

DIOPHANTINE ANALYSIS OF CUBIC FORMS

Jörg Brüdern, Stuttgart, Germany

Let $s \geq 2$ be fixed. For given coefficients $\lambda_i \in \mathbb{N}$ we shall be concerned with the diagonal cubic equation

$$\lambda_1 x_1^3 + \lambda_2 x_2^3 + \dots + \lambda_s x_s^3 = 0. \quad (1)$$

When $s \leq 6$ it may happen that (1) admits only the trivial integer solution $x_1 = x_2 = \dots = x_s = 0$. A simple example is given by

$$F(\underline{x}) = x_1^3 - Hx_2^3 + p(x_3^3 - Hx_4^3) + p^2(x_5^3 - Hx_6^3) \quad (2)$$

where $p \equiv 1 \pmod{3}$ is a prime, and H is a cubic non-residue (mod p)

For $s \geq 7$, however, it is known that (1) has non-trivial solutions, and one may ask for the size of the smallest solution. In this spirit,

Pitman and Ridout (1966) showed that for $s = 9$, there are solutions of (1) satisfying

$$0 < \sum_{i=1}^9 |\lambda_i x_i^3| \ll (\lambda_1 \lambda_2 \dots \lambda_9)^{\frac{3}{2} + \varepsilon},$$

for any set of coefficients $\lambda_1, \dots, \lambda_9$. To simplify the statement of such results, we define, for $s \geq 7$,

$$B(s) := \inf_{g > 0} \left\{ \text{For any } \lambda_1, \dots, \lambda_s \in \mathbb{N}, \text{ there are solutions of (1) with } 0 < \sum_{i=1}^s |\lambda_i x_i^3| \ll (\lambda_1 \dots \lambda_s)^g \right\}$$

The result of Pitman and Ridout can then be restated as $B(9) \leq \frac{3}{2}$.

Later work implies that $B(s)$ is finite if and only if $s \geq 7$. The following explicit inequalities were obtained:

$$\text{Leung (1983)} \quad B(8) \leq \frac{35}{8}$$

$$\text{Baker (1989)} \quad B(7) \leq 61, \quad B(8) \leq 2 + \frac{5}{149}, \quad B(9) \leq \frac{7}{6}$$

$$\text{Brüderin (1992)} \quad B(8) \leq \frac{5}{3}$$

$$\text{Li Hongze (1997)} \quad B(7) \leq 14$$

It may be of interest to compare this with known lower bounds. If $F(x_1, \dots, x_6)$ is the form given by (2), we may consider the equation

$$F(x_1, \dots, x_6) + \lambda (x_7^3 + \dots + x_s^3) = 0 \quad (\lambda \in \mathbb{N} \text{ fixed})$$

For any non-trivial solution, we must have $x_j \neq 0$ for at least one $7 \leq j \leq s$ (since $F(\underline{x}) = 0$ implies $\underline{x} = 0$) whence $|\lambda (x_7^3 + \dots + x_s^3)| \geq 1$.

This easily yields

$$B(s) \geq \frac{1}{s-6} \quad (3)$$

Maybe this holds with equality. We can now improve the upper estimates.

THEOREM 1. We have $B(7) \leq 5 + \frac{18}{35}$, $B(8) \leq \frac{7}{5}$.

As all earlier writers, we apply the circle method to obtain the Theorem, but there is an important difference. To describe this in more detail, we briefly describe the approach of Pitman-Ridout. Let

$$\Lambda := \lambda_1 \lambda_2 \dots \lambda_s, \quad P^3 = \Lambda^g, \quad P_j = P \lambda_j^{-1/3}$$

and

$$f_j(x) := \sum_{|x| \leq P_j} e(\lambda_j x^3)$$

The integral

$$R(\underline{\lambda}) := \int_0^1 f_1(x) f_2(x) \dots f_s(x) dx$$

counts solutions of (1) with $|x_j| \leq P_j$, that is, $\lambda_j |x_j^3| \leq \Lambda^g$.

Hence, if we can show that $R(\underline{\lambda}) \geq 2$ for any $\underline{\lambda} = (\lambda_1, \dots, \lambda_s)$, then $B(s) \leq g$ follows.

Now define the "major arcs" $\mathcal{M}$ as the union of all intervals $|qx - a| \leq P^{-2}$ with $1 \leq a \leq q \leq P$, $(a, q) = 1$, and the "minor arcs" as $[0, 1] \setminus \mathcal{M} = m \pmod{1}$. It is easy to obtain an asymptotic formula for the contribution from $\mathcal{M}$, this takes the shape

$$\int_{\mathcal{M}} f_1 f_2 \dots f_s dx \sim \mathcal{G}(\underline{\lambda}) P^{s-3} \Lambda^{-\frac{1}{3}}, \quad (4)$$

here $\mathcal{G}(\underline{\lambda})$ is the formal singular series, and (4) will hold for $s \geq 7$, $g > 1$, and one also has $\Lambda^{-\varepsilon} \ll \mathcal{G}(\underline{\lambda}) \ll \Lambda^{\varepsilon}$ subject to mild conditions.

Take $s = 9$ for a typical example of a standard treatment of the minor arcs. One starts with

$$\int_m \leq \left(\sup_{\alpha \in m} |f_9(\alpha)| \right) \int_0^1 |f_1(\alpha) \dots f_8(\alpha)| d\alpha \quad (5)$$

Pitman and Ridout now applied Hölder's inequality to the integral in (5), followed by Hua's inequality in the form

$$\int_0^1 \left| \sum_{|x| \leq X} e(\alpha x^3) \right|^8 dx \ll X^{5+\varepsilon} \quad (6)$$

This yields

$$\int_0^1 |f_1 \dots f_8| dx \leq \prod_{j=1}^8 \left(\int_0^1 |f_j|^8 dx \right)^{\frac{1}{8}} \ll (P_1 P_2 \dots P_8)^{\frac{5}{8} + \varepsilon} \quad (7)$$

By Weyl's inequality, one may hope for $|f_9(\alpha)| \ll P_9^{\frac{3}{4} + \varepsilon}$ for $\alpha \in m$, and a comparison of (4) and (5) produces $B(9) \leq \frac{3}{2}$. However, although (6) is essentially best possible (can be replaced by an asymptotic formula $\sim CX^5$ with $C > 0$), the use of Hölder's inequality is highly inefficient, at least in some cases. For example, we can also use Cauchy's inequality to produce

$$\int_0^1 |f_1 \dots f_8| dx \leq \left(\int_0^1 |f_1 f_2 f_3 f_4|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |f_5 \dots f_8|^2 dx \right)^{\frac{1}{2}},$$

and we have

$$\int_0^1 |f_1 \dots f_4|^2 dx = \# \left\{ |x_i| \leq P_i, |y_i| \leq P_i : \sum_{i=1}^4 \lambda_i (x_i^3 - y_i^3) = 0 \right\}$$

This is again a counting problem of the type considered earlier, and we may expect at least an upper bound of the form

$$\int_0^1 |f_1 \dots f_4|^2 dx \ll (\lambda_1 \lambda_2 \lambda_3 \lambda_4) P_1^2 P_2^2 P_3^2 P_4^2 P^{-3} + P_1 P_2 P_3 P_4 \quad (8)$$

(the second term accounts for diagonal solutions $x_i = y_i$). For coprime values of $\lambda_1, \dots, \lambda_4$ and $\lambda_5, \dots, \lambda_8$ we get a considerable improvement of (7), namely

$$\int_0^1 |f_1 \dots f_8| dx \ll (P_1 \dots P_8) P^{-3}$$

which, if true, is best possible. We have not attempted to verify (8), but the above discussion clearly shows that the arithmetic nature of the coefficients should help to estimate $B(s)$. In fact, the use of Hölder's inequality in (7) makes such an interplay of various coefficients impossible.

When $s=7$ or 8 , it is more advisable to work with 6-th moments. Here it is possible to establish a partial analogue of (8).

THEOREM 2. Let $\lambda, \mu \in \mathbb{N}$ be cube-free, $X \geq 1$, $Y \geq 1$. Let $S_{\lambda, \mu}(X, Y)$ denote the number of solutions of

$$\lambda(x_1^3 - x_2^3) = \mu(y_1^3 + y_2^3 - y_3^3 - y_4^3) \quad (9)$$

subject to

$$|x_j| \leq X; \quad y_j \in \mathcal{A}$$

where

$$\mathcal{A} = \{y \leq Y : p|y \Rightarrow p \leq Y^\gamma\}.$$

Then, for sufficiently small $\gamma > 0$, one has

$$S_{\lambda, \mu}(X, Y) \ll XY^2 \left(1 + (XY)^{\frac{1}{3}} \frac{(\lambda, \mu)^{\frac{1}{2}}}{(\lambda\mu)^{\frac{1}{2}}} \right)$$

When $\lambda = \mu = 1$ and $X = Y$, this is contained in Wooley (1992).

The proof follows Wooley's variant of Vaughan's new iterative method,

with the simple observation that (9) implies $x_1^3 \equiv x_2^3 \pmod{\mu}$.

If μ is not too small, this allows an "efficient differencing" to be performed. Theorem 1 is derived from Theorem 2 along the patterns outlined above, but the details are very complicated.