

Spherical functions on $U(2n)/(U(n) \times U(n))$ and hermitian Siegel series

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§0 Introduction

For each nondegenerate hermitian matrix T of size n with respect to an unramified quadratic extension k'/k of non-archimedean local fields of characteristic 0, we consider the space X_T which is equivalent to $U(2n)/(U(n) \times U(n))$ over the algebraic closure of k and study spherical functions on X_T .

In §1, we construct X_T which is an homogeneous space of $G = U(H_n)$ with stabilizer isomorphic to $U(T) \times U(T)$ over k' , and define the spherical function $\omega_T(x; z)$ on X_T ($x \in X_T$, $z \in \mathbb{C}^n$), which means that $\omega_T(x; z)$ is K -invariant and a common eigenfunction for the action of the Hecke algebra $\mathcal{H}(G, K)$, K being the maximal compact subgroup of G . By a general theory, $\omega_T(x; z)$ is continued to a rational function on $q^{z_1}, \dots, q^{z_n}$, where q is the cardinality of the residue class field of k .

The Weyl group W of G acts on $z \in \mathbb{C}^n$ via rational characters of the Borel group of G , and we show functional equations with respect to W and locations of possible poles and zeros of $\omega_T(x; z)$ by giving an explicit rational function $G(z)$ of $q^{z_1}, \dots, q^{z_n}$ for which $G(z) \cdot \omega_T(x; z)$ is holomorphic in $z \in \mathbb{C}^n$ and W -invariant, in §2.

Using the functional equations, we give an explicit expression of $\omega_T(x; z)$ at many points in X_T in §3, define the spherical Fourier transform on the Schwartz space $\mathcal{S}(K \backslash X_T)$ and show the image is a free $\mathcal{H}(G, K)$ -module of rank 2^{n-1} in §4. In §5, as an application, we consider hermitian Siegel series $b_\pi(T; t)$ and prove their functional equations by use of results in §2.

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§1 Spaces $\mathfrak{X}_T$ and X_T , and spherical functions $\omega_T(x; s)$

Let k'/k be an unramified quadratic extension of p -adic fields with involution $*$, and for each $A = (a_{ij}) \in M_{mn}(k')$, we denote by A^* the matrix $(a_{ji}^*) \in M_{nm}(k')$. We fix a unit $\epsilon \in \mathcal{O}_k^\times$ such that $k' = k(\sqrt{\epsilon})$ and $\epsilon - 1 \in 4\mathcal{O}_k^\times$ (cf. [Om], 63.3 and 63.4), and set

$$\xi = \frac{1 + \sqrt{\epsilon}}{2}. \quad (1.1)$$

Then $\{1, \xi\}$ forms an $\mathcal{O}_k$ -basis for $\mathcal{O}_{k'}$, and $\{\alpha \in \mathcal{O}_{k'} \mid \alpha^* = -\alpha\} = \sqrt{\epsilon}\mathcal{O}_k$. We fix a prime element π of k , and denote by $v_\pi(\cdot)$ the additive value on k , by $||$ the normalized absolute value on $k^\times$ with $|\pi|^{-1} = q$ being the cardinality of the residue class field of k .

We set

$$\mathcal{H}_m = \{A \in M_m(k') \mid A^* = A\}, \quad \mathcal{H}_m^{nd} = \mathcal{H}_m \cap GL_m(k').$$

For $A \in \mathcal{H}_m$ and $X \in M_{mn}(k')$, we write

$$A[X] = X^* \cdot A = X^*AX \in \mathcal{H}_n,$$

and define the unitary group

$$U(A) = \{g \in GL_m(k') \mid A[g] = A\}.$$

In particular we set

$$G = U(H_n) \quad \text{with } H_n = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \quad U(m) = U(1_m).$$

For $T \in \mathcal{H}_n^{nd}$, we set

$$\begin{aligned} \mathfrak{X}_T &= \{x \in M_{2n,n}(k') \mid H_n[x] = T\} \ni x_T = \begin{pmatrix} \xi T \\ 1_n \end{pmatrix}, \\ X_T &= \mathfrak{X}_T / U(T). \end{aligned} \quad (1.2)$$

The group G acts on $\mathfrak{X}_T$, as well as on X_T , through left multiplication, which is transitive by Witt's theorem for hermitian matrices (cf. [Sch], Ch.7, §9).

Lemma 1.1 *The stabilizer G_0 of G at $x_T U(T) \in X_T$ is isomorphic to $U(T) \times U(T)$:*

$$U(T) \times U(T) \xrightarrow{\sim} G_0, \quad (h_1, h_2) \longmapsto \tilde{T}^{-1} \begin{pmatrix} h_1^{*-1} & 0 \\ 0 & h_2^{*-1} \end{pmatrix} \tilde{T},$$

where

$$\tilde{T} = \begin{pmatrix} 1_n & \xi^* T \\ 1_n & -\xi T \end{pmatrix} \in GL_{2n}(k').$$

We fix the Borel subgroup B of G as

$$B = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{*-1} \end{pmatrix} \begin{pmatrix} 1_n & a \\ 0 & 1_n \end{pmatrix} \mid \begin{array}{l} b \text{ is upper triangular of size } n, \\ a + a^* = 0 \end{array} \right\}. \quad (1.3)$$

For each element $x \in \mathfrak{X}_T$, we denote by x_2 the lower half n by n block of x . We define relative B -invariants on $\mathfrak{X}_T$ by

$$f_{T,i}(x) = d_i(x_2 \cdot T^{-1}) = d_i(x_2 T^{-1} x_2^*), \quad 1 \leq i \leq n, \quad (1.4)$$

where $d_i(y)$ is the determinant of the upper left i by i block of a matrix y . It is easy to see, for $b \in B$,

$$f_{T,i}(bx) = \psi_i(b) f_{T,i}(x), \quad \psi_i(b) = \prod_{j=1}^i N(b_j)^{-1}, \quad (1.5)$$

where b_j is the j -th diagonal component of b and $N = N_{k'/k}$. Hence $f_{T,i}(x)$, $1 \leq i \leq n$ are relative B -invariants on $\mathfrak{X}_T$ associated with the rational characters ψ_i of B , and we may regard them as relative B -invariants on X_T , since $f_{T,i}(xh) = f_{T,i}(x)$ for any $h \in U(T)$. We set

$$\mathfrak{X}_T^{\text{op}} = \{x \in X_T \mid f_{T,i}(x) \neq 0, 1 \leq i \leq n\}, \quad X_T^{\text{op}} = \mathfrak{X}_T^{\text{op}}/U(T). \quad (1.6)$$

Remark 1.2 Though we may realize above objects as the sets of k -rational points of algebraic sets defined over k and develop the arguments, we take down to earth way for simplicity of notations. We only note here that X_T^{op} (resp. $\mathfrak{X}_T^{\text{op}}$) becomes a Zariski open B -orbit in X_T (resp. $B \times U(T)$ -orbit in $\mathfrak{X}_T^{\text{op}}$) over the algebraic closure of k .

We introduce a spherical function $\omega_T(x; s)$ on $\mathfrak{X}_T$ as well as on $X_T = \mathfrak{X}_T/U(T)$. For $x \in \mathfrak{X}_T$ and $s \in \mathbb{C}^n$, set

$$\omega_T(x; s) = \omega_T^{(n)}(x; s) = \int_K |f_T(kx)|^{s+\varepsilon} dk, \quad (1.7)$$

where $K = G \cap GL_{2n}(\mathcal{O}_{k'})$, dk is the normalized Haar measure on K and k runs over the set $\{k \in K \mid kx \in \mathfrak{X}_T^{\text{op}}\}$,

$$\begin{aligned} \varepsilon &= \varepsilon_0 + \left(\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q} \right), \quad \varepsilon_0 = (-1, \dots, -1, -\frac{1}{2}) \in \mathbb{C}^n, \\ |f_T(x)|^s &= \prod_{i=1}^n |f_{T,i}(x)|^{s_i}. \end{aligned}$$

The right hand side of (1.7) is absolutely convergent if $\text{Re}(s_i) \geq 1$, $1 \leq i \leq n-1$, and $\text{Re}(s_n) \geq \frac{1}{2}$, and continued to a rational function of $q^{s_1}, \dots, q^{s_n}$. We note here that

$$|\psi(p)|^{\varepsilon_0} \left(= \prod_{i=1}^n |\psi_i(p)|^{\varepsilon_{0,i}} \right) = \delta^{\frac{1}{2}}(p),$$

where δ is the modulus character on B (i.e., $d(pp') = \delta(p')^{-1}d(p)$).

We denote by $\mathcal{C}^\infty(K \backslash X_T)$ the space of left K -invariant functions on X_T , which can be identified with the space $\mathcal{C}^\infty(K \backslash \mathfrak{X}_T / U(T))$ of left K -invariant right $U(T)$ -invariant functions on $\mathfrak{X}_T$. The function $\omega_T(x; z)$ can be regarded as a function in $\mathcal{C}^\infty(K \backslash X_T)$ and becomes a common eigenfunction for the action of the Hecke algebra $\mathcal{H}(G, K)$ (cf. [H2] §1, or [H4] §1). In detail, the Hecke algebra $\mathcal{H}(G, K)$ is the commutative $\mathbb{C}$ -algebra consisting of compactly supported two-sided K -invariant functions on G , acting on $\mathcal{C}^\infty(K \backslash X_T)$ by the convolution product

$$(\phi * \Psi)(x) = \int_G \phi(g) \Psi(g^{-1}x) dg, \quad (\phi \in \mathcal{H}(G, K), \Psi \in \mathcal{C}^\infty(K \backslash X_T)), \quad (1.8)$$

and we see

$$(\phi * \omega_T(\cdot; s))(x) = \lambda_s(\phi) \omega_T(x; s), \quad (\phi \in \mathcal{H}(G, K)) \quad (1.9)$$

where λ_s is the $\mathbb{C}$ -algebra homomorphism defined by

$$\begin{aligned} \lambda_s : \mathcal{H}(G, K) &\longrightarrow \mathbb{C}(q^{s_1}, \dots, q^{s_n}), \\ \phi &\longmapsto \int_B \phi(p) |\psi(p)|^{-s+\varepsilon} dp, \quad (|\psi(p)|^{-s+\varepsilon} = |\psi(p)|^{-s+\varepsilon_0}), \end{aligned}$$

with dp being the left invariant measure on B normalized by $\int_{K \cap B} dp = 1$.

We introduce a new variable z which is related to s by

$$s_i = -z_i + z_{i+1} \quad (1 \leq i \leq n-1), \quad s_n = -z_n \quad (1.10)$$

and write $\omega_T(x; z) = \omega_T(x; s)$. The Weyl group W of G relative to the maximal k -split torus in B acts on rational characters of B as usual (i.e., $\sigma(\psi)(b) = \psi(n_\sigma^{-1} b n_\sigma)$ by taking a representative n_σ of σ), so W acts on $z \in \mathbb{C}^n$ and on $s \in \mathbb{C}^n$ as well. We will determine the functional equations of $\omega_T(x; s)$ with respect to this Weyl group action. The group W is isomorphic to $S_n \ltimes C_2^n$, S_n acts on z by permutation of indices, and W is generated by S_n and $\tau : (z_1, \dots, z_n) \longmapsto (z_1, \dots, z_{n-1}, -z_n)$. Keeping the relation (1.10), we also write $\lambda_z(\phi) = \lambda_s(\phi)$; then λ_z gives a $\mathbb{C}$ -algebra isomorphism (Satake isomorphism)

$$\begin{aligned} \lambda_z : \mathcal{H}(G, K) &\xrightarrow{\sim} \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W, \\ \phi &\longmapsto \int_B \phi(p) \prod_{i=1}^n |N(p_i)|^{-z_i} \delta^{\frac{1}{2}}(p) dp, \end{aligned} \quad (1.11)$$

where p_i is the i -th diagonal component of $p \in B$.

Proposition 1.3 *Set $\mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ and*

$$\tilde{u} = (u_1 \frac{\pi\sqrt{-1}}{\log q}, \dots, u_{n-1} \frac{\pi\sqrt{-1}}{\log q}, 0) \in \mathbb{C}^n, \quad u = (u_1, \dots, u_{n-1}) \in \mathcal{U}.$$

Then $\omega_T(x; z + \tilde{u})$, $u \in \mathcal{U}$, are linearly independent for generic $z \in \mathbb{C}^n$ and correspond to the same eigenvalue through $\lambda_z : \mathcal{H}(G, K) \longrightarrow \mathbb{C}$.

Proof. The set $\mathfrak{X}_T^{\text{op}}$ is decomposed into the disjoint union of B -orbits as follows:

$$\begin{aligned}\mathfrak{X}_T^{\text{op}} &= \bigsqcup_{u \in \mathcal{U}} \mathfrak{X}_{T,u}, \\ \mathfrak{X}_{T,u} &= \{x \in \mathfrak{X}_T^{\text{op}} \mid v_\pi(f_{T,i}(x)) \equiv u_1 + \cdots + u_i \pmod{2}, 1 \leq i \leq n-1\}.\end{aligned}$$

We consider finer spherical functions

$$\omega_{T,u}(x; s) = \int_K |f_T(kx)|_u^{s+\varepsilon} dk, \quad |f_T(y)|_u^{s+\varepsilon} = \begin{cases} |f_T(y)|^{s+\varepsilon} & \text{if } y \in \mathfrak{X}_{T,u} \\ 0 & \text{otherwise,} \end{cases} \quad (1.12)$$

then $\{\omega_{T,u}(x; s) \mid u \in \mathcal{U}\}$ are linearly independent for generic s associated with the same λ_s . For each character χ of $\mathcal{U}$, we have

$$\sum_{u \in \mathcal{U}} \chi(u) \omega_{T,u}(x; s) = \omega_T(x; s + \tilde{v}),$$

for some $v \in \mathcal{U}$, and the result follows from this. ■

We note here the relation between $\omega_T(x; s)$ and $\omega_{T'}(y; s)$ when T and T' are equivalent under the action of $GL_n(k')$, which is easy to see.

Proposition 1.4 *For $T \in \mathcal{H}_n^{\text{nd}}$ and $h \in GL_n(k')$, we set $T' = T[h] (= h^*Th)$. Then*

$$\mathfrak{X}_{T'} = (\mathfrak{X}_T)h, \quad X_{T'} = \mathfrak{X}_T h / U(T') \quad \text{and} \quad f_{T',i}(xh) = f_{T,i}(x) \quad (x \in \mathfrak{X}_T),$$

and

$$\omega_{T'}(xh; s) = \omega_T(x; s), \quad (x \in \mathfrak{X}_T).$$

By use of some results on spherical functions on the space $\mathcal{H}_n^{\text{nd}}$ of hermitian forms, we obtain the following.

Theorem 1.5 *For any $T \in \mathcal{H}_n^{\text{nd}}$, the function*

$$\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i-1}} \times \omega_T(x; z)$$

is holomorphic for any z in $\mathbb{C}^n$ and S_n -invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n}$.

Outline of a proof. By the embedding

$$K_0 = GL_n(\mathcal{O}_{k'}) \longrightarrow K, \quad h \longmapsto \tilde{h} = \begin{pmatrix} h^{*-1} & 0 \\ 0 & h \end{pmatrix},$$

we obtain

$$\omega_T(x; z) = \int_{K_0} dh \int_K |f_T(\tilde{h}kx)|^{s+\varepsilon} dk = \int_K \zeta^{(n)}(D(kx); s) dk.$$

Here $D(kx) = (kx)_2 \cdot T^{-1} \in \mathcal{H}_n$, $\zeta^{(n)}(y; s)$ is a spherical function on $\mathcal{H}_n^{nd}$ defined by

$$\zeta^{(n)}(y; s) = \int_{K_0} \prod_{i=1}^n |d_i(h \cdot y)|^{s_i + \varepsilon_i} dh, \quad (h \cdot y = hyh^*),$$

and we have already known (cf. [H1], or [H3]) that

$$\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i-1}} \times \zeta^{(n)}(y; s) \in \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^{S_n},$$

the result follows from this. ■

Remark 1.6 For the transposition $\tau_i = (i \ i+1) \in W$, $1 \leq i \leq n-1$, the following functional equation holds by Theorem 1.5

$$\omega_T(x; z) = \frac{1 - q^{z_i - z_{i+1} - 1}}{q^{z_i - z_{i+1}} - q^{-1}} \times \omega_T(x; \tau_i(z)), \quad 1 \leq i \leq n-1. \quad (1.13)$$

On the other hand, one may obtain (1.13) directly in the similar way to the case of τ in § 2, where the sufficient condition in [H4]-§3 for having a functional equation with respect to τ_i is satisfied and the Gamma factor in (1.13) is essentially the same to that of the zeta function of prehomogeneous vector space $(U \times GL_1(k'), (k')^2)$, where $U \cong U(2)$ or $U\left(\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}\right)$. Then Theorem 1.5 follows from (1.13).

§2 Functional equations of $\omega_T(x; z)$

We calculate the functional equation for $\tau \in W$, and give the functional equations with respect to the whole W .

2.1.

Theorem 2.1 *For any $T \in \mathcal{H}_n^{nd}$, the spherical function satisfies*

$$\omega_T(x; z) = \omega_T(x; \tau(z)),$$

where $\tau(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, -z_n)$.

For $n = 1$, we have the following by a direct calculation, where we set $K_1 = U(H_1) \cap GL_2(\mathcal{O}_{k'})$.

Proposition 2.2 (i) *The set*

$$\left\{ x_e = \begin{pmatrix} \pi^e \\ \xi \pi^{t-e} \end{pmatrix} \mid e \in \mathbb{Z}, 2e \leq t \right\}, \quad \left(\xi = \frac{1 + \sqrt{\epsilon}}{2} \right)$$

forms a set of complete representatives of $K_1 \backslash \mathfrak{X}_T$ for $T = \pi^t$.

(ii) *For $x_e \in \mathfrak{X}_T$ with $T = \pi^t$ as in (i), one has*

$$\omega_T^{(1)}(x_e; s) = \frac{(-1)^t q^{e-\frac{1}{2}t}}{1 + q^{-1}} \times \frac{q^{(t-2e+1)s}(1 - q^{-2s-1}) - q^{-(t-2e+1)s}(1 - q^{2s-1})}{q^s - q^{-s}}.$$

(iii) *For any $T \in \mathcal{H}_1^{nd}$, $\omega_T^{(1)}(x; s)$ is holomorphic for all $s \in \mathbb{C}$ and satisfies the functional equation*

$$\omega_T^{(1)}(x; s) = \omega_T^{(1)}(x; -s).$$

Until the end of this subsection we assume $n \geq 2$. The parabolic subgroup P attached to τ , in the sense of [Bo] §21.11, is given as follows:

$$\begin{aligned} P &= B \cup Bw_\tau B \\ &= \left\{ \left(\begin{array}{c|c} q & \\ \hline a & b \\ c & q^{*-1}d \end{array} \right) \left(\begin{array}{c|c} 1_{n-1} & \alpha \\ \hline & 1 \\ & 1_{n-1} & -\alpha^* \\ & & 1 \end{array} \right) \left(\begin{array}{c|c} 1_n & \gamma \quad \beta \\ \hline & -\beta^* & 0 \\ & & 1_n \end{array} \right) \in G \mid \right. \\ &\quad \left. \begin{array}{l} q \text{ is upper triangular in } GL_{n-1}(k'), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1, 1), \alpha, \beta \in M_{n-1,1}(k'), \\ \gamma \in M_{n-1}(k'), \gamma + \gamma^* = 0 \end{array} \right\}, \end{aligned} \quad (2.1)$$

where each empty place in the above expression means zero-entry.

Since it suffices to show Theorem 2.1 for diagonal T 's (cf. Proposition 1.4), we fix a diagonal $T \in \mathcal{H}_n^{nd}$ and write $f_i(x) = f_{T,i}(x)$ for simplicity of notations. We consider the following action of $\tilde{P} = P \times GL_1$ on $\tilde{\mathfrak{X}}_T = \mathfrak{X}_T \times V$ with $V = M_{21}(k')$:

$$(p, r) \star (x, v) = (px, \rho(p)vr^{-1}), \quad (p, r) \in \tilde{P}, (x, v) \in \tilde{\mathfrak{X}}_T,$$

where $\rho(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by the decomposition of $p \in P$ as in (2.1). We define

$$g(x, v) = \det \left[\left(\begin{array}{c|c} 1_{n-1} & \\ \hline & t_v \end{array} \right) \begin{pmatrix} x_2 \\ -y \end{pmatrix} \cdot T^{-1} \right], \quad (x, v) \in \tilde{\mathfrak{X}}_T, \quad (2.2)$$

where x_2 is the lower half n by n block of x (the same before) and y is the n -th row of x . Then we have

Lemma 2.3 (i) *$g(x, v)$ is a relative $\tilde{P}$ -invariant on $\tilde{\mathfrak{X}}_T$ associated with character $\tilde{\psi}$:*

$$\tilde{\psi}(p, r) = \psi_{n-1}(p)N(r)^{-1}, \quad (p, r) \in \tilde{P} = P \times GL_1,$$

where ψ_{n-1} is given in (1.5) and well-defined on P , and satisfies

$$g(x, v_0) = f_n(x), \quad v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$$

(ii) $g(x, v)$ is expressed as

$$g(x, v) = D(x)[v], \quad (2.3)$$

with some hermitian matrix

$$D(x) = \begin{pmatrix} a(x) & \beta(x) \\ \beta(x)^* & d(x) \end{pmatrix} \quad (a(x), d(x) \in k, \beta(x) \in k'), \quad (2.4)$$

such that $\det(D(x)) = 0$ and $\text{Tr}(\beta(x)) = -f_{n-1}(x)$, where Tr is the trace $\text{Tr}_{k'/k}$.

For $A \in \mathcal{H}_2$ and $s \in \mathbb{C}$, we define

$$\zeta_{K_1}(A; s) = \int_{K_1} |d_1(h \cdot A)|^{s-\frac{1}{2}} dh,$$

where dh is the normalized Haar measure on K_1 , which is absolutely convergent if $\text{Re}(s) \geq \frac{1}{2}$ and continued to the whole $\mathbb{C}$. Then we obtain

Lemma 2.4 Assume $x \in \mathfrak{X}_T^{\text{op}}$ and $D(x)$ is given by (2.3). Set $m = \min\{v_\pi(a(x)), v_\pi(d(x))\}$ and $t = v_\pi(\beta(x)) - m$ for the expression of $D(x)$ as in (2.4). Then $t \geq 0$ and

$$\zeta_{K_1}(D(x); s) = \frac{q^{\frac{m}{2}}}{1+q^{-1}} \cdot |f_{n-1}(x)|^s \cdot \frac{q^{(t+1)s}(1-q^{-2s-1}) - q^{-(t+1)s}(1-q^{2s-1})}{q^s - q^{-s}}.$$

In particular, one has the functional equation

$$\zeta_{K_1}(D(x); s) = |f_{n-1}(x)|^{2s} \cdot \zeta_{K_1}(D(x); -s). \quad (2.5)$$

We give a sketch of a proof of Theorem 2.1. By the embedding

$$K_1 \longrightarrow K = K_n, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \tilde{h} = \left(\begin{array}{c|c} 1_{n-1} & \\ \hline a & b \\ \hline c & d \end{array} \right),$$

we have

$$\begin{aligned} \omega_T(x; s) &= \int_{K_1} dh \int_K |f(kx)|^{s+\varepsilon} dk \\ &= \int_K \chi_\pi \left(\prod_{i < n} f_i(kx) \right) \prod_{i < n} |f_i(kx)|^{s_i-1} \zeta_{K_1}(D(kx); s_n + \frac{\pi\sqrt{-1}}{\log q}) dk. \end{aligned}$$

By Lemma 2.4, we see

$$\omega_T(x; s) = \omega_T(x; s_1, \dots, s_{n-2}, s_{n-1} + 2s_n, -s_n),$$

and, in variable z , we have

$$\omega_T(x; z) = \omega_T(x; \tau(z)), \quad \tau(z) = (z_1, \dots, z_{n-1}, -z_n).$$

■

2.2. In order to describe functional equations of $\omega_T(x; z)$ with respect to W , we prepare some notations. We denote by Σ the set of roots of G with respect to the k -split torus of G contained in B and by Σ^+ the set of positive roots with respect to B . We may understand Σ as a subset in $\mathbb{Z}^n$, and set

$$\Sigma^+ = \Sigma_s^+ \cup \Sigma_\ell^+, \quad \Sigma_s^+ = \{e_i - e_j, e_i + e_j \mid 1 \leq i < j \leq n\}, \quad \Sigma_\ell^+ = \{2e_i \mid 1 \leq i \leq n\},$$

where e_i is the i -th unit vector in $\mathbb{Z}^n$, $1 \leq i \leq n$. The set

$$\Delta = \{\tau_i = (i \ i+1) \in S_n \mid 1 \leq i \leq n-1\} \cup \{\tau\},$$

is associated with the set of simple roots and generates W . For each $\sigma \in W$, we set

$$\Sigma_s^+(\sigma) = \{\alpha \in \Sigma_s^+ \mid -\sigma(\alpha) \in \Sigma^+\}.$$

The pairing on $\mathbb{Z}^n \times \mathbb{C}^n$

$$\langle t, z \rangle = \sum_{i=1}^n t_i z_i, \quad (t \in \mathbb{Z}^n, z \in \mathbb{C}^n),$$

is W -invariant on $\Sigma \times \mathbb{C}^n$, i.e.,

$$\langle \alpha, z \rangle = \langle \sigma(\alpha), \sigma(z) \rangle, \quad (\alpha \in \Sigma, z \in \mathbb{C}^n, \sigma \in W). \quad (2.6)$$

Theorem 2.5 *For $T \in \mathcal{H}_n^{nd}$ and $\sigma \in W$, the spherical function $\omega_T(x; z)$ satisfies the following functional equation*

$$\omega_T(x; z) = \Gamma_\sigma(z) \cdot \omega_T(x; \sigma(z)), \quad (2.7)$$

where

$$\Gamma_\sigma(z) = \prod_{\alpha \in \Sigma_s^+(\sigma)} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}}.$$

In particular, the Gamma factor $\Gamma_\sigma(z)$ does not depend on x nor T .

Outline of a proof. We determine $\Gamma_\sigma(z)$ by the equation (2.7), which is a rational function of $q^{z_1}, \dots, q^{z_n}$. We set for $\alpha \in \Sigma$ and $z \in \mathbb{C}^n$

$$f_\alpha(\langle \alpha, z \rangle) = \begin{cases} 1 & \text{if } \alpha = \pm 2e_i, (1 \leq i \leq n) \\ \frac{1 - q^{\langle \alpha, z \rangle - 1}}{q^{\langle \alpha, z \rangle} - q^{-1}} & \text{otherwise} \end{cases}$$

We see $\Gamma_\sigma(z)$ for $\sigma \in \Delta$ by (1.13) and Theorem 2.1. For general $\sigma \in W$, we obtain the result by cocycle relations of $\Gamma_\sigma(z)$ and W -invariance of the inner product (2.6). ■

We will use the following explicit value $\Gamma_\rho(z)$ in §5.

Corollary 2.6 *Set $\rho \in W$ by*

$$\rho(z_1, \dots, z_n) = (-z_n, -z_{n-1}, \dots, -z_1).$$

Then

$$\Gamma_\rho(z) = \prod_{1 \leq i < j \leq n} \frac{1 - q^{z_i + z_j - 1}}{q^{z_i + z_j} - q^{-1}}.$$

Remark 2.7 The above ρ gives the functional equation of the hermitian Siegel series (cf. §5), and it is interesting that such ρ corresponds to the unique automorphism of the extended Dynkin diagram of the root system of type (C_n) , which was pointed out by Y. Komori.

By Theorem 1.5 and Theorem 2.5, we obtain the following theorem.

Theorem 2.8 *Set*

$$G(z) = \prod_{\alpha \in \Sigma_+^*} \frac{1 + q^{\langle \alpha, z \rangle}}{1 - q^{\langle \alpha, z \rangle - 1}}.$$

Then, for any $T \in \mathcal{H}_n^{nd}$, the function $G(z) \cdot \omega_T(x; z)$ is holomorphic for all z in $\mathbb{C}^n$ and W -invariant. In particular it is an element in $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$.

§3 Explicit formula for $\omega_T(x; z)$

3.1. Set

$$\Lambda_n^+ = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}, \quad (3.1)$$

and, for each $\lambda \in \Lambda_n^+$,

$$\begin{aligned} \pi^\lambda &= \text{Diag}(\pi^{\lambda_1}, \dots, \pi^{\lambda_n}) \in \mathcal{H}_n^{nd}, & x_\lambda &= \begin{pmatrix} \xi \pi^\lambda \\ 1_n \end{pmatrix} \in \mathfrak{X}_{\pi^\lambda}, \\ \omega_\lambda(x; z) &= \omega_T(x; z) \quad \text{for } T = \pi^\lambda. \end{aligned} \quad (3.2)$$

Theorem 3.1 *For $\lambda \in \Lambda_n^+$, one has the following explicit expression:*

$$\begin{aligned} \omega_\lambda(x_\lambda; z) &= \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})} (1 - q^{-2})^n}{\prod_{i=1}^{2n} (1 - (-q^{-1})^i)} \times \frac{1}{G(z)} \times \sum_{\sigma \in W} q^{-\langle \lambda, \sigma(z) \rangle} H(\sigma(z)), \end{aligned}$$

where $G(z)$ is given in Theorem 2.8 and

$$H(z) = \prod_{\alpha \in \Sigma_s^+} \frac{1 + q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}} \prod_{\alpha \in \Sigma_t^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}.$$

Remark 3.2 By Theorem 2.8, the main part

$$H_\lambda(z) = \sum_{\sigma \in W} \sigma(q^{-\langle \lambda, z \rangle} H(z)) = \sum_{\sigma \in W} q^{-\langle \lambda, \sigma(z) \rangle} H(\sigma(z))$$

of $\omega_\lambda(x_\lambda; z)$ belongs to $\mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W$. Further we see in a standard way that the set $\{H_\lambda(z) \mid \lambda \in \Lambda_n^+\}$ forms its $\mathbb{C}$ -basis. On the other hand, $H_\lambda(z)$ is a special case of P_λ (up to a scalar factor) introduced by Macdonald [Mac] §10 in a generous context of orthogonal polynomials associated with root systems.

We will prove the above theorem by using a general expression formula (Theorem 2.6 in [H4], or in [H2]) of spherical functions on homogeneous spaces, which is based on functional equations of finer spherical functions and some data depending only on the group G . We explain about the proof in the next subsection.

By Theorem 3.1 and Proposition 1.4, we may have the explicit formula of $\omega_T(x; s)$ at many points. For $T \in \mathcal{H}_n^{nd}$ and $\lambda \in \Lambda_n^+$, it is known that T and π^λ belong to the same $GL_n(k')$ -orbit in $\mathcal{H}_n^{nd}$ if and only if

$$v_\pi(\det(T)) \equiv |\lambda| \pmod{2},$$

where $|\lambda| = \sum_{i=1}^n \lambda_i$. Thus we obtain

Theorem 3.3 *Let $T \in \mathcal{H}_n^{nd}$ and $\lambda \in \Lambda_n^+$ and assume that $v_\pi(\det(T)) \equiv |\lambda| \pmod{2}$. Taking $h_\lambda \in GL_n(k')$ for which $\pi^\lambda[h_\lambda] = T$, one has $x_\lambda h_\lambda \in \mathfrak{X}_T$ and*

$$\begin{aligned} \omega_T(x_\lambda h_\lambda; z) &= \omega_\lambda(x_\lambda; z) \\ &= \frac{(-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})} (1-q^{-2})^n}{\prod_{i=1}^{2n} (1-(-q^{-1})^i)} \cdot \frac{1}{G(z)} \cdot \sum_{\sigma \in W} \sigma(q^{-\langle \lambda, z \rangle} H(z)). \end{aligned}$$

Further, each such a λ gives a different K -orbit

$$Kx_\lambda h_\lambda U(T) \text{ in } K \backslash X_T \left(= K \backslash \mathfrak{X}_T / U(T) \right).$$

3.2. In order to apply Theorem 2.6 in [H4], we need to check the assumptions there. Let $\mathbb{G}$ be a connected reductive linear algebraic group and $\mathbb{X}$ be an affine algebraic variety which is $\mathbb{G}$ -homogeneous, where everything is assumed to be defined over a p -adic field k . For an algebraic set, we use the same ordinary letter to indicate the set of k -rational points. Let K be a maximal compact open subgroup of G , and $\mathbb{B}$ a minimal parabolic subgroup of $\mathbb{G}$ defined over k satisfying $G = KB = BK$. We denote by $\mathfrak{X}(\mathbb{B})$ the group of rational character of $\mathbb{B}$ defined over k and by $\mathfrak{X}_0(\mathbb{B})$ the subgroup consisting of those characters associated with some relative $\mathbb{B}$ -invariant on $\mathbb{X}$ defined over k . In these situation, the assumptions are the following:

(A1) $\mathbb{X}$ has only a finite number of $\mathbb{B}$ -orbits.

(A2) A basic set of relative $\mathbb{B}$ -invariants on $\mathbb{X}$ defined over k can be taken by regular functions on $\mathbb{X}$.

(A3) For $y \in \mathbb{X}$ not contained in the open orbit, there exists some ψ in $\mathfrak{X}_0(\mathbb{B})$ whose restriction to the identity component of the stabilizer $\mathbb{H}_y$ of $\mathbb{G}$ at y is not trivial.

(A4) The rank of $\mathfrak{X}_0(\mathbb{B})$ coincides with that of $\mathfrak{X}(\mathbb{B})$.

In the present situation, as is noted in Remark 1.2, we may understand $\mathbb{G} = U(H_n)$ defined over k , $G = \mathbb{G}(k)$, $B = \mathbb{B}(k)$ for the Borel subgroup defined over k , and $X = X_T$ as the set of k -rational points of the affine algebraic variety $\mathbb{X} = \mathfrak{X}_T / U(T)$, and we recall that relative invariants $f_{T,i}(x)$ and the spherical function $\omega_T(x; s)$ can be regarded as functions on X_T .

It is easy to see the present $(\mathbb{X}, \mathbb{B})$ satisfies the conditions (A1), (A2) and (A4) (cf. Lemma 1.1, (1.4) and (1.5)), in particular, the unique Zariski open $\mathbb{B}$ -orbit is given by $\mathbb{X}^{op} = \{x \in \mathbb{X} \mid f_{T,i}(x) \neq 0, 1 \leq i \leq n\}$ (cf. (1.6)).

First we give an outline of a proof of Theorem 3.1, admitting the condition (A3). By Theorem 2.5, we obtain vector-wise functional equations for finer spherical functions $\omega_{T,u}(x; z)$, $u \in \mathcal{U} = (\mathbb{Z}/2\mathbb{Z})^{n-1}$ (cf. (1.12))

$$(\omega_{T,u}(x; z))_{u \in \mathcal{U}} = A^{-1} \cdot G(\sigma, z) \cdot \sigma A \cdot (\omega_{T,u}(x; \sigma(z)))_{u \in \mathcal{U}}, \quad \sigma \in W, \quad (3.3)$$

where

$$A = (\chi(u))_{\chi, u}, \quad \sigma A = (\sigma(\chi)(u))_{\chi, u} \in GL_{2^{n-1}}(\mathbb{Z}),$$

χ runs over characters of $\mathcal{U}$, $u \in \mathcal{U}$, and $G(\sigma, z)$ is the diagonal matrix of size 2^{n-1} whose (χ, χ) -component is $\Gamma_\sigma(z_\chi)$. Here $\Gamma_\sigma(z)$ is given in Theorem 2.5 and z_χ is determined by the identity

$$\sum_{u \in \mathcal{U}} \chi(u) \omega_{T,u}(x; z) = \omega_T(x; z_\chi).$$

We denote by U the Iwahori subgroup of K compatible with B , take the normalized Haar measure du on U , and set

$$\begin{aligned} \delta_u(x_\lambda, z) &= \int_U |f_T(ux_\lambda)|_u^{s+\varepsilon} du \\ &= \begin{cases} (-1)^{\sum_i \lambda_i(n-i+1)} q^{-\sum_i \lambda_i(n-i+\frac{1}{2})} q^{-\langle \lambda, z \rangle} & \text{if } x_\lambda \in \mathfrak{X}_{T,u} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Applying Theorem 2.6 in [H4] to our present case, we obtain

$$(\omega_{T,u}(x_\lambda; z))_{u \in \mathcal{U}} = \frac{1}{Q} \sum_{\sigma \in W} \gamma(\sigma(z)) (A^{-1} \cdot G(\sigma, z) \cdot \sigma A) (\delta_u(x_\lambda, \sigma(z)))_{u \in \mathcal{U}}, \quad (3.4)$$

where

$$\begin{aligned} Q &= \sum_{\sigma \in W} [U\sigma U : U]^{-1} = \prod_{i=1}^{2n} (1 - (-1)^i q^{-i}) / (1 - q^{-2})^n, \\ \gamma(z) &= \prod_{\alpha \in \Sigma_s^+} \frac{1 - q^{2\langle \alpha, z \rangle - 2}}{1 - q^{2\langle \alpha, z \rangle}} \cdot \prod_{\alpha \in \Sigma_t^+} \frac{1 - q^{\langle \alpha, z \rangle - 1}}{1 - q^{\langle \alpha, z \rangle}}. \end{aligned}$$

Since $\omega_T(x_\lambda; z) = \sum_{u \in \mathcal{U}} \mathbf{1}(u) \omega_u(x_\lambda; z)$, we obtain the explicit formula for $\omega_\lambda(x_\lambda; z)$ from (3.4). $\blacksquare$

Now we explain about the condition (A3). We consider the action of $G \times U(T)$ on $\mathfrak{X}_T$ by $(g, h) \circ x = gxh^{-1}$. Then, the stabilizer B_y of B at $yU(T) \in X_T$ coincides with the image $B_{(y)}$ of the projection to B of the stabilizer $(B \times U(T))_y$ at $y \in \mathfrak{X}_T$ to B . Hence, in our case, the condition (A3) is equivalent to the following:

(C) : For each $y \in \mathfrak{X}_T$ not contained in $\mathfrak{X}_T^{\text{op}}$, there exists $\psi \in \mathfrak{X}(\mathbb{B})$ whose restriction to the identity component of $B_{(y)}$ is not trivial.

It suffices to prove the condition (A3) (or (C)) over the algebraic closure $\bar{k}$ of k , hence we may assume that $T = 1_n$; for simplicity of notation, we write $f_i(x)$ instead of $f_{T,i}(x)$. Until the end of this subsection, we consider algebraic sets over $\bar{k}$, extend the involution $*$ on k' to $\bar{k}$, indicate it by $-$, and write $\bar{x} = (\bar{x}_{ij}) \in M_{\ell m}(\bar{k})$ for $x = (x_{ij}) \in M_{\ell m}(k')$.

Then, our situation is the following:

$$\begin{aligned} \mathfrak{X} &= \mathfrak{X}_{1_n} = \{x \in M_{2n,n} \mid H_n[x] = 1_n\}, \\ (U(H_n) \times U(1_n)) \times \mathfrak{X} &\longrightarrow \mathfrak{X}, \quad ((g, h), x) \longmapsto (g, h) \circ x = gxh^{-1}, \end{aligned}$$

and B is the Borel subgroup of $U(H_n)$ (as in (1.3)). We introduce a $(GL_{2n} \times GL_n)$ -set $\tilde{\mathfrak{X}}$ as follows:

$$\begin{aligned} \tilde{\mathfrak{X}} &= \{ (x, y) \in M_{2n,n} \oplus M_{2n,n} \mid {}^t y H_n x = 1_n \} \\ (g, h) \star (x, y) &= (gxh^{-1}, \dot{g}y^t h), \quad ((g, h) \in GL_{2n} \times G_n, \dot{g} = H_n {}^t g^{-1} H_n), \end{aligned} \quad (3.5)$$

and we write an element of $\tilde{\mathfrak{X}}$ as $(x, y) = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$ with $x_i, y_i \in M_n$. We take the Borel subgroup P of GL_{2n} by

$$P = \left\{ \begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \in GL_{2n} \mid p, {}^t q \in B_n, r \in M_n \right\},$$

where B_n is the Borel subgroup of GL_n consisting of the upper triangular matrices. The involution $g \mapsto \dot{g} = H_n {}^t g^{-1} H_n$ on GL_{2n} induces an involution on P :

$$\begin{pmatrix} p & r \\ 0 & q \end{pmatrix} \mapsto \begin{pmatrix} {}^t q^{-1} & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & -{}^t r \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & {}^t p^{-1} \end{pmatrix}. \quad (3.6)$$

The embedding $\iota : \mathfrak{X} \hookrightarrow \tilde{\mathfrak{X}}, x \mapsto (x, \bar{x})$ is compatible with action, i.e., we have the commutative diagram

$$\begin{array}{ccccc} (U(H_n) \times U(1_n)) & \times & \mathfrak{X} & \xrightarrow{\circ} & \mathfrak{X} \\ \downarrow \text{incl.} & & \downarrow \iota & \circlearrowleft & \downarrow \iota \\ (GL_{2n} \times GL_n) & \times & \tilde{\mathfrak{X}} & \xrightarrow{\star} & \tilde{\mathfrak{X}}. \end{array}$$

For $(x, y) \in \tilde{\mathfrak{X}}$ and $p \in P$, set

$$\tilde{f}_i(x, y) = d_i(x_2 {}^t y_2), \quad \tilde{\psi}_i(p) = \prod_{1 \leq j \leq i} p_j^{-1} p_{n+j}, \quad (1 \leq i \leq n), \quad (3.7)$$

where p_j is the j -th diagonal component of p . Then $\tilde{f}_i(x, y)$'s are basic relative P -invariants on $\tilde{\mathfrak{X}}$ associated with characters $\tilde{\psi}_i$, $\tilde{f}_i(x, \bar{x}) = f_i(x)$ for $x \in \mathfrak{X}$, and $\tilde{\psi}_i|_B = \psi_i$. We set

$$\mathcal{S} = \left\{ (x, y) \in \tilde{\mathfrak{X}} \cap (P \times GL_n) \star \mathfrak{X} \mid \prod_{i=1}^n \tilde{f}_i(x, y) = 0 \right\}.$$

For $\alpha = (x, y) \in \tilde{\mathfrak{X}}$, we denote by H_α the stabilizer of $P \times GL_n$ at α , and by P_α the identity component of the image of H_α by the projection to P . In order to prove the condition (C), it is sufficient to show the following:

($\tilde{C}$) : For each $\alpha \in \mathcal{S}$, there exists some $\psi \in \langle \tilde{\psi}_i \mid 1 \leq i \leq n \rangle$ whose restriction to P_α is not trivial.

We have only to consider ($\tilde{C}$) for representatives under the action of $P \times GL_n$. In the following we consider the case $n \geq 2$, since $\mathfrak{X}_T = \mathfrak{X}_T^{\text{op}}$ for $n = 1$ and there is nothing to prove. We denote by $\delta_i(a) \in GL_n$ the diagonal matrix whose j -th entry is 1 except the i -th which is $a \in GL_1$. We show ($\tilde{C}$) according to the type of $\alpha \in \mathcal{S}$.

The case $\alpha = (x, y) \in \mathcal{S}$ with $\det(x_2) \neq 0$: Under $(P \times GL_n)$ -action, we may assume that

$$\alpha = \left(\begin{pmatrix} 0 \\ 1_n \end{pmatrix}, \begin{pmatrix} 1_n \\ h \end{pmatrix} \right),$$

where $h = 1_r \perp h_1$, $0 \leq r < n$, and h_1 is a hermitian matrix such that

the first row and column are zero, or
for some i , $(1 < i \leq n - r)$, each entry in the first row and column or in the i -th row and column is 0 except at $(1, i)$ or $(i, 1)$ which are 1.

Then H_α contains the following elements, according to the above type of h_1 ,

$$\left(\left(\frac{\delta_{r+1}(a)}{1_n} \right), 1_n \right) \quad \text{or} \quad \left(\left(\frac{\delta_{r+1}(a)}{\delta_{r+i}(a)} \right), \delta_{r+i}(a) \right) \quad (a \in GL_1),$$

and we see $\tilde{\psi}_{r+1} \not\equiv 1$ on P_α .

The case $\alpha = (x, y) \in \mathcal{S}$ with $\det(y_2) \neq 0$ is reduced to the case $\det(x_2) \neq 0$.

The remaining case is $\alpha \in \mathcal{S}$ with $\det(x_2) = \det(y_2) = 0$. We set $J(i_1, i_2, \dots, i_t)$ the matrix of size $n \times t$ such that $1 \leq i_1 < i_2 < \dots < i_t \leq n$ and whose (i_j, j) -entry is 1, $1 \leq j \leq t$, and all the other entries are 0.

Under $(P \times GL_n)$ -action, we may assume that

$$\alpha = \left(\left(\frac{0}{J_2} \middle| \frac{J_1}{0} \right), \left(\frac{z_1}{z_2} \middle| \frac{0}{z_3} \right) \right), \quad (J_1, z_3 \in M_{n\ell}, J_2, z_1, z_2 \in M_{nk}),$$

where

$$J_1 = J(r_1, r_2, \dots, r_\ell), \quad J_2 = J(e_1, e_2, \dots, e_k), \quad 1 \leq \ell, k < n, \quad \ell + k = n,$$

and

$$\begin{aligned} &\text{the } e_j\text{-th row of } z_1 \text{ is the same as in } J_2 \text{ and } (i, j)\text{-entry is 0 if } i < e_j, \quad 1 \leq j \leq k, \\ &\text{the } r_j\text{-th row of } z_2 \text{ is 0, } \quad 1 \leq j \leq \ell, \\ &\text{the } r_j\text{-th row of } z_3 \text{ is the same as in } J_1 \text{ and } (i, j)\text{-entry is 0 if } i > r_j, \quad 1 \leq j \leq \ell. \end{aligned} \tag{3.8}$$

We see, for any $a \in GL_1$,

$$\begin{aligned} &\left(\left(\frac{1_n}{0} \middle| \frac{0}{\delta_1(a)} \right), 1_n \right) \in H_\alpha \quad \text{if } e_1 > 1, \\ &\left(\left(\frac{\delta_1(a)}{0} \middle| \frac{0}{1_n} \right), \delta_{k+1}(a) \right) \in H_\alpha \quad \text{if } r_1 = 1, \\ &\left(\left(\frac{a1_n}{0} \middle| \frac{0}{1_n} \right), a1_n \right) \in H_\alpha \quad \text{if } z_2 = 0. \end{aligned}$$

If $e_1 = 1, r_1 > 1$ and $z_2 \neq 0$, we modify z_i -part of α to satisfy not only (3.8) but also the following

if the i -th row of z_2 is nonzero, then the i -th row of z_3 is 0,

and we still call it α . Then H_α contains the following (A_1, A_2) for any $a \in GL_1$

$$A_1 = \text{Diag}(a_1, \dots, a_n) \perp 1_n, \quad a_i = \begin{cases} a & \text{if the } i\text{-th row of } z_2 \text{ is } 0 \\ 1 & \text{if the } i\text{-th row of } z_2 \text{ is not } 0, \end{cases}$$

$$A_2 = 1_k \perp a 1_\ell.$$

Hence $\tilde{\psi}_n \neq 1$ on P_α for $\alpha \in \mathcal{S}$ with $\det(x_2) = \det(y_2) = 0$. ■

Thus we have shown the condition $(\tilde{C})$ is satisfied for every $(x, y) \in \mathcal{S}$, which shows that our $(\mathbb{X}, \mathbb{B})$ satisfies the condition (A3) and Theorem 3.1 is established.

§4 Spherical Fourier transform on $\mathcal{S}(K \backslash X_T)$

We consider the space $\mathcal{S}(K \backslash X_T)$ consisting of functions in $C^\infty(K \backslash \mathfrak{X}_T / U(T))$ compactly supported modulo $U(T)$, which is an $\mathcal{H}(G, K)$ -submodule (cf. (1.8)). We define the spherical Fourier transform F_T on $\mathcal{S}(K \backslash X_T)$ as follows

$$F_T : \mathcal{S}(K \backslash X_T) \longrightarrow \mathbb{C}(q^{z_1}, \dots, q^{z_n}),$$

$$\xi \longmapsto F_T(\xi)(z) = \hat{\xi}_T(z) = \int_X \xi(x) \Psi_T(x; z) dx, \quad (4.1)$$

where $\Psi_T(x; z) = G(z) \cdot \omega_T(x; z)$ and dx is the G -invariant measure on X . By Theorem 2.8, we see the image of F_T is contained in

$$\mathcal{R} = \mathbb{C}[q^{\pm z_1}, \dots, q^{\pm z_n}]^W.$$

We decompose $\mathcal{R}$ as follows

$$\mathcal{R} = \bigoplus_{\mathbf{e} \in \{0,1\}^n} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0,$$

where

$$\mathcal{R}_0 = \mathbb{C}[q^{\pm 2z_1}, \dots, q^{\pm 2z_n}]^W = \mathbb{C}[q^{2z_1} + q^{-2z_1}, \dots, q^{2z_n} + q^{-2z_n}]^{S_n},$$

and $s_i = s_i(z)$ is the i -th fundamental symmetric polynomial of $\{q^{z_j} + q^{-z_j} \mid 1 \leq j \leq n\}$; $\mathcal{R}$ is a free $\mathcal{R}_0$ -module of rank 2^n . We set

$$\mathcal{R}_{\text{even}} = \bigoplus_{\mathbf{e}:\text{even}} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0, \quad \mathcal{R}_{\text{odd}} = \bigoplus_{\mathbf{e}:\text{odd}} s_1^{e_1} \cdots s_n^{e_n} \mathcal{R}_0,$$

where $\mathbf{e} \in \{0,1\}^n$ is even (resp. odd) if $\sum_{i=1}^n i e_i$ is even (resp. odd), and for each $T \in \mathcal{H}_n^{\text{nd}}$, and define

$$\mathcal{R}_{\langle T \rangle}$$

to be $\mathcal{R}_{\text{even}}$ or $\mathcal{R}_{\text{odd}}$ according to the parity of $v_\pi(\det(T))$.

Theorem 4.1 For each $T \in \mathcal{H}_n^{nd}$, one has a surjective $\mathcal{H}(G, K)$ -module homomorphism

$$F_T : \mathcal{S}(K \backslash X_T) \longrightarrow \mathcal{R}_{\langle T \rangle},$$

and a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(G, K) & \times & \mathcal{S}(K \backslash X_T) & \xrightarrow{*} & \mathcal{S}(K \backslash X_T) \\ \downarrow \wr & & \downarrow F_T & \circlearrowleft & \downarrow F_T \\ \mathcal{R}_0 & \times & \mathcal{R}_{\langle T \rangle} & \longrightarrow & \mathcal{R}_{\langle T \rangle}, \end{array} \quad (4.2)$$

where the upper horizontal arrow is given by the action of $\mathcal{H}(G, K)$ on $\mathcal{S}(K \backslash X_T)$, the left end vertical isomorphism is given by Satake isomorphism (1.11)

$$\mathcal{H}(G, K) \xrightarrow{\sim} \mathcal{R}_0, \quad \phi \longmapsto \lambda_z(\check{\phi}), \quad (\check{\phi}(g) = \phi(g^{-1})),$$

and the lower horizontal arrow is given by the ordinal multiplication in $\mathcal{R}$.

Outline of a proof. For $\phi \in \mathcal{H}(G, K)$ and $\xi \in \mathcal{S}(K \backslash X_T)$, it is easy to see

$$F_T(\phi * \xi)(z) = \lambda_z(\check{\phi}) F_T(\xi)(z).$$

We may expand $\omega_T(x; z)$ in a region of absolute convergence of the integral (1.7)

$$\omega_T(x; z) = \sum_{\mu \in \mathbb{Z}^n} a_\mu q^{\langle \mu, z \rangle},$$

where $a_\mu = 0$ unless $|\mu| (= \sum_{i=1}^n \mu_i) \equiv v_\pi(\det(T)) \pmod{2}$. Further we may expand $G(z)$ also in terms $q^{\langle \nu, z \rangle}$ with $|\nu|$ is even. Hence we see that $\text{Im}(F_T) \subset \mathcal{R}_{\langle T \rangle}$. On the other hand, by Remark 3.2 and Theorem 3.3 we see

$$\text{Im}(F_T) \supset \{ H_\lambda(z) \mid \lambda \in \Lambda_n^+, |\lambda| \equiv v_\pi(\det(T)) \pmod{2} \},$$

and the image of F_T coincides with $\mathcal{R}_{\langle T \rangle}$. ■

Remark 4.2 We expect that the spherical Fourier transform F_T is injective, which is equivalent to the identity

$$\mathfrak{X}_T = \bigcup_{\substack{\lambda \in \Lambda_n^+ \\ |\lambda| \equiv v_\pi(\det(T)) \pmod{2}}} Kx_\lambda h_\lambda U(T), \quad (4.3)$$

where disjointness in the right hand side is known by Theorem 3.3. If it is true, then $\mathcal{S}(K \backslash X_T)$ would be a free $\mathcal{H}(G, K)$ -module of rank 2^{n-1} and the set $\{ \Psi_T(x; z + \tilde{u}) \mid u \in \mathcal{U} \}$ would form a basis of spherical functions on X_T corresponding to $z \in \mathbb{C}^n$ through λ_z (cf. Proposition 1.3). This is true when $n = 1$ by Proposition 2.1, and we have the following.

Proposition 4.3 *Assume $n = 1$. Then the spherical transform F_T is injective and $\mathcal{S}(K \backslash X_T)$ is a free $\mathcal{H}(G, K)$ -module of rank 1, in fact the image coincides with*

$$\mathbb{C}[q^{2z} + q^{-2z}] \text{ if } v_\pi(T) \text{ is even, } (q^z + q^{-z})\mathbb{C}[q^{2z} + q^{-2z}] \text{ if } v_\pi(T) \text{ is odd.}$$

Any spherical function on X_T corresponding to $z \in \mathbb{C}$ through λ_z is a constant multiple of $\omega_T(x; z)$.

§5 Hermitian Siegel series

We recall p -adic hermitian Siegel series, and give those integral representation and a new proof of the functional equation as an application of spherical functions.

Let ψ be an additive character of k of conductor $\mathcal{O}_k$. For $T \in \mathcal{H}_n(k')$, the hermitian Siegel series $b_\pi(T; s)$ is defined by

$$b_\pi(T; t) = \int_{\mathcal{H}_n(k')} \nu_\pi(R)^{-t} \psi(\text{tr}(TR)) dR, \quad (5.1)$$

where $\text{tr}(\)$ is the trace of matrix and $\nu_\pi(R)$ is defined as follows: if the elementary divisors of R with negative π -powers are $\pi^{-e_1}, \dots, \pi^{-e_r}$, then $\nu_\pi(R) = q^{e_1 + \dots + e_r}$, and $\nu_\pi(R) = 1$ otherwise (cf. [Sh]-§13). The right hand side of (5.1) is absolutely convergent if $\text{Re}(t)$ is sufficiently large.

In the following we assume that T is nondegenerate, since the properties of $b_\pi(T; t)$ can be reduced to the nondegenerate case. We give an integral expression of $b_\pi(T; t)$ in a similar argument for Siegel series in [HS]-§2. We recall the set $\mathfrak{X}_T$ for $T \in \mathcal{H}_n^{nd}$ and take the measure $|\Theta_T|$ on it simultaneously as the fibre space of T by the polynomial map $M_{2n,n}(k') \longrightarrow \mathcal{H}_n(k')$, $x \longmapsto H_n[x]$.

Theorem 5.1 *If $\text{Re}(t) > 2n$, we have*

$$b_\pi(T; t) = \zeta_n(k'; \frac{t}{2})^{-1} \times \int_{\mathfrak{X}_T(\mathcal{O}_{k'})} |N(\det(x_2))|^{\frac{t}{2}-n} |\Theta_T|(x),$$

where $\zeta_n(k'; t)$ is the zeta function of the matrix algebra $M_n(k')$

$$\zeta(k'; t) = \int_{M_n(\mathcal{O}_{k'})} |\det(x)|_{k'}^{t-n} dx = \prod_{i=1}^n \frac{1 - q^{-2i}}{1 - q^{-2(t-i+1)}},$$

and

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \{x \in M_{2n,n}(\mathcal{O}_{k'}) \mid H_n[x] = T\},$$

Since $\mathfrak{X}_T(\mathcal{O}_{k'})$ is compact, we obtain

Proposition 5.2 Denote the K -orbit decomposition of $\mathfrak{X}_T(\mathcal{O}_{k'})$ as

$$\mathfrak{X}_T(\mathcal{O}_{k'}) = \sqcup_{i=1}^r Kx_i.$$

Then one has

$$b_\pi(T; t) = \zeta_n(k'; \frac{t}{2})^{-1} |\det(T)|^{\frac{t}{2}-n} \times \sum_{i=1}^r c_i \cdot \omega_T(x_i; s_t),$$

where c_i is the volume of Kx_i and

$$s_t = (1, \dots, 1, \frac{t}{2} - n + \frac{1}{2}) + (\frac{\pi\sqrt{-1}}{\log q}, \dots, \frac{\pi\sqrt{-1}}{\log q}) \in \mathbb{C}^n.$$

Then, by Corollary 2.6, we obtain the functional equation of $b_\pi(T; t)$.

Theorem 5.3 For any $T \in \mathcal{H}_n^{nd}$, one has

$$b_\pi(T; t) = \chi_\pi(\det(T))^{n-1} |\det(T)|^{t-n} \times \prod_{i=0}^{n-1} \frac{1 - (-1)^i q^{-t+i}}{1 - (-1)^i q^{-(2n-t)+i}} \times b_\pi(T; 2n - t),$$

where $\chi_\pi(a) = (-1)^{v_\pi(a)}$ for $a \in k^\times$.

Remark 5.4 The above functional equation is related to an element of the Weyl group of $U(H_n)$, which is not the case for Siegel series when n is odd. In [HS], even n is odd, we needed some harmonic analysis on $O(H_n)$ to establish the functional equation.

The existence of the functional equation of $b_\pi(T; t)$ was known in an abstract form as functional equations of Whittaker functions of p -adic groups by Karel [Kr]. Recently Ikeda [Ik] has given explicit functional equations on the basis of the results of Kudla-Sweet [KS] for all quadratic extensions over $\mathbb{Q}_p$ containing split cases. There is an error in the range of i in the definition of $t_p(K/\mathbb{Q}; X)$ in [Ik] p.1112, and it is better to refer the original $f_\zeta(t)$ in [Sh] Theorem 13.6; if $K/\mathbb{Q}$ is unramified at p , $t_p(K/\mathbb{Q}; X)$ is the product of $1 - (-p)^i X$ from $i = 0$ to $n - 1$, and coincides with our case by taking $X = p^{-t}$.

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