

Some remarks on locally o-minimal structures

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概要

abstract Locally o-minimal structures are some local adaptation from o-minimal structures. They were treated, e.g. in [1], [2]. We try to characterize types of definably complete locally o-minimal structures. And we argue about the dp-rank of them.

1. Introduction

We recall some definitions and fundamental facts at first.

Definition 1. Let M be a densely linearly ordered structure without endpoints.

M is *o-minimal* if every definable subset of M^1 is a finite union of points and intervals.

M is *locally o-minimal* if for any element $a \in M$ and any definable subset $X \subset M^1$, there is an open interval $I \subset M$ such that $I \ni a$ and $I \cap X$ is a finite union of points and intervals.

M is *definably complete* if any definable subset X of M^1 has the supremum and infimum in $M \cup \{\pm\infty\}$.

Here we consider densely linearly ordered structures only.

Example 2. [1], [2]

$(\mathbf{R}, +, <, \mathbf{Z})$ where $\mathbf{Z}$ is the interpretation of a unary predicate, and $(\mathbf{R}, +, <, \sin)$ are definably complete locally o-minimal structures.

Fact 3. [1] *Definably complete local o-minimality is preserved under elementary equivalence.*

Thus we argue in a sufficiently large saturated model $\mathcal{M}$ as usual.

O-minimal structures are characterized by means of behavior of 1-types. They consider two kinds of 1-types by the way to cut linear orders of parameter sets, e.g. in [5]. Here we consider nonisolated types only.

Definition 4. Let M be a densely linearly ordered structure and $A \subset M$.

And let $p(x) \in S_1^{or}(A)$, that is, $p(x)$ is complete over A w.r.t. the order relation.

We say that $p(x)$ is *cut over* A if for any $a \in A$, if $a < x \in p(x)$, then there is $b \in A$ such that $a < b < x \in p(x)$, and similarly if $x < a \in p(x)$, then there is $c \in A$ such that $x < c < a \in p(x)$.

We say that $q(x) \in S_1^{or}(A)$ is *noncut over* A if $q(x)$ is not a cut type.

And sometimes we call $q(x) \in S_1(A)$ *cut (noncut)* over A if $q(x)$ contains a cut (noncut) $p(x) \in S_1^{or}(A)$.

Remark 5. Let M be a densely linearly ordered structure and $A \subset M$. And let $p(x) \in S_1^{or}(A)$ be noncut.

There are four kinds of noncut types.

$p(x) = \{b < x < a : b < a \in A\}$ for some fixed a , or $\{a < x < b : a < b \in A\}$ for some fixed a .

Here we call these types *bounded noncut types*.

And $p(x) = \{b < x : b \in A\}$ or $\{x < b : b \in A\}$.

We call these types *unbounded noncut types*.

2. Characterization of definably complete locally o-minimal structures

In o-minimal structures, types of the order relation are complete. Similar argument hold in definably complete locally o-minimal structures to some extent.

Lemma 6. Let M be a definably complete locally o-minimal structure and $A \subset M$.

Then any bounded noncut type $p(x) \in S_1^{or}(dcl(A))$ is complete over $dcl(A)$.

Proof ;

Let $p(x) \in S_1^{or}(dcl(A))$ be bounded noncut, that is, $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some fixed $a \in dcl(A)$ (Another case is proved similarly). By local o-minimality, for any formula $\varphi(x, \bar{b})$ over $dcl(A)$, there is an interval $I \subset M$ such that $a \in I$ and $I \cap \varphi(M, \bar{b})$ is a union of finite points and intervals. Thus there is $c \in I$ such that either for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \bar{b})$, or for any $d \in I$ with $c < d < a$, $M \models \neg \varphi(d, \bar{b})$. Now we assume that for any $d \in I$ with $c < d < a$, $M \models \varphi(d, \bar{b})$. We consider the formula $\forall y (x < y < a \rightarrow \varphi(y, \bar{b}))$. By definably completeness, there is the infimum $e \in M$ such that for any f with $e < f < a$, $M \models \varphi(f, \bar{b})$ (If $e = -\infty$, then " $x < a$ " implies $\varphi(x, \bar{b})$). And $e \in dcl(A)$. ■

Notation 7. In the lemma above, for $p(x) = \{c < x < a : c < a \in dcl(A)\}$ for some

fixed $a \in \text{dcl}(A)$, there is the infimum $e \in M$ such that for any f with $e < f < a$ satisfies the formula $\varphi(y, \bar{b})$. We denote " b_φ " such boundary point in the following.

Next we characterize about the definability of types. We recall the definition.

Definition 8. Let M be a structure.

A type $p(\bar{x}) \in S_n(M)$ is *definable* if for any L -formula $\varphi(\bar{x}, \bar{y})$, there is an $L(M)$ -formula $d\varphi(\bar{y})$ such that for all $\bar{a} \in M$, $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$ if and only if $M \models d\varphi(\bar{a})$.

We can prove the next fact.

Fact 9. Let M be a definably complete locally o-minimal structure and let $p(x) \in S_1(M)$. Then $p(x)$ is definable if and only if $p(x)$ is noncut.

We can generalize the fact above for n -types to a certain extent.

Definition 10. [6] Let $\mathcal{M}$ be a sufficiently large saturated densely linearly ordered structure and $A \subset \mathcal{M}$.

We say that $p(\bar{x}) \in S_n(A)$ is *noncut over A* if ;

- 1) $n = 1$, we define it by the same way as above, and
- 2) $n \geq 2$, let $\bar{x} = (x_1, x_2, \dots, x_n)$, we define inductively,

$q(\bar{x}') := tp(x_1 \cdots x_{n-1}/A)$ is noncut over A and $tp(x_n/Aa_1 \cdots a_{n-1})$ is noncut over $A \cup \{a_1, \dots, a_{n-1}\}$ for any realizations $a_1 \cdots a_{n-1}$ of $q(\bar{x}')$.

Fact 11. Let M be a locally o-minimal structure and let $p(\bar{x}) \in S_n(M)$ be noncut. Then $p(\bar{x})$ is definable.

Next we characterize definably complete locally o-minimal structures by the notion of forking. We recall some definitions.

Definition 12. A formula $\varphi(\bar{x}, \bar{a})$ *divides* over a set A if there is a sequence $\{\bar{a}_i : i \in \omega\}$ with $tp(\bar{a}_i/A) = tp(\bar{a}/A)$ such that $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is k -inconsistent for some $k \in \omega$.

A formula $\phi(\bar{x}, \bar{a})$ *forks* over A if $\phi(\bar{x}, \bar{a}) \vdash \bigvee_{i < n} \psi_i(\bar{x}, \bar{b}_i)$ and each $i < n$, $\psi_i(\bar{x}, \bar{b}_i)$ divides over A .

In some papers, they consider the notion of dimension for (definably complete) locally o-minimal structures, e.g. in [3].

Definition 13. Let M be a densely linearly ordered structure without endpoints and let $X \subset M^n$ be a nonempty definable subset.

The dimension $\dim(X)$ of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coordinate projection $\pi : M^n \rightarrow M^d$.

According to the argument in [7], we recall some lemmas.

Lemma 14. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $p(x), q(x) \in S_1(A)$ (with $\dim(p) = \dim(q) = 1$).*

Then either

- (a) (i) *all A -definable $f : p(\mathcal{M}) \rightarrow q(\mathcal{M})$ are increasing, or*
- (ii) *all A -definable $f : p(\mathcal{M}) \rightarrow q(\mathcal{M})$ are decreasing.*
- (b) *In case (i), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a > dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$,*

In case (ii), whenever $B \supset A$, $a \in p(\mathcal{M})$ and $a < dcl(B) \cap p(\mathcal{M})$, then $dcl(aA) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

In the lemma above, we just say that if there is a function f between $p(\mathcal{M})$ and $q(\mathcal{M})$, then f has these properties. There is no definable function between a cut type and a noncut type. By this lemma, they consider characteristic extensions of complete types in o-minimal structures. Here we adapt the argument for noncut types.

Definition 15. Let $p(x_1, \dots, x_n) \in S_n(A)$ of dimension n and $A \subset B$.

Fix some sequence $\eta = (\eta(1), \dots, \eta(n))$ where each $\eta(i)$ is 1 or 0.

For $1 \leq i \leq n$, let $p_i(x_1, \dots, x_i)$ be the restriction of p to the variables $x_1, \dots, x_i$.

We define an extension $p_B^\eta \in S_n(B)$ of p . Choose a realization $(b_1, \dots, b_n)$ of p_B^η inductively as follows ;

$b_1 \in p_1(\mathcal{M})$ and if $\eta(1) = 1$, then $b_1 > dcl(B) \cap p_1(\mathcal{M})$,
while if $\eta(1) = 0$, then $b_1 < dcl(B) \cap p_1(\mathcal{M})$.

For some realization $b_1, \dots, b_i$ of $p_i(x_1, \dots, x_i)$, let b_{i+1} be a realization of $p_{i+1}(b_1, \dots, b_i, x_{i+1})$ such that :

- if $\eta(i+1) = 1$, then $b_{i+1} > dcl(B, b_1, \dots, b_i) \cap p_{i+1}(b_1, \dots, b_i, \mathcal{M})$ and
- if $\eta(i+1) = 0$, then $b_{i+1} < dcl(B, b_1, \dots, b_i) \cap p_{i+1}(b_1, \dots, b_i, \mathcal{M})$.

Lemma 16. [7] *Let $p(\bar{x}) \in S_n(A)$ of dimension n and let $q(y) \in S_1(A)$ of dimension 1.*

Then there is $\eta \in {}^n 2$ as in the definition above such that ;

for any $B \supset A$ and any realization $\bar{a}$ of $p_B^\eta(\bar{x})$, $dcl(\bar{a}A) \cap q(\mathcal{M}) > dcl(B) \cap q(\mathcal{M})$.

By the lemmas above, we can prove the next fact.

Proposition 17. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure. And let $c \in \mathcal{M}$ and $A \subset \mathcal{M}$ with $\dim(\text{tp}(c/A)) = 1$, and $A \cup \{c\} \subset B$.*

Moreover let $r(x) \in S_1(B)$ be a bounded noncut type of c over $dcl(B)$ satisfying $r \restriction Ac \not\vdash r$.

Then $r(x)$ divides over A .

Sketch of proof ;

Let $r(x) := \{c < x < d : d \in \text{dcl}(B)\}$ (Another case is proved similarly). As $r \upharpoonright Ac \not\vdash r$, there is an $L(B)$ -formula $\varphi(x, \bar{b}) \in r(x)$ such that for any $L(Ac)$ -formula $\psi(x) \in r(x)$, $b_\varphi < b_\psi$.

Case 1. $c \equiv_A b_\varphi$.

We consider an automorphism $\sigma \in \text{Aut}_A(\mathcal{M})$ such that $\sigma(c) = b_\varphi$. And let $b_\varphi' = \sigma(b_\varphi)$. Thus $c < b_\varphi < b_\varphi'$.

Case 2. $c \not\equiv_A b_\varphi$.

Now there is an $L(A)$ -formula $\psi(x)$ such that $\neg\psi(x) \in \text{tp}(c/A)$ and $\psi(x) \in \text{tp}(b_\varphi/A)$. As for any $L(A)$ -formula $\psi(x) \in r(x)$, $b_\varphi < b_\psi$, so for any d with $c < d < b_\varphi$, $\models \psi(d)$.

In the lemmas above, let $p(\bar{x}) = \text{tp}(cb_\varphi/A)$, $q(x) = r(x) \upharpoonright A$, and the noncut extension $p'(\bar{x}) \in S^\eta(\bar{b}cA)$ of p such that for any $c'b_\varphi' \models p'(\bar{x})$, $\text{dcl}(c'b_\varphi'A) \cap q(\mathcal{M}) > \text{dcl}(\bar{b}cA) \cap q(\mathcal{M})$. If $c < c' \leq b_\varphi$, then $\models \psi(c')$, a contradiction. Thus $c < b_\varphi < c' < b_\varphi'$.

We iterate this construction infinitely many times and prove that the formula $c < x < b_\varphi$ divides over A . ■

Corollary 18. Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure. And let $A \subset B \subset C \subset \mathcal{M}$ and $p(x) \in S_1(C)$ be a cut type over C .

Suppose that there is $d \in B$ with $\dim(\text{tp}(d/A)) = 1$ and a bounded noncut type $q(x) \in S_1(\text{dcl}(B))$ of d such that $q \upharpoonright Ad \not\vdash q$ and $p \vdash q$.

Then $p(x)$ divides over A .

3. Dp-rank of locally o-minimal structures

We recall some definitions.

Definition 19. An independent partition pattern of a partial type $p(\bar{x})$ is a sequence of formulas $(\varphi^\alpha(\bar{x}, \bar{y}^\alpha))_{\alpha < \kappa}$ and tuples $\bar{b}_i^\alpha$ for $\alpha < \kappa$ and $i < \omega$ satisfying that ;

for any $\alpha < \kappa$, $\{\varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid i < \omega\}$ is k^α -inconsistent for some $k^\alpha < \omega$, and for any $\eta \in \omega^\kappa$, $\{\varphi^\alpha(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \kappa\}$ is consistent with $p(\bar{x})$.

For a theory T , the invariant $\kappa_{inp}^n(T)$ is the smallest infinite cardinal κ such that no n -type has an inp -pattern of cardinality κ .

A formula $\varphi(\bar{x}, \bar{y})$ has the *independence property* if there are sequences $(\bar{a}_i : i < \omega)$ and $(\bar{b}_I : I \subset \omega)$ such that $\models \varphi(\bar{a}_i, \bar{b}_I)$ if and only if $i \in I$.

A formula $\varphi(\bar{x}, \bar{y})$ has the *tree property of the second kind* (TP_2) if there are tuples

$(\bar{b}_i^\alpha)_{\alpha, i < \omega}$ such that $\{\varphi(\bar{x}, \bar{b}_i^\alpha) \mid i < \omega\}$ is 2-inconsistent for any $\alpha < \omega$, and for any $\eta \in \omega^\omega$, $\{\varphi(\bar{x}, \bar{b}_{\eta(\alpha)}^\alpha) \mid \alpha < \omega\}$ is consistent.

We call the *burden* of $p(\bar{x})$ the supremum of the cardinalities κ of all *inp*-patterns for p . It is known that the burden of p is equal to the (classical) weight of p in simple theories.

Theorem 20. *e.g. [15]*

For any theory T ,

any formula $\varphi(\bar{x}, \bar{y})$ with $|\bar{x}| = n$ is NTP_2 if and only if $\kappa_{inp}^n(T) \leq |T|^+ (< \infty)$.

Definition 21. An *independent contradictory types pattern* is a sequence of formulas $(\varphi^\alpha(\bar{x}, \bar{y}^\alpha))_{\alpha < \kappa}$ and tuples $\bar{b}_i^\alpha$ for $\alpha < \kappa$ and $i < \omega$ satisfying that ;

for any $\eta \in \omega^\kappa$, the following set of formulas is consistent,

$$\Gamma_\eta(\bar{x}) := \{\varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid \alpha < \kappa, i < \omega, \eta(\alpha) = i\} \cup \{\neg\varphi^\alpha(\bar{x}, \bar{b}_i^\alpha) \mid \alpha < \kappa, i < \omega, \eta(\alpha) \neq i\}.$$

Theorem 22. *e.g. [11]*

If T is NIP, then $\kappa_{ict}(T) = \kappa_{inp}(T)$. Otherwise, $\kappa_{ict}(T) = \infty$.

And we recall the definition of dp-rank from [16].

Definition 23. Let $p(\bar{x})$ be a partial type over a set $A \subset \mathcal{M}$. We define the *dp-rank* of $p(\bar{x})$ as follows.

The dp-rank of $p(\bar{x})$ is always greater than or equal to 0. Let μ be a cardinal.

We say that $p(\bar{x})$ has dp-rank $\leq \mu$ if given any realization a of p and any $1 + \mu$ mutually A -indiscernible sequences, at least one of them is indiscernible over Aa .

And we say that p has dp-rank $= \mu$ if it has dp-rank $\leq \mu$, but it is not the case that it has dp-rank $\leq \lambda$ for any $\lambda < \mu$.

We call p *dp-minimal* if it has dp-rank 1, we denote $\text{rk-dp}(p) = 1$.

We call p *dependent* if it has an ordinal dp-rank, that is, $\text{rk-dp}(p) < \infty$.

And we call p *strongly dependent* if $\text{rk-dp}(p) \leq \omega$.

There exists many examples whose theories are dp-minimal. For example, structures of superstable with U-rank = 1, C-minimal, p-adics, ordered set with finite width, tree, and so on. Here we recall the next fact. It is proved by the argument about the notion of Vapnik-Chervonenkis density (or VC-minimality).

Theorem 24. [14], [17]

Weakly o-minimal theories are dp-minimal.

But there are examples of locally o-minimal structures whose theories have the independence property.

Many results are proved under the strong assumption that structures $M \prec \mathcal{M}$ are dp-minimal.

Proposition 25. [13]

Let M be an inp-minimal linearly ordered structure without endpoints and $A \subset M$. And let M be $|A|^+$ -saturated and $p(x) \in S_1(M)$.

Then the following are equivalent ;

1. $p(x)$ divides over A .
2. There exist $a, b \in M$ such that $p \vdash a < x < b$ and $a \equiv_A b$.

We can prove the next fact under the same assumption.

Fact 26. *Let $\mathcal{M}$ be a sufficiently large saturated definably complete locally o-minimal structure and $A \subset \mathcal{M}$. And let $Th(\mathcal{M})$ be inp-minimal.*

Moreover let $p(x) \in S_1(dcl(A))$ be an unbounded noncut type.

Then $p(x)$ does not fork over $\emptyset$.

4. Further problems

For definably complete locally o-minimal structures, we can prove the next fact easily.

Lemma 27. *Let M be a definably complete locally o-minimal structure and $A \subset M$ with $dcl(A) \neq \emptyset$.*

Then the isolated 1-types of $Th(M, a)_{a \in A}$ are dense.

Thus I will try to characterize definably complete locally o-minimal structures by means of prime models.

And I will try to characterize locally o-minimal structures satisfying some additional conditions of their theories. The additional conditions are ; definably complete, dp-minimal, strongly dependent, *NIP* or *NTP₂*.

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