

On dense subsets of the boundary of a Coxeter system

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The purpose of this note is to introduce main results of my recent paper [10] about dense subsets of the boundary of a Coxeter system.

A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \rangle,$$

where S is a finite set and $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

- (1) $m(s, t) = m(t, s)$ for each $s, t \in S$,
- (2) $m(s, s) = 1$ for each $s \in S$, and
- (3) $m(s, t) \geq 2$ for each $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T , and called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group. A subset $T \subset S$ is called a *spherical subset* of S , if the parabolic subgroup W_T is finite. For each $w \in W$, we define $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in S which represents w . For a subset $T \subset S$, we also define $W^T = \{w \in W \mid S(w) = T\}$.

Let (W, S) be a Coxeter system and let $\mathcal{S}^f$ be the family of spherical subsets of S . We denote $W\mathcal{S}^f$ as the set of all cosets of the form wW_T , with $w \in W$ and $T \in \mathcal{S}^f$. The sets $\mathcal{S}^f$ and $W\mathcal{S}^f$ are partially ordered by inclusion. Contractible simplicial complexes $K(W, S)$ and $\Sigma(W, S)$ are

defined as the geometric realizations of the partially ordered sets $\mathcal{S}^f$ and $W\mathcal{S}^f$, respectively ([7, §3], [5]). The natural embedding $\mathcal{S}^f \rightarrow W\mathcal{S}^f$ defined by $T \mapsto W_T$ induces an embedding $K(W, S) \rightarrow \Sigma(W, S)$ which we regard as an inclusion. The group W acts on $\Sigma(W, S)$ via simplicial automorphism. Then $\Sigma(W, S) = WK(W, S)$ ([5], [7]). For each $w \in W$, $wK(W, S)$ is called a *chamber* of $\Sigma(W, S)$. If W is infinite, then $\Sigma(W, S)$ is noncompact. In [12], G. Moussong proved that a natural metric on $\Sigma(W, S)$ satisfies the CAT(0) condition. Hence, if W is infinite, $\Sigma(W, S)$ can be compactified by adding its ideal boundary $\partial\Sigma(W, S)$ ([6, §4], [8]). This boundary $\partial\Sigma(W, S)$ is called the *boundary of (W, S)* . We note that the natural action of W on $\Sigma(W, S)$ is properly discontinuous and cocompact ([5], [6]), and this action induces an action of W on $\partial\Sigma(W, S)$.

A subset A of a space X is said to be *dense* in X , if $\overline{A} = X$. A subset A of a metric space X is said to be *quasi-dense*, if there exists $N > 0$ such that each point of X is N -close to some point of A .

Let (W, S) be a Coxeter system. Then W has the *word metric* d_ℓ defined by $d_\ell(w, w') = \ell(w^{-1}w')$ for each $w, w' \in W$.

Here we obtained the following theorems in [10].

Theorem 1. *Let (W, S) be a Coxeter system. Suppose that $W^{\{s_0\}}$ is quasi-dense in W with respect to the word metric and $m(s_0, t_0) = \infty$ for some $s_0, t_0 \in S$. Then there exists $\alpha \in \partial\Sigma(W, S)$ such that the orbit $W\alpha$ is dense in $\partial\Sigma(W, S)$.*

Suppose that a group Γ acts properly and cocompactly by isometries on a CAT(0) space X . Every element $\gamma \in \Gamma$ such that the order $o(\gamma) = \infty$ is a hyperbolic transformation of X , i.e., there exists a geodesic axis $c : \mathbb{R} \rightarrow X$ and a real number $a > 0$ such that $\gamma \cdot c(t) = c(t + a)$ for each $t \in \mathbb{R}$ ([3]). Then, for all $x \in X$, the sequence $\{\gamma^i x\}$ converges to $c(\infty)$ in $X \cup \partial X$. We denote $\gamma^\infty = c(\infty)$.

Theorem 2. *Let (W, S) be a Coxeter system. If the set*

$$\bigcup \{W^{\{s\}} \mid s \in S \text{ such that } m(s, t) = \infty \text{ for some } t \in S\}$$

is quasi-dense in W , then $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial\Sigma(W, S)$.

Remark. For a negatively curved group G and the boundary ∂G of G ,

- (1) we can show that $G\alpha$ is dense in ∂G for each $\alpha \in \partial G$ by an easy argument, and
- (2) it is known that $\{g^\infty \mid g \in G \text{ such that } o(g) = \infty\}$ is dense in ∂G ([2]).

Example. Let $S = \{s, t, u\}$ and let

$$W = \langle S \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$

Then (W, S) is a Coxeter system and $W^{\{s\}}$ is quasi-dense in W . On the other hand, for any $\alpha \in \partial\Sigma(W, S)$, $W\alpha$ is a finite-points set and not dense in $\partial\Sigma(W, S)$ which is a circle. Thus we can not omit the assumption " $m(s_0, t_0) = \infty$ " in Theorem 1.

We showed the following lemma in [10].

Lemma 3. *Let (W, S) be a Coxeter system. Suppose that there exist a maximal spherical subset T of S and $s_0 \in S$ such that $m(s_0, t) \geq 3$ for each $t \in T$ and $m(s_0, t_0) = \infty$ for some $t_0 \in T$. Then $W^{\{s_0\}}$ is quasi-dense in W .*

As an application of Theorems 1 and 2, we can obtain the following corollary from Lemma 3.

Corollary 4. *Let (W, S) be a Coxeter system. Suppose that there exist a maximal spherical subset T of S and an element $s_0 \in S$ such that $m(s_0, t) \geq 3$ for each $t \in T$ and $m(s_0, t_0) = \infty$ for some $t_0 \in T$. Then*

- (1) $W\alpha$ is dense in $\partial\Sigma(W, S)$ for some $\alpha \in \partial\Sigma(W, S)$, and
- (2) $\{w^\infty \mid w \in W \text{ such that } o(w) = \infty\}$ is dense in $\partial\Sigma(W, S)$.

Example. The Coxeter system defined by the diagram in Figure 1 is not hyperbolic in Gromov sense, since it contains a copy of $\mathbb{Z}^2$, and it satisfies the condition of Corollary 4.

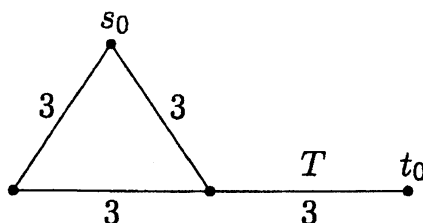


FIGURE 1

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