

4.1 Exponential Stability

We show the exponential stability of the closed-loop system using the frequency domain method. According to this method, we need to show the following two facts to prove the exponential stability of a C_0 -semigroup of contractions in a Hilbert space [31]:

$$(i) \quad \rho(\mathbb{A}) \supset \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \quad (27)$$

$$(ii) \quad \overline{\lim}_{\beta \rightarrow \infty} \|(i\beta - \mathbb{A})^{-1}\|_{\mathbb{H}} < \infty. \quad (28)$$

Here, the exponential stability of a C_0 -semigroup of contractions, which satisfies (27) and (28), in a Hilbert space is obtained from the well-known Gearhart-Prüss-Greiner theorem [32]. In the following, we demonstrate fact (ii) in Lemma 2, and (i) in Lemma 3.

Lemma 2. Condition (28) holds for the operator $\mathbb{A}$, defined by (13).

Proof: According to the contradiction argument method [31], if (28) is false, then there exists a sequence $\beta_n \in \mathbb{R}$ with $\beta_n \rightarrow \infty$ and a

sequence $z_n \in D(\mathbb{A})$ with $\|z_n\|_{\mathbb{H}} = 1$ such that:

$$(i\beta_n - \mathbb{A})z_n \equiv \varphi_n \rightarrow 0 \text{ in } \mathbb{H}, \quad (29)$$

where $z_n = [u_{1n}, v_{1n}, u_{2n}, v_{2n}, \eta_{1n}, \eta_{2n}, \xi_n^T]^T$, $\xi_n = [\xi_{1n}, \xi_{2n}, \xi_{3n}, \xi_{4n}]^T$, $\varphi_n = [\varphi_{11n}, \varphi_{21n}, \varphi_{12n}, \varphi_{22n}, \varphi_{31n}, \varphi_{32n}, \phi_n^T]^T$, and $\phi_n = [\varphi_{4n}, \varphi_{5n}, \varphi_{6n}, \varphi_{7n}]^T$. Here, (29) means the followings:

$$i\beta_n u_{in} - v_{in} = \varphi_{1in}, \quad (30)$$

$$i\beta_n v_{in} + \frac{1}{\rho_i} (EI_i u_{in}'''' + \mu_i v_{in}) = \varphi_{2in}, \quad (31)$$

$$i\beta_n \eta_{in} + k_{pi} u_{in}'(0) + D_i v_{in}'(0) - k_{pi}^s EI_i u_{in}''(0) = \varphi_{3in}, \quad (32)$$

$$i\beta_n \xi_n - \mathbf{G}\mathbf{F} = \phi_n. \quad (33)$$

We show the contradictions of $\|z_n\|_{\mathbb{H}} = 1$, i.e., $\|z_n\|_{\mathbb{H}} \rightarrow 0$. For clarity, we divide the proof into four steps.

(1st step) We derive the required estimations.

(i) From the fact that $\varphi_n \rightarrow 0$ in $\mathbb{H}$, we obtain the followings:

$$\begin{cases} \|\varphi_{1in}\|_{L^2} \rightarrow 0, \|\varphi_{2in}\|_{L^2} \rightarrow 0, |\varphi_{1in}'(0)| \rightarrow 0, \\ |\varphi_{3in}| \rightarrow 0, \|\phi_n\|_{H_b} \rightarrow 0. \end{cases} \quad (34)$$

(ii) From the equation $u_{in}(x_i) = \int_0^{x_i} u_{in}'(s) ds$ and the Cauchy-Schwartz inequality, we obtain $\|u_{in}\|_{L^2}^2 \leq l_i^2 \|u_{in}'\|_{L^2}^2$. Further, combining the equation $u_{in}'(x_i) = \int_0^{x_i} u_{in}''(s) ds + u_{in}'(0)$, the Cauchy-Schwartz inequality, and (25) gives

$$\|u_{in}'\|_{L^2}^2 \leq \kappa_5 \left[\|u_{in}''\|_{L^2}^2 + |u_{in}'(0)|^2 \right], \quad (35)$$

where κ_5 is a positive constant. Thus, we obtain the following: For $p_i \in \{p_i \in H^2(0, l_i) : p_i(0) = 0\}$, there exists a positive constant κ_6 such that

$$\begin{aligned} EI_i \|p_i''\|_{L^2}^2 + \frac{k_{pi}}{k_{pi}^s} |p_i'(0)|^2 \\ \geq \kappa_6 \left[\|p_i\|_{L^2}^2 + \|p_i'\|_{L^2}^2 + \|p_i''\|_{L^2}^2 \right]. \end{aligned} \quad (36)$$

(iii) From (34) and (36), we obtain

$$\|\varphi_{1in}\|_{L^2} \rightarrow 0, \quad \|\varphi_{1in}'\|_{L^2} \rightarrow 0. \quad (37)$$

Further, combining the equation $\varphi_{1in}(l_i) = \int_0^{l_i} \varphi_{1in}'(s) ds$, the Cauchy-Schwartz inequality, and (37) leads to

$$|\varphi_{1in}(l_i)| \rightarrow 0. \quad (38)$$

(iv) Equation (29) implies $\text{Re}\langle (i\beta_n - \mathbb{A})z_n, z_n \rangle_{\mathbb{H}} = -\text{Re}\langle \mathbb{A}z_n, z_n \rangle_{\mathbb{H}} \rightarrow 0$. This together with (16) and the boundary conditions in (14) yields

$$\begin{cases} |v_{in}'(0)| \rightarrow 0, |k_{pi} u_{in}'(0) - k_{pi}^s EI_i u_{in}''(0)| \rightarrow 0, \\ \|v_{in}\|_{L^2} \rightarrow 0, |\eta_{in}| \rightarrow 0. \end{cases} \quad (39)$$

In addition, from (30), (34), and (39), we have

$$\begin{cases} |\beta_n u_{in}'(0)| \rightarrow 0, |u_{in}'(0)| \rightarrow 0, |u_{in}''(0)| \rightarrow 0, \\ \|\beta_n u_{in}\|_{L^2} \rightarrow 0, \|u_{in}\|_{L^2} \rightarrow 0. \end{cases} \quad (40)$$

On the other hand, we obtain

$$\|u_{in}'\|_{L^2} \rightarrow 0, \quad (41)$$

from $\|z_n\|_{\mathbb{H}} = 1$, (40), and the following Gagliardo-Nirenberg inequality [31]: For $p(x) \in H^2(0, l_i)$

$$\|p'(x)\|_{L^2} \leq \kappa_7 \|p''(x)\|_{L^2}^{\frac{1}{2}} \|p(x)\|_{L^2}^{\frac{1}{2}} + \kappa_8 \|p(x)\|_{L^2}, \quad (42)$$

where κ_7 and κ_8 are positive constants.

(v) For $v_{in}(l_i)$, we obtain the following:

$$|v_{in}(l_i)| \leq \|v_{in}\|_{L^\infty} \leq \kappa_9 \|v'_{in}\|_{L^2}^{\frac{1}{2}} \|v_{in}\|_{L^2}^{\frac{1}{2}} + \kappa_{10} \|v_{in}\|_{L^2}, \quad (43)$$

where κ_9 and κ_{10} are positive constants, and we used (42) in the derivation of the last inequality. Thus, combining (43), (39), and the fact $v_{in} \in H^2(0, l_i)$ gives

$$|v_{in}(l_i)| \rightarrow 0. \quad (44)$$

In addition, from (30), (44), and (38), we obtain

$$|\beta_n u_{in}(l_i)| \rightarrow 0, \quad |u_{in}(l_i)| \rightarrow 0. \quad (45)$$

Finally, by using the same procedure as (43), we obtain

$$|u'_{in}(l_i)| \leq \kappa_{11} \|u''_{in}\|_{L^2}^{\frac{1}{2}} \|u'_{in}\|_{L^2}^{\frac{1}{2}} + \kappa_{12} \|u_{in}\|_{L^2}, \quad (46)$$

where κ_{11} and κ_{12} are positive constants. This together with $u_{in} \in H^4(0, l_i)$, (40), and (41) yields

$$|u'_{in}(l_i)| \rightarrow 0. \quad (47)$$

(2nd step) Combining (30) and (31) leads to

$$\beta_n^2 u_{in} - \frac{EI_i}{\rho_i} u''''_{in} = - \left(i\beta_n + \frac{\mu_i}{\rho_i} \right) \varphi_{1in} - \varphi_{2in} + i\beta_n \frac{\mu_i}{\rho_i} u_{in}. \quad (48)$$

Multiplying (48) by $\rho_i a_i(x_i) \overline{u'_{in}}$, and integrating it gives

$$\begin{aligned} & \int_0^{l_i} a_i(x_i) \left(\rho_i \beta_n^2 u_{in} - EI_i u''''_{in} \right) \overline{u'_{in}} dx_i \\ &= -\rho_i \int_0^{l_i} a_i(x_i) \left[\left(i\beta_n + \frac{\mu_i}{\rho_i} \right) \varphi_{1in} + \varphi_{2in} \right] \overline{u'_{in}} dx_i \\ &+ \mu_i \int_0^{l_i} a_i(x_i) i\beta_n u_{in} \overline{u'_{in}} dx_i, \end{aligned} \quad (49)$$

where $a_i(x_i)$ is a function to be determined in the latter. Using integration by parts, (14), and the Cauchy-Schwarz inequality, the right-hand side of (49) can be estimated as

$$\begin{aligned} & -\rho_i \int_0^{l_i} a_i(x_i) \left[\left(i\beta_n + \frac{\mu_i}{\rho_i} \right) \varphi_{1in} + \varphi_{2in} \right] \overline{u'_{in}} dx_i \\ &+ \mu_i \int_0^{l_i} a_i(x_i) i\beta_n u_{in} \overline{u'_{in}} dx_i \\ &\leq |a_i(l_i)| |\varphi_{1in}(l_i)| \left[\rho_i |\beta_n u_{in}(l_i)| + \frac{\mu_i}{\rho_i} |u_{in}(l_i)| \right] \\ &+ [\kappa_{13} \|\varphi_{1in}\|_{L^2} + \kappa_{14} \|\varphi'_{1in}\|_{L^2}] \\ &\times \left[\rho_i \|\beta_n u_{in}\|_{L^2} + \frac{\mu_i}{\rho_i} \|u_{in}\|_{L^2} \right] \\ &+ [\rho_i \kappa_{14} \|\varphi_{2in}\|_{L^2} + \mu_i \kappa_{14} \|\beta_n u_{in}\|_{L^2}] \|u'_{in}\|_{L^2}, \end{aligned} \quad (50)$$

where $\kappa_{13} = \max_{x_i \in [0, l_i]} |a'_i(x_i)|$ and $\kappa_{14} = \max_{x_i \in [0, l_i]} |a_i(x_i)|$. This together with (34), (37), (38), (40), (41), and (45) gives the fact that the right-hand side in (50) goes to zero. Thus, (49) can be

rewritten as

$$\int_0^{l_i} a_i(x_i) \left(\rho_i \beta_n^2 u_{in} - EI_i u''''_{in} \right) \overline{u'_{in}} dx_i \rightarrow 0. \quad (51)$$

In this equation, a simple calculation using the integration by parts and (14) leads to

$$\begin{aligned} & \operatorname{Re} \int_0^{l_i} a_i(x_i) \left(\rho_i \beta_n^2 u_{in} - EI_i u''''_{in} \right) \overline{u'_{in}} dx_i \\ &= \frac{\rho_i a_i(l_i)}{2} |\beta_n u_{in}(l_i)|^2 - \frac{\rho_i}{2} \int_0^{l_i} a'_i(x_i) |\beta_n u_{in}|^2 dx_i \\ &+ \operatorname{Re} EI_i \left[a'_i(x_i) u''_{in}(x_i) \overline{u'_{in}(x_i)} - a_i(x_i) u'''_{in}(x_i) \overline{u'_{in}(x_i)} \right]_0^{l_i} \\ &+ \frac{EI_i}{2} \left[a_i(x_i) |u''_{in}(x_i)|^2 - a'_i(x_i) |u'_{in}(x_i)|^2 \right]_0^{l_i} \\ &+ \frac{EI_i}{2} \int_0^{l_i} \left[a'''_{in}(x_i) |u'_{in}|^2 - a'_i(x_i) |u''_{in}|^2 \right] dx_i \rightarrow 0. \end{aligned} \quad (52)$$

Here, we set $a_i(x_i) = l_i - x_i$. In addition, from the equation $u'''_{in}(0) = -\int_0^{x_i} u''''_{in}(s) ds + u''_{in}(0)$, the Cauchy-Schwarz inequality, and (25), we obtain $|u'''_{in}(0)|$ is bounded because of $u_{in} \in H^4(0, l_i)$. Thus, (52) with (40) and (47), and the fact that $|u'''_{in}(0)|$ is bounded, can be estimated as

$$-\frac{3}{2} EI_i \|u''_{in}\|_{L^2}^2 \rightarrow 0. \quad (53)$$

Thus, from (35), (40), and (53), we obtain

$$\|u'_{in}\|_{L^2} \rightarrow 0, \quad \|u''_{in}\|_{L^2} \rightarrow 0. \quad (54)$$

(3rd step) Setting $a_i(x_i) = x_i$ in (52) yields

$$\begin{aligned} & \operatorname{Re} \int_0^{l_i} a_i(x_i) \left(\rho_i \beta_n^2 u_{in} - EI_i u''''_{in} \right) \overline{u'_{in}} dx_i \\ &= \frac{EI_i l_i}{2} |u''_{in}(l_i)|^2 \rightarrow 0. \end{aligned} \quad (55)$$

In this derivation, we used (40), (45), (47), (54), the fact that $|u'''_{in}(0)|$ is bounded, and the followings: (i) From the equation $u'''_{in}(l_i) = \int_0^{l_i} u''''_{in}(s) ds + u''_{in}(0)$, the Cauchy-Schwarz inequality, (25), (40), and $u_{in} \in H^4(0, l_i)$, $|u'''_{in}(l_i)|$ is bounded; (ii) Similarly, we can obtain the boundedness of $|u'''_{in}(l_i)|$. Thus, from (55), we obtain

$$|u''_{in}(l_i)| \rightarrow 0. \quad (56)$$

(4th step) From the equation $\int_0^{l_i} u'''_{in}(x_i) dx_i = u''_{in}(l_i) - u''_{in}(0)$, the Cauchy-Schwarz inequality, (25), (40), and (56), we obtain

$$\|u'''_{in}\|_{L^2} \rightarrow 0. \quad (57)$$

In addition, from (42), we obtain the following

$$|u'''_{in}(l_i)| \leq \|u'''_{in}\|_{L^\infty} \leq \kappa_{15} \|u''''_{in}\|_{L^2}^{\frac{1}{2}} \|u'''_{in}\|_{L^2}^{\frac{1}{2}} + \kappa_{16} \|u'''_{in}\|_{L^2}, \quad (58)$$

where κ_{15} and κ_{16} are positive constants. This together with (57) and the fact $u_{in} \in H^4(0, l_i)$ yields

$$|u'''_{in}(l_i)| \rightarrow 0. \quad (59)$$

Thus, the combining (33), (34), (56), and (59) gives

$$\|\xi_n\|_{H_b} \rightarrow 0. \quad (60)$$

Finally, from (39), (40), (54), and (60), we obtain $\|z_n\|_{\mathbb{H}} \rightarrow 0$, and this is the contradiction of $\|z_n\|_{\mathbb{H}} = 1$. Thus, the claim is proved. $\square$

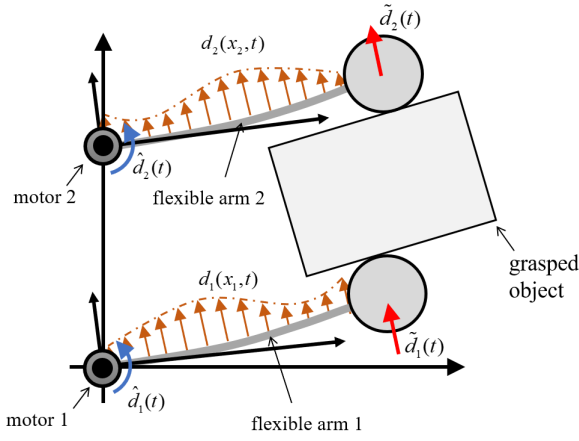


Fig. 2: Disturbances acting on the controlled system.

Lemma 3. The operator $\mathbb{A}$, defined by (13), satisfies condition (27).

Proof: If (27) is false, then there exists $\beta \in \mathbb{R}$ and a sequence $z_n \in D(\mathbb{A})$ with $\|z_n\|_{\mathbb{H}} = 1$ such that $(i\beta - \mathbb{A})z_n \equiv \varphi_n \rightarrow 0$ in $\mathbb{H}$ [12]. By replacing β_n with β in the proof of Lemma 2, we obtain the contradiction of $\|z_n\|_{\mathbb{H}} = 1$. Here, note that we did not use the fact that $\beta_n \rightarrow \infty$ in the proof of Lemma 2. Thus, the claim is proved. $\square$

Combining Lemma 2 and 3 gives the following theorem about the exponential stability of (15).

Theorem 1. The closed-loop system (15) is exponentially stable.

Proof: According to the frequency domain method [31] with Lemmas 2 and 3, the closed-loop system (15) is exponentially stable. $\square$

4.2 Disturbance Responses

The influences of the disturbances are an unavoidable issue in real situations, and the robustness with respect to the disturbances are an important property. Here, we investigate the response of the closed-loop system to the disturbances. As shown in Fig. 2, we consider the following three disturbances: the disturbance $d_i(x_i, t)$ distributed over the flexible arm i , the disturbance $\hat{d}_i(t)$ at the control input $\tau_i(t)$, and disturbance $\tilde{d}_i(t)$ at the arm-tip i . Here, we assume that the disturbances are continuous and bounded; that is, $d_i(x_i, t) \in C^1([0, \tau]; \mathbb{H})$, $\hat{d}_i(t) \in C^1$, $\tilde{d}_i(t) \in C^1$, and there exist positive constants D_i , $\bar{D}_i$, and $\bar{D}_i$ such that $\|d_i(x_i, t)\|_{L^2} < D_i$, $|\hat{d}_i(t)| < \bar{D}_i$, and $|\tilde{d}_i(t)| < \bar{D}_i$.

When these disturbances exist, the closed-loop system (15) becomes the following:

$$\dot{z}(t) = \mathbb{A}z(t) + \mathbb{D}, \quad z(0) = z_0, \quad (61)$$

where $\mathbb{D} = [0, d_1(x_1, t)/\rho_1, 0, d_2(x_2, t)/\rho_2, \hat{d}_1(t), \hat{d}_2(t), [\tilde{d}_1(t), \tilde{d}_2(t), 0, 0]G^T]^T$. For example, $\tilde{d}_i(t)$ is corresponding to the contact force arisen at the arm-tip i when the arm-tip i makes contact with the environment during transportation. In addition, $d_i(x_i, t)$ is considered as distributed disturbance due to the bending of the beam caused by this contact force or wind. Further, $\hat{d}_i(t)$ can be caused by static friction. When we consider these disturbances, the disturbances are separable as in (61) and it is not an unrealistic situation.

Since $\mathbb{A}$ is the infinitesimal generator of a C_0 -semigroup of contractions $\mathbb{T}(t)$, the system (61) has a unique classical solution for

$z_0 \in D(\mathbb{A})$:

$$z(t) = \mathbb{T}(t)z_0 + \int_0^t \mathbb{T}(t-s)\mathbb{D}(s)ds, \quad (62)$$

where $\mathbb{T}(t)$ is an exponential stable C_0 -semigroup, and thus $\mathbb{T}(t)$ satisfies $\|\mathbb{T}(t)\|_{\mathbb{H}} \leq Ke^{-\sigma t}$ for some positive constants K and σ . Taking norm in (62) gives

$$\|z(t)\|_{\mathbb{H}} \leq Ke^{-\sigma t}\|z_0\|_{\mathbb{H}} + \frac{K_1K}{\sigma}(1 - e^{-\sigma t}) \leq K_2,$$

where K_1 and K_2 are some positive constants, and K_1 has the following relation: $\sum_{i=1}^2 (D_i^2 + C_i \hat{D}_i^2 + \lambda_{\max}(G) \bar{D}_i^2) \leq K_1$. Thus, the solution (62) is bounded, and the closed-loop system is robust with respect to disturbances including the disturbances distributed over the arms and disturbances acting at both boundaries.

5 Numerical Simulations

To investigate the theoretical results, we carried out numerical simulations, and we show that the controller (7) has higher control performance than the controller, which can asymptotically stabilize the system but cannot exponentially stabilize the system. For the simulation, we used the finite dimensional approximated modal model derived by the Galerkin approximations, and the model consists of first ten modes. The physical parameters were as follows: For $i = 1, 2$, $l_i = 1.0$ (m), $\rho_i = 0.69$ (kg/m), $EI_i = 8.95$ (Nm²), $\mu_i = 0.3$ (kg/(ms)), $m_i = 0.1$ (kg), $r_i = 0.05$ (m), $I_i = 5 \times 10^{-4}$ (kgm²), $J_i = 15 \times 10^{-4}$ (kgm²), $M = 0.5$ (kg), $I = 5.4 \times 10^{-3}$ (kgm²), $L_i = 0.1$ (m), and $L = 0.56$ (m). In this simulation, we compared the boundary controller (7), named PDPD, which can exponentially stabilize the system and can be written as

$$\begin{aligned} \tau_i(t) = & (k_{pi}^s - 1)EI_i [w_i''(0, t) - (w_i^d)''(0)] + k_{di}^s EI_i \dot{w}_i''(0, t) \\ & - k_{pi} [\theta_{ie}(t) - \theta_{ie}^d] - k_{di} \dot{\theta}_{ie}(t) - EI_i (w_i^d)''(0), \end{aligned} \quad (63)$$

with the PDS controller in [27], which can asymptotically stabilize the system but cannot exponentially stabilize it,

$$\begin{aligned} \tau_i(t) = & (k_{pi}^s - 1)EI_i [w_i''(0, t) - (w_i^d)''(0)] - EI_i (w_i^d)''(0) \\ & - k_{pi} [\theta_{ie}(t) - \theta_{ie}^d] - k_{di} \dot{\theta}_{ie}(t). \end{aligned} \quad (64)$$

In both controllers, we set the feedback gains as $k_{pi} = 1.0$, $k_{di} = 0.4$, $k_{pi}^s = 0.4/EI_i$, and $k_{di}^s = 0.1/EI_i$.

First, we investigated the step responses of the desired grasping force $f^d = 3$ (N) and orientation of the grasped object $q_e^d = -5$ (deg). The initial values were set as follows: $\bar{q} = 5$ (deg), $\bar{\theta}_1 = 5$ (deg), $\bar{\theta}_2 = -10$ (deg), $\bar{x}_m = 1.14$ (m), and $\bar{y}_m = 0.25$ (m). These values lead to the followings: $EI_1(w_1^d)''(0) = -2.99$ (Nm), $EI_2(w_2^d)''(0) = 2.97$ (Nm), $\theta_{1e}^d = 9.1$ (deg), and $\theta_{2e}^d = -3.6$ (deg). For more details, see Appendix B.

Fig. 3 shows the step responses of the PDPD and PDS controller. As an example, we show the responses of $f_1(t)$, $q_e(t)$, $\theta_{1e}(t)$, and $EI_1 w_1'''(0, t)$. In the figures, the black solid line, the light black solid line, and the dotted line show the response of the PDPD controller, the response of the PDS controller, and the desired value of the corresponding value, respectively. From the figures, we found that each response of the PDPD controller converged to the desired value. On the other hand, the responses of the PDS controller left some little vibrations. In fact, the error values of $f_1(t)$, $q_e(t)$, $\theta_{1e}(t)$, and $EI_1 w_1'''(0, t)$ in the responses of the PDPD controller were $E_f(4, 10) = 0.103 \times 10^{-1}$ (N·s), $E_q(4, 10) = 0.011 \times 10^{-1}$ (rad·s), $E_{\theta}(4, 10) = 4.010 \times$

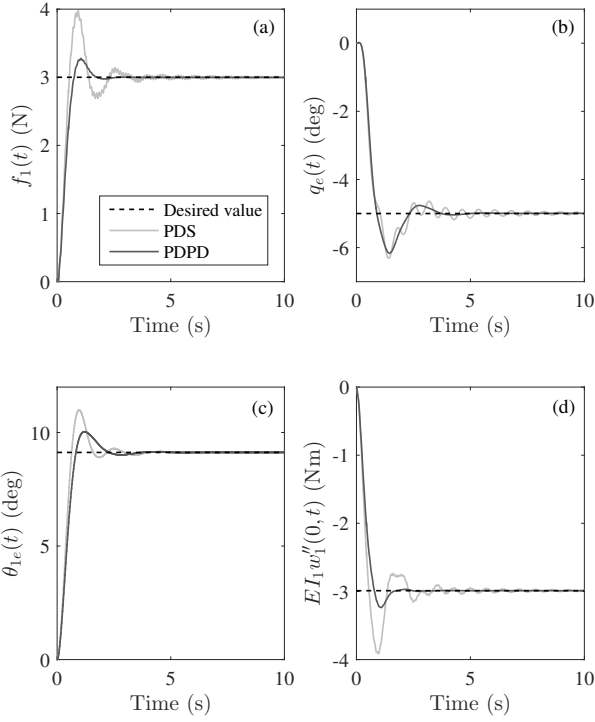


Fig. 3: Step responses: (a) the contact force of arm 1; (b) the rotational angle of the object; (c) the rotational angle of motor 1; (d) the bending moment at the root of arm 1.

10^{-4} (rad·s), and $E_w(4, 10) = 0.022 \times 10^{-1}$ (Nm·s), respectively, where

$$\begin{cases} E_f(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{(f_1 - f^d)^2} dt, \\ E_q(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{(q_e - q_e^d)^2} dt, \\ E_\theta(t_1, t_2) = \int_{t_1}^{t_2} \sqrt{(\theta_{1e} - \theta_{1e}^d)^2} dt, \\ E_w(t_1, t_2) = \int_{t_1}^{t_2} EI_1 \sqrt{[w_1''(0, t) - (w_1^d)''(0)]^2} dt, \end{cases}$$

and these values are average error values of the corresponding variables. On the other hand, these values in the PDS controller were as follows: $E_f(4, 10) = 0.676 \times 10^{-1}$ (N·s), $E_q(4, 10) = 0.055 \times 10^{-1}$ (rad·s), $E_\theta(4, 10) = 6.321 \times 10^{-4}$ (rad·s), and $E_w(4, 10) = 0.876 \times 10^{-1}$ (Nm·s). Thus, we consider the controller (7) worked well in the step response and had a higher performance than the PDS controller, which could not exponentially stabilize the system.

Next, we investigate the disturbance response of the closed-loop system. The feedback gains in both controllers (63) and (64) are the same as in the simulation of the step responses. In fact, the disturbance response and step response simulations were a series, and the disturbances shown in Fig. 2 were applied from $t = 11$ (s) to 11.05 (s) in the step-response simulation. The applied disturbances were the following pulse disturbances: $d_i(x_i, t) = 2x_i$, $\dot{d}_i(t) = 0.5$, and $\ddot{d}_i(t) = 1$. Fig. 4 shows the disturbance responses of the PDPD and PDS controller. In the responses of the PDPD controller, some vibrations excited by the disturbances were not converged to zeros. In contrast, the PDS controller were not able to suppress the excited vibrations in a satisfactory manner, and the response of the PDS controller left some vibrations. The error values in the PDPD controller were $E_f(10, 20) = 0.125$ (N·s), $E_q(10, 20) = 0.088$ (rad·s), $E_\theta(10, 20) = 0.043$ (rad·s), and $E_w(10, 20) = 0.175$ (Nm·s), and the error values in the PDS controller were as follows: $E_f(10, 20) = 0.481$ (N·s), $E_q(10, 20) = 0.093$ (rad·s), $E_\theta(10, 20) = 0.042$ (rad·s), and $E_w(10, 20) =$

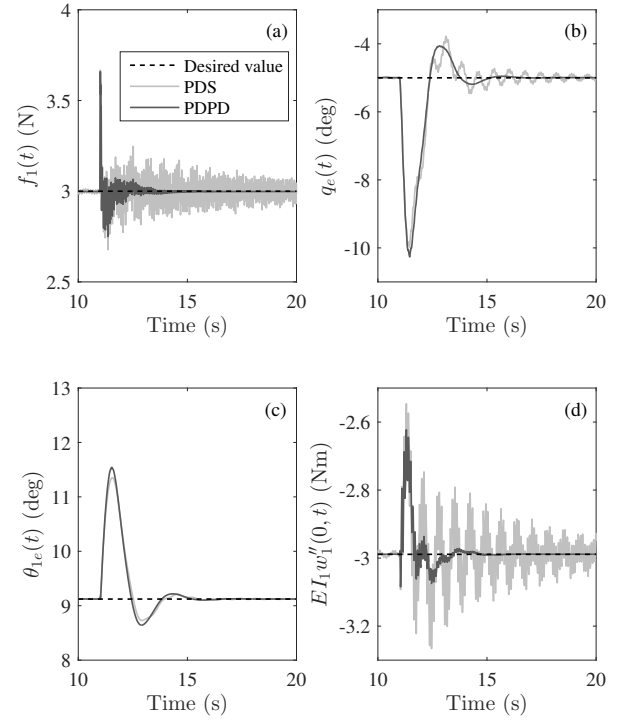


Fig. 4: Disturbance responses: (a) the contact force of arm 1; (b) the rotational angle of the object; (c) the rotational angle of motor 1; (d) the bending moment at the root of arm 1.

0.586 (Nm·s). From these results, we found that the closed-loop system (15) is robust with respect to the disturbances by comparing with the asymptotic stabilizing controller such as the PDS controller.

6 Conclusions

This paper described well-posedness and exponential stability of two one-link flexible arms for grasping and orientation control of an object. Using the frequency domain method, exponential stability of the closed-loop system under our boundary controller was proved. In addition, we discussed the robustness of the closed-loop system with respect to the several disturbances including disturbances distributed over the arm and disturbances acting at boundaries, and the boundedness of the solution of the closed-loop system was shown under these disturbances. Finally, we carried out numerical simulations to investigate the theoretical results.

In this paper, we included the viscous damping in the mathematical model of the system. Then we proved the exponential stability of the closed-loop system by utilizing this damping as well as the controller (7). However, we could not show the exponential stability of the closed-loop system without the damping, and this is a problem to be tackled in the future. Further, while this paper considered a small motion around a stationary state; that is, we considered a linear motion, it is desired to take the nonlinear motion to realize the large motion. In addition, the design of the boundary controller, that keeps the gripping force in the friction cone, and the investigation of the system motion in three-dimensional space are the future research issues.

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8 Appendix A: Derivation of Equations of Motion

Although the derivation of the equations of motion is same as [27], we present it for the sake of completeness. First, we derive the geometric constraints between the arm-tips and the grasped object based on [26–28]. The contact between the arm-tip and the surface of the object is a point contact, and there is no contact break. Thus, the following geometric constraint along the Y_m -axis in Fig. 1 (a) is

obtained: For $i = 1, 2$,

$$\begin{aligned} \Phi_i &:= (-1)^{i+1} \{x_{0i}(t) - [\bar{x}_m + x_{me}(t)]\} \sin[\bar{q} + q_e(t)] \\ &\quad + (-1)^i \{y_{0i}(t) - [\bar{y}_m + y_{me}(t)]\} \cos[\bar{q} + q_e(t)] \\ &\quad - r_i - L_i = 0, \end{aligned} \quad (65)$$

where the positions of the center of mass $[x_{0i}(t), y_{0i}(t)]$ of arm-tip i are expressed as $x_{0i}(t) = l_i [\cos \bar{\theta}_i - \theta_{ie}(t) \sin \bar{\theta}_i] - w_i(l_i, t) \sin \bar{\theta}_i$, and $y_{0i}(t) = l_i [\sin \bar{\theta}_i + \theta_{ie}(t) \cos \bar{\theta}_i] + w_i(l_i, t) \cos \bar{\theta}_i + L(i - 1)$.

On the other hand, the arm-tip rolls on the surface of the object, and there is no slippage. Thus, the distance $X_i(t)$, which is a distance between the contact point of arm i and axis Y_m , can be written as follows: For $i = 1, 2$, $X_i(t) = [x_{0i}(t) - x_m(t)] \cos q(t) + [y_{0i}(t) - y_m(t)] \sin q(t) = c_i - r_i [\alpha_i(t) - \pi/2]$, where c_i is the value of $X_i(t)$ when $\alpha_i(t) = \pi/2$, and $\alpha_i(t)$, which is the angle between the extended line of arm i and its contact point, can be written as $\alpha_i(t) = \pi/2 + (-1)^{i+1} q(t) + (-1)^i \theta_{ie}(t) + (-1)^i w'_i(l_i, t)$. Thus, we can obtain the rolling constraint along the X_m -axis as follows: For $i = 1, 2$,

$$\begin{aligned} R_i &:= \{x_{0i}(t) - [\bar{x}_m + x_{me}(t)]\} \cos[\bar{q} + q_e(t)] \\ &\quad + \{y_{0i}(t) - [\bar{y}_m + y_{me}(t)]\} \sin[\bar{q} + q_e(t)] \\ &\quad + r_i \left[\alpha_i(t) - \frac{\pi}{2} \right] - c_i = 0. \end{aligned} \quad (66)$$

The kinetic energy K and the potential energy P of the system are expressed as follows: $K = \sum_{i=1}^2 (K_i^a + K_i^\theta) + K^m$, and $P = \sum_{i=1}^2 EI_i \int_0^{l_i} [w''_i(x_i, t)]^2 dx_i / 2$, where K_i^a , K_i^θ , and K^m are the kinetic energy of arm i , motor i , and the grasped object, respectively, and they are written as follows: $2K_i^a = \rho_i \int_0^{l_i} [x_i \dot{\theta}_{ie}(t) + \dot{w}_i(x_i, t)]^2 dx_i + m_i [l_i \dot{\theta}_{ie}(t) + \dot{w}_i(l_i, t)]^2 + I_i [\dot{\theta}_{ie}(t) + \dot{w}'_i(l_i, t)]^2$, $2K_i^\theta = J_i \dot{\theta}_{ie}^2(t)$, and $2K^m = M[\dot{x}_{me}^2(t) + \dot{y}_{me}^2(t)] + I \dot{q}_e^2(t)$. Further, the virtual work is given by $\delta W = \sum_{i=1}^2 \tau_i(t) \delta \theta_{ie}(t)$, where δ indicates a variation of the corresponding term.

Let $f_i(t)$ and $\lambda_i(t)$ be the Lagrange multiplier corresponding to (65) and (66), respectively. Then, the Hamilton's principle gives $\int_{t_1}^{t_2} [\delta K + \delta P + \sum_{i=1}^2 (f_i \delta \Phi_i + \lambda_i \delta R_i) + \delta W] dt = 0$, where t_1 and t_2 are time. From this variational principle, the following equations of motion can be obtained by considering the viscous damping in the flexible arms: For $(x_i, t) \in (0, l_i) \times \mathbb{R}^+$ and $i = 1, 2$,

$$\begin{cases} \rho_i [x_i \ddot{\theta}_{ie} + \ddot{w}_i(x_i, t)] + \mu_i \dot{w}_i(x_i, t) + EI_i w_i''''(x_i, t) = 0, \\ w_i(0, t) = w'_i(0, t) = 0, \\ m_i [l_i \ddot{\theta}_{ie} + \ddot{w}_i(l_i, t)] \\ \quad = EI_i w_i'''(l_i, t) + (-1)^i f_i \cos \bar{\theta}_i - \lambda_i \sin \bar{\theta}_i, \\ I_i [\ddot{\theta}_{ie} + \ddot{w}'_i(l_i, t)] = -EI_i w_i''(l_i, t) + (-1)^i \lambda_i r_i, \\ M \begin{bmatrix} \ddot{x}_{me} \\ \ddot{y}_{me} \end{bmatrix} = (f_1 - f_2) \begin{bmatrix} -\sin \bar{q} \\ \cos \bar{q} \end{bmatrix} - (\lambda_1 + \lambda_2) \begin{bmatrix} \cos \bar{q} \\ \sin \bar{q} \end{bmatrix}, \\ I \ddot{q}_e(t) = \sum_{i=1}^2 \left\{ (-1)^{i+1} f_i [l_i \cos \bar{\theta}_i - \bar{x}_m \cos \bar{q} + g_i \sin \bar{q}] \right. \\ \quad \left. + \lambda_i [l_i \sin \bar{\theta}_i + \bar{x}_m \sin \bar{q} + g_i \cos \bar{q} + (-1)^{i+1} r_i] \right\}, \\ \bar{\theta}_i = \bar{\theta}_i - \bar{q}, \quad g_i = L(i - 1) - \bar{y}_m, \\ J_i \ddot{\theta}_{ie}(t) = \tau_i(t) + EI_i w_i''(0, t), \end{cases} \quad (67)$$

with constraints (65) and (66).

Next, to eliminate $f_i(t)$ and $\lambda_i(t)$ in (67), we consider the parts of the ODEs in (67) and constraints (65) and (66). The first-time derivatives of the constraints ($\dot{\Phi}_i = 0$ and $\dot{R}_i = 0$) are combined as

the following matrix relation:

$$\mathbf{A}_1 \dot{\mathbf{z}}_1 + \mathbf{A}_2 \dot{\mathbf{z}}_2 = \mathbf{0}, \quad (68)$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined in (3), and

$$\begin{cases} \mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T]^T, & \mathbf{z}_1 = [x_{me}(t), y_{me}(t), q_e(t)]^T, \\ \mathbf{z}_2 = [l_1 \theta_{1e}(t) + w_1(l_1, t), l_2 \theta_{2e}(t) + w_2(l_2, t), \\ \theta_{1e}(t) + w'_1(l_1, t), \theta_{2e}(t) + w'_2(l_2, t)]^T. \end{cases} \quad (69)$$

In addition, the ODEs from the third to the sixth equation in (67) are combined as the following matrix relation:

$$\hat{\mathbf{M}}_o \ddot{\mathbf{z}} = \hat{\mathbf{F}}_o + \mathbf{A}^T [f_1(t), f_2(t), \lambda_1(t), \lambda_2(t)]^T, \quad (70)$$

where $\mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2]$, $\hat{\mathbf{M}}_o$ is defined by (3), and $\hat{\mathbf{F}}_o = [0, 0, 0, EI_1 w_1'''(l_1, t), EI_2 w_2'''(l_2, t), -EI_1 w_1''(l_1, t), -EI_2 w_2''(l_2, t)]^T$.

Now, multiplying both sides of (70) by $\mathbf{L}^T$ defined in (3) from the left leads to $\mathbf{M}_o \ddot{\mathbf{z}}_1 = \mathbf{F}_o$, where $\mathbf{F}_o = \mathbf{L}^T \hat{\mathbf{F}}_o$. Here, note that it is easy to show that $\mathbf{M}_o$ is a positive definite, and $\mathbf{M}_o^{-1}$ exists. Thus, introducing the fact $\dot{\mathbf{z}}_2 = \mathbf{B}_o \dot{\mathbf{z}}_1$ leads to $\ddot{\mathbf{z}}_2 = \mathbf{B}_o \mathbf{M}_o^{-1} \mathbf{B}_o^T [EI_1 w_1'''(l_1, t), EI_2 w_2'''(l_2, t), -EI_1 w_1''(l_1, t), -EI_2 w_2''(l_2, t)]^T$.

Finally, introducing variable (2) and noting the fact, that the relation $[y_1(l_1, t), y_2(l_2, t), y'_1(l_1, t), y'_2(l_2, t)]^T = \mathbf{B}_o [x_{me}(t) - x_{me}^d, y_{me}(t) - y_{me}^d, q_e(t) - q_e^d]^T$ is obtained from (65) and (66), gives the equations of motion (1).

9 Appendix B: Equilibrium States

We present the equilibrium states derived in [27] for the understanding of reader. In the equilibrium states, the flexible arms grasp the object, and the arms and the object are in the rest state. At the equilibrium satisfying $(\dot{x}_{me}(t) = \dot{y}_{me}(t) = \dot{q}_e(t) = \dot{w}_i(x_i, t) = \dot{\theta}_{ie}(t) = \dot{f}_i(t) = \dot{\lambda}_i(t) = 0)$, $w_i(x_i, t)$ becomes a function of x_i , and the others become constants as follows: $w_i^s(x_i)$, x_{me}^s , y_{me}^s , q_e^s , θ_{ie}^s , f_i^s , and λ_i^s . Substituting these into (67) with (65) and (66) leads to the following relations:

$$f^s := f_1^s = f_2^s, \quad \lambda^s := \lambda_1^s = -\lambda_2^s, \quad (71)$$

$$\lambda^s = \frac{-l_1 \cos \bar{\Theta}_1 + l_2 \cos \bar{\Theta}_2 + L \sin \bar{q}}{l_1 \sin \bar{\Theta}_1 - l_2 \sin \bar{\Theta}_2 - L \cos \bar{q} + r_1 + r_2} f^s, \quad (72)$$

$$\begin{aligned} w_i^s(x_i) = & -\frac{\lambda^s r_i x_i^2}{2EI_i} \\ & + \frac{(-1)^{i+1}}{6EI_i} x_i^2 (x_i - 3l_i) (f^s \cos \bar{\Theta}_i + \lambda^s \sin \bar{\Theta}_i), \end{aligned} \quad (73)$$

$$\mathbf{M}_s [x_{me}^s, y_{me}^s, \theta_{1e}^s, \theta_{2e}^s]^T = \mathbf{F}_s. \quad (74)$$

Here, $w_i^s(x_i)$ in (73) is a static transverse displacement of arm i at the equilibrium. Further, (74) is derived from (65) and (66), $\mathbf{M}_s \in \mathbb{R}^{4 \times 4}$ is a matrix, and $\mathbf{F}_s \in \mathbb{R}^{4 \times 1}$ is a vector. Here, $\mathbf{F}_s$ contains q_e^s , initial parameters, and physical parameters, and thus we can derive each element of $\mathbf{F}_s$. Further, a simple calculation shows $\det \mathbf{M}_s = -l_1 r_2 \cos \bar{\Theta}_1 - l_2 r_1 \cos \bar{\Theta}_2 - l_1 l_2 \sin(\bar{\Theta}_1 - \bar{\Theta}_2) \neq 0$, and thus x_{me}^s , y_{me}^s , and θ_{ie}^s can be uniquely determined if q_e^s is given. Therefore, for any given $f^s = f^d$ and $q_e^s = q_e^d$, the values of λ^s , $w_i^s(x_i)$, x_{me}^s , y_{me}^s , and θ_{ie}^s are uniquely determined. Here, we set them as λ^d , $w_i^d(x_i)$, x_{me}^d , y_{me}^d , and θ_{ie}^d , and note that f^d is a desired grasping force and q_e^d is a desired orientation of the grasped object.