

Compact energy surface of a Hamiltonian system

by Kiyoshi Hayashi

(Keio University)

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be points of $\mathbb{R}^n$ and

$$H = H(x, y) : \mathbb{R}^{2n} \rightarrow \mathbb{R}$$

a smooth function.

We consider a Hamiltonian system

$$(H) \quad \dot{x}_k = H_{y_k}, \quad \dot{y}_k = -H_{x_k}, \quad k = 1, \dots, n.$$

Along a solution $(x(t), y(t))$ of (H), $H(x(t), y(t))$ is a constant, so, for fixed $e \in \mathbb{R}$, the set

$$H^{-1}(e) = \{ (x, y) ; H(x, y) = e \}$$

is an invariant set of the system (H), called an energy surface.

We assume that

(A) e is a regular value of H , that is, there are no critical points of H on $H^{-1}(e)$.

Then $H^{-1}(e)$ is a smooth submanifold of $\mathbb{R}^{2n}$.

If $H^{-1}(e)$ is not compact, there is not necessarily periodic orbit on it (for example $H = \frac{1}{2} |y|^2 + x_n$).

Rabinowitz [1] proved that, if $H^{-1}(e)$ is star-shaped, then there exists at least one periodic orbit on it.

Whether " star-shaped " can be replaced by " homeomorphic to the

sphere " (or more optimistically " compact ") is not known.

Classically, H is the sum of the kinetic energy T and the potential U , that is,

$$(1) \quad H = \sum_{i,j=1}^n a^{ij}(x) y_i y_j + U(x) ,$$

where (a^{ij}) is symmetric and positive definite.

We have

Theorem. Assume that $H = H(x, y)$ is given by (1). If for some $e \in \mathbb{R}$, H satisfies (A) and $H^{-1}(e)$ is compact, then there exists at least one periodic orbit on $H^{-1}(e)$.

In this case, (H) is equivalent to the Lagrangian system

$$(L) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_k} = \frac{\partial}{\partial x_k} (T - U) ,$$

where $T = \sum a_{ij}(x) \dot{x}_i \dot{x}_j$, $a_{ij} = (a^{ij})^{-1}$.

We consider solutions $x = x(t)$ of (L) with $T(x, \dot{x}) + U(x) = e$. Since $T \geq 0$, the solution $x(t)$ lies in

$$M = \{ x \in \mathbb{R}^n ; U(x) \leq e \} .$$

M is a compact manifold with boundary $\partial M = \{U = e\}$. In the case $M \approx D^n$, the theorem is proved by H. Seifert [2].

We prove this theorem by the principle of least action of Maupertuis - Jacobi.

We consider a Riemannian metric

$$(2) \quad ds^2 = (e - U) a_{ij} dx_i dx_j ,$$

called Jacobi-metric for e . This is positive on $M - \partial M$ and degenerates on ∂M .

A smooth curve

$$\gamma = \gamma(s) : [0, 1] \rightarrow M$$

with $\gamma(0), \gamma(1) \in \partial M$ and $\gamma(s)$, $0 < s < 1$, being a geodesic by the metric (2) on $M - \partial M$, gives a desired periodic solution of (L) after proper time change (see [2]).

As usual [3], we seek such a geodesic as a critical point of the functional

$$(3) \quad E(\lambda) = \int_0^1 (e^{-U(\lambda(t))}) T(\lambda(t), \dot{\lambda}(t)) dt.$$

As in [2], for small $\delta > 0$, we define a set $M_\delta \subset M$ as follows.

For $b \in \partial M$, let $x_b(t)$ be the solution of (L) with $x_b(0) = b$ and $\dot{x}_b(0) = 0$.

We put $F_b = \{ x_b(t_1) \in M : t_1 \geq 0 \text{ and the length of the curve } x_b(t), 0 \leq t \leq t_1, \text{ by the metric (2) is less than } \delta \}$ and define

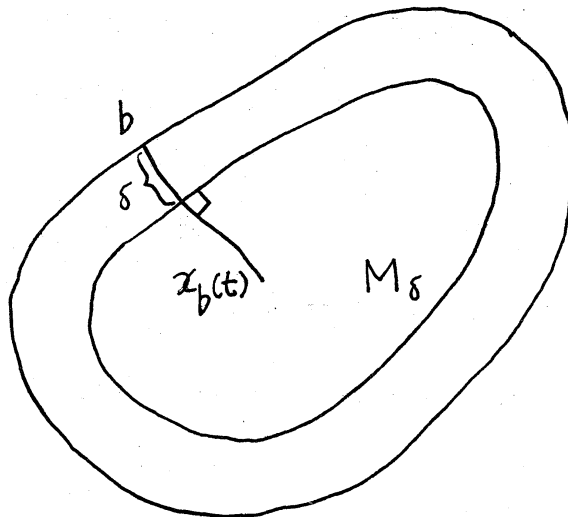
$$M_\delta = M - \bigcup_{b \in \partial M} F_b$$

Put $B_\delta = \partial M_\delta$. For small $\delta > 0$, $M_\delta \approx M$ and

$\dot{x}_b(t) \perp T_{x_b(t)} B_\delta$
if $x_b(t) \in B_\delta$. (t : small)

So, a geodesic $\gamma = \gamma(s) : [0, 1] \rightarrow M_\delta$ with $\gamma(i) \in B_\delta$ and $\dot{\gamma}(i) \perp T_{\gamma(i)} B_\delta$ ($i = 0, 1$)

gives a desired solution.



In general, let $\Omega(X; A, B)$ be the set of continuous curves $\omega = \omega(t) : [0, 1] \rightarrow X$ with $\omega(0) \in A$ and $\omega(1) \in B$, endowed with the compact open topology.

Consider a compact connected smooth manifold M with boundary $\partial M = B$, and put $Y = \Omega(M; B, B)$. We identify $b \in B$ with the constant curve $\tilde{b}$ whose image is b , so $B \subset Y$.

Then we have

Lemma 1. $H_0(Y, B) \neq 0$ or $\pi_k(Y, B) \neq 0$ for some $k \geq 1$.

(proof) It is easily proved that, if B is not arcwise connected then $H_0(Y, B) \neq 0$; moreover, if B is arcwise connected and Y is not arcwise connected, then $H_0(Y, B) \neq 0$.

So we assume that B and Y are arcwise connected and $\pi_k(Y, B) = 0$ for all $k \geq 1$.

We put

$$Y_0 = \Omega(B; B, B), \quad B \subset Y_0 \stackrel{j}{\hookrightarrow} Y.$$

Since $B \simeq Y_0$, we have

$$\pi_k(Y, B) \cong \pi_k(Y, Y_0) = 0 \quad \text{for } k \geq 1.$$

Let $\pi : Y \rightarrow B \times B$ be the fibration.

$$\omega \mapsto (\omega(0), \omega(1))$$

We put $F = \pi^{-1}(*) = \Omega M$; the loop space, $\pi_0 = \pi|_{Y_0} : Y_0 \rightarrow B \times B$ and $F_0 = \pi_0^{-1}(*) = \Omega B$.

Then we have a commutative diagram of fibrations

$$\begin{array}{ccccc}
\Omega B & \longrightarrow & Y_0 & \xrightarrow{\pi_0} & B \times B \\
\cap \Omega i & & \cap j & & \parallel \\
\Omega M & \longrightarrow & Y & \xrightarrow{\pi} & B \times B .
\end{array}$$

This derives the following commutative diagram of long exact sequence of homotopy groups of fibrations

$$\begin{array}{ccccccccc}
\pi_k(Y_0) & \longrightarrow & \pi_k(B \times B) & \longrightarrow & \pi_{k-1}(\Omega B) & \longrightarrow & \pi_{k-1}(Y_0) & \longrightarrow & \pi_{k-1}(B \times B) \\
\downarrow j_* & & \parallel & & \downarrow (\Omega i)_* & & \downarrow j_* & & \parallel \\
\pi_k(Y) & \longrightarrow & \pi_k(B \times B) & \longrightarrow & \pi_{k-1}(\Omega M) & \longrightarrow & \pi_{k-1}(Y) & \longrightarrow & \pi_{k-1}(B \times B) .
\end{array}$$

Since $\pi_k(Y, Y_0) = 0$, we have $j_* : \pi_k(Y_0) \cong \pi_k(Y)$. Hence by the 5 lemma, we have

$$\begin{array}{ccc}
(\Omega i)_* : \pi_{k-1}(\Omega B) & \cong & \pi_{k-1}(\Omega M) \\
\parallel & \circlearrowleft & \parallel \\
\pi_k(B) & \xrightarrow{i_*} & \pi_k(M)
\end{array}$$

Therefore $i_* : \pi_k(B) \cong \pi_k(M)$ for $k \geq 1$.

B and M are arcwise connected and CW complexes, hence

$$i : B \subset M$$

is homotopy equivalence.

But, on the other hand, $H_m(M, B; \mathbb{Z}_2) \neq 0$ ($m = \dim M$).

This is a contradiction. Q.E.D.

Now we define

$$\Lambda_\delta = \{ \lambda : [0, 1] \rightarrow M_\delta ; \text{ piecewise smooth with } \lambda(0), \lambda(1) \in B_\delta \}$$

with a distance as §16 in [4]. Then, as Theorem 17.1 in [4], we can prove

$$\Lambda_\delta \cong \Omega(M_\delta; B_\delta, B_\delta) \approx \Omega(M; B, B) .$$

From Lemma 1, we have $H_0(\Lambda_\delta, B_\delta) \neq 0$ or $\pi_k(\Lambda_\delta, B_\delta) \neq 0$.

For example, let $\alpha \in \pi_k(\Lambda_\delta, B_\delta)$ be the nontrivial element.

A representative $f \in \alpha$ is a continuous function $D^k \rightarrow \Lambda_\delta$ with $f(S^{k-1}) \subset B_\delta$.

We define

$$(4) \quad c_\delta = \inf_{f \in \alpha} \text{Max } E(\text{Im } f)$$

For the case of homology, take a component A with $A \cap B_\delta = \emptyset$ and define $c_\delta = \inf_{a \in A} E(a)$.

The following lemma is easily proved.

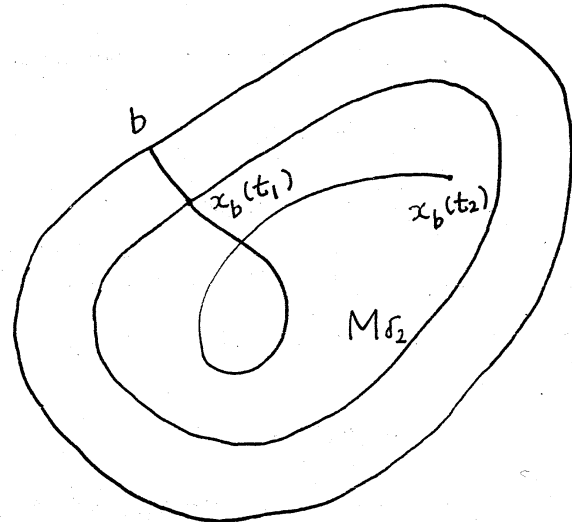
Lemma 2. There exist $\delta_1 > 0$ and $K \geq 1$ such that

$$c_\delta + 1 \leq K \quad \text{if} \quad 0 < \delta \leq \delta_1 .$$

(proof of Theorem)

Assume that there are no periodic orbit on $H^{-1}(e)$. Then any solution $x_b(t)$, $b \in \partial M$, of (L) does not reach at the boundary.

Hence we can choose δ_2 , $0 < \delta_2 < \delta_1$ in Lemma 2, such that any solution $x_b(t)$ lies in M_{δ_2} for $t_1 \leq t \leq t_2$, where the length of $x_b(t)$, $0 \leq t \leq t_1$, by the metric



(2) is δ_2 and the length of $x_b(t)$, $t_1 \leq t \leq t_2$, by (2) is $K^{1/2}$.

Then we change the metric ds to $d\tilde{s}$ so that

$$(5) \quad ds = d\tilde{s} \quad \text{on } M_{\delta_2},$$

$$(6) \quad ds \geq d\tilde{s} \quad \text{on } M_{\delta_3} - M_{\delta_2} \quad \text{for some } 0 < \delta_3 < \delta_2,$$

$$(7) \quad M_{\delta_3} \text{ is geodesically convex w.r.t. } d\tilde{s}.$$

This is done as in [2]. The condition (6) is fulfilled if we modify the function λ used in [2] so as to $\lambda \leq 1$ but $|\lambda'(\delta)|$ sufficiently large.

$$0 \leq t \leq t_i$$

Remark that then $x_b(t)$ is also a geodesic w.r.t. $d\tilde{s}$ after a time change, because $d\tilde{s}$ is a conformal transformation of ds by the function λ depending only on y_n in [2].

Now let $d(,)$ be the Riemannian distance on $M - \partial M$ w.r.t. $d\tilde{s}$.

We choose $\eta > 0$ so that

$$(8) \quad \text{two points } x, y \in M_{\delta_3}, \text{ with } d(x, y) \leq \eta, \text{ is uniquely combined by the shortest geodesic in } M_{\delta_3},$$

$$(9) \quad \text{for } x \in M_{\delta_3} \text{ with } d(x, B_{\delta_3}) \leq \eta, \text{ there is the unique } r(x) \in B_{\delta_3} \text{ such that } d(x, r(x)) = d(x, B_{\delta_3}).$$

We put $N = (K/\eta)^2$.

Then for $\lambda \in \Lambda_{\delta_3}$ with $\tilde{E}(\lambda) \leq K$, where $\tilde{E}$ is defined by (3) replacing ds with $d\tilde{s}$, we have

$$\begin{aligned} d(\lambda(t_1), \lambda(t_2)) &\leq \int_{t_1}^{t_2} |\dot{\lambda}(t)|_{d\tilde{s}} dt \\ &\leq (t_2 - t_1)^{1/2} \tilde{E}(\lambda) \leq \eta, \end{aligned}$$

if $0 \leq t_2 - t_1 \leq 1/N$.

We put $\tilde{\Lambda} = \{ \lambda \in \Lambda_{\delta_3} ; \tilde{E}(\lambda) \leq K \}$.

For $\lambda \in \widetilde{\Lambda}$, we join $r(\lambda(1/N))$, $\lambda(1/N)$, $\lambda(2/N)$, ..., $\lambda(1-1/N)$, $r(\lambda(1-1/N))$ by the shortest geodesics, mark the centers of the geodesics and join them by another geodesics (see [2]).

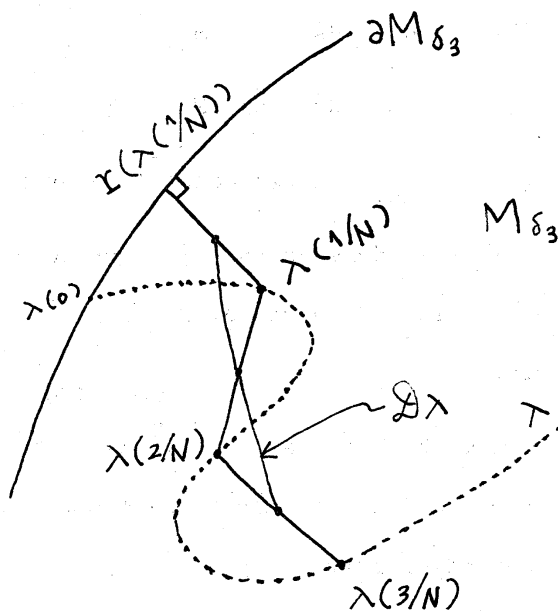
Thus we deform λ to the new curve $\mathcal{S}\lambda$.

$$\mathcal{L} : \tilde{\Lambda} \rightarrow \tilde{\Lambda}$$

is continuous and

$$(10) \quad \mathcal{Q} \simeq i_d \quad ,$$

(11)  : E - decreasing.



Let c be defined by (4) putting $\delta = \delta_3$ and replacing E with $\tilde{E}$. We have

(12) $c > 0$.

Because α is nontrivial in the relative sense (if $c = 0$, $\text{Im } f$ is deformed into B_{δ_3}) .

Then $c \leq c_{\delta_3} \leq K - 1$ by (6) and Lemma 2.

Now for a natural number j , we choose $f \in \alpha$ with

$$c \leq \max E(\operatorname{Im} f) \leq c + 1/j.$$

By (10) and (11), $\mathcal{D} \circ f \in \alpha$ and $\text{Max } E(\text{Im } \mathcal{D} \circ f) \leq c + 1/j$.

So we have $\lambda_j \in \tilde{\Lambda}$ with

$$(13) \quad c \leq \tilde{E}(\lambda_j) \leq \tilde{E}(\lambda_j) \leq c + 1/j .$$

For this sequence $\{\lambda_j\}_{j=1, 2, \dots}$, we can assume

$$\lambda_j(k/N) \rightarrow p_k \quad ; \quad k = 0, 1, \dots, N .$$

Consider the curve λ_∞ given by combining $p_0, p_1, \dots, p_N$ by the shortest geodesic. Then we can prove that λ_∞ is a smooth geodesic w.r.t. $d\tilde{s}$ with

$$\tilde{E}(\lambda_\infty) = c \quad \text{and} \quad \dot{\lambda}_\infty(i) \perp T_{\lambda_\infty(i)} B_{\delta_3} \quad (i = 0, 1) .$$

This corresponds to the condition (C) of Palais-Smale [3] .

Consider the point $p \in B_{\delta_2}$, at which λ_∞ encounter M_{δ_2} for the first time. ; $p = \lambda_\infty(s_1)$. Then

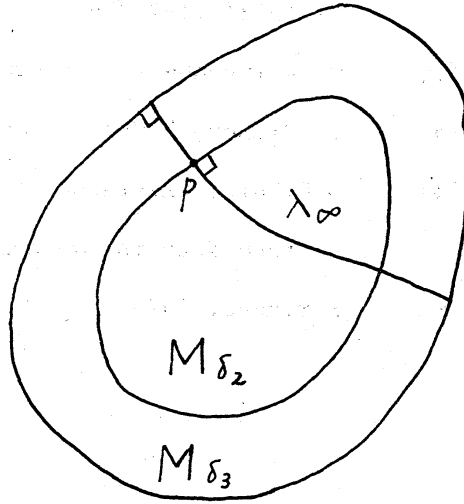
$$\dot{\lambda}_\infty(s_1) \perp T_p B_{\delta_2} .$$

But, by the construction of δ_2 , the geodesic $\lambda_\infty(s)$; $s_1 \leq s \leq 1$, is contained in M_{δ_2} , because the length of the curve $\lambda_\infty(s)$; $s_1 \leq s \leq 1$, w.r.t. $ds (= d\tilde{s})$ as long as $\lambda_\infty(s) \in M_{\delta_2}$ by (5) is less than $K^{1/2}$.

($\tilde{E}(\lambda_\infty) = c \leq K$ implies that the length of λ_∞ w.r.t. $d\tilde{s} \leq K^{1/2}$.)

This is a contradiction, proving the theorem. Q.E.D.

[2] treats the analytic system, but it is not essential for our argument. In [5], Seifert's result is proved for C^3 - Finsler systems.



For the case $M \approx D^n$, Seifert conjectured that there may be at least n periodic orbits.

For counting the number of critical points, we use the homology group (pairwise subordinated homology classes [3]) instead of the homotopy group

But I don't know whether Lemma 1 is valid, replacing π with H .

References

- [1] P. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math., 31(1978), 157-184.
- [2] H. Seifert, Periodische Bewegungen mechanischer Systeme, Math. Z., 51(1948), 197-216.
- [3] W. Klingenberg, "Lectures on Closed Geodesics", Springer, 1978.
- [4] J. Milnor, "Morse Theory", Princeton Univ. Press, 1974.
- [5] O.R. Ruiz, Existence of brake orbits in Finsler mechanical systems, Lecture Note in Mathematics, 597, "Geometry and Topology", Springer, 1976.