

ON SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE AND FIXED FINITELY MANY COEFFICIENTS

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ABSTRACT. The object of the present paper is to derive several interesting properties of the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ consisting of regular and univalent meromorphic functions with positive and fixed finitely many coefficients. These include coefficient estimates, closure theorems, and radius of convexity for functions belonging to the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$.

KEY WORDS- Regular , univalent, meromorphic .

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1.1)$$

which are regular in $U^* = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. And let Σ_p denote the subclass of Σ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0) \quad (1.2)$$

that are analytic and univalent in U^* . Recently Nunokawa, Aouf and Owa [2] investigated the class $\Sigma_p^*(\alpha, \beta, \mu)$ which is a subclass of Σ_p , defined as follows :

A function $f(z) \in \Sigma_p$ is in the class $\Sigma_p^*(\alpha, \beta, \mu)$ if it satisfies the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} + 1}{\mu \frac{zf'(z)}{f(z)} - 1 + (1+\mu)\alpha} \right| < \beta \quad (z \in U^*) \quad (1.3)$$

for some α ($0 \leq \alpha < 1$), β ($0 < \beta \leq 1$), and μ ($0 \leq \mu \leq 1$).

For the class $\Sigma_p^*(\alpha, \beta, \mu)$ Nunokawa, Aouf and Owa [2] showed the following lemma.

LEMMA 1. Let the function $f(z)$ be defined by (1.2). Then $f(z)$ is in the class $\Sigma_p^*(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=1}^{\infty} \left\{ (n+1) + \beta [\mu n + (1+\mu)\alpha - 1] \right\} a_n \leq (1+\mu)\beta(1-\alpha). \quad (1.4)$$

The result is sharp.

In view of Lemma 1, all functions belonging to the class $\Sigma_p^*(\alpha, \beta, \mu)$ satisfy the coefficient inequality

$$a_n \leq \frac{(1+\mu)\beta(1-\alpha)}{(n+1) + \beta [\mu n + (1+\mu)\alpha - 1]} \quad (n \geq 1). \quad (1.5)$$

Making use of (1.5), we now introduce the following class of functions:

Let $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ denote the subclass of $\Sigma_p^*(\alpha, \beta, \mu)$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta [\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} a_n z^n, \quad (1.6)$$

where

$$a_n \geq 0, \quad 0 \leq c_i \leq 1, \quad \text{and} \quad 0 \leq \sum_{i=1}^k c_i \leq 1.$$

For $k=1$, the class $\Sigma_{p, c_1}^*(\alpha, \beta, \mu) = \Sigma_{p, c}^*(\alpha, \beta, \mu)$ was introduced by Aouf and Nunokawa [1].

In this paper, we obtain coefficient estimates and closure theorems for the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. Further the radius of convexity is obtained for the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. Techniques used are similar to those of Silverman and Silvia [4], Uralegaddi [5] and Owa and Srivastava [3].

2. Coefficient Estimates

THEOREM 1. Let the function $f(z)$ be defined by (1.6). Then $f(z)$ is in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ if and only if

$$\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta [\mu n + (1+\mu)\alpha - 1] \right\} a_n \leq (1+\mu)\beta(1-\alpha) \left(1 - \sum_{i=1}^k c_i \right), \quad (2.1)$$

where

$$0 \leq c_i \leq 1 \text{ and } 0 \leq \sum_{i=1}^k c_i \leq 1.$$

The result (2.1) is sharp.

Proof. Putting

$$a_i = \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} \quad (i=1, 2, \dots, k), \quad (2.2)$$

in Lemma 1, we have

$$\sum_{i=1}^k (1+\mu)\beta(1-\alpha)c_i + \sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} a_n \leq (1+\mu)\beta(1-\alpha), \quad (2.3)$$

which clearly implies (2.1). Further, by taking the function $f(z)$ of the form

$$f(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \frac{(1+\mu)\beta(1-\alpha)(1 - \sum_{i=1}^k c_i)}{(n+1) + \beta[\mu n + (1+\mu)\alpha - 1]} z^n \quad (2.4)$$

for $n \geq k+1$, we can see that the result (2.1) is sharp.

COROLLARY 1. Let the function $f(z)$ defined by (1.6) be in the

class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. Then

$$a_n \leq \frac{(1+\mu)\beta(1-\alpha)(1 - \sum_{i=1}^k c_i)}{(n+1) + \beta[\mu n + (1+\mu)\alpha - 1]} \quad (n \geq k+1). \quad (2.5)$$

The result (2.5) is sharp for the function $f(z)$ given by (2.4).

3. Closure Theorems

THEOREM 2. Let the functions

$$f_j(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} a_{n,j} z^n$$

($a_{n,j} \geq 0$) (3.1)

be in the class $\Sigma_{p,c_k}^*(\alpha, \beta, \mu)$ for every $j=1, 2, \dots, m$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^m d_j f_j(z) \quad (d_j \geq 0) \quad (3.2)$$

is also in the same class $\Sigma_{p,c_k}^*(\alpha, \beta, \mu)$, where

$$\sum_{j=1}^m d_j = 1. \quad (3.3)$$

PROOF. Combining the definitions (3.1) and (3.2), we have

$$F(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} \left(\sum_{j=1}^m d_j a_{n,j} \right) z^n, \quad (3.4)$$

where we have also used the relationship (3.3). Since $f_j(z) \in \Sigma_{p,c_k}^*(\alpha, \beta, \mu)$ for every $j=1, 2, \dots, m$, Theorem 1 yields

$$\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} a_{n,j} \leq (1+\mu)\beta(1-\alpha) \left(1 - \sum_{i=1}^k c_i \right), \quad (3.5)$$

for every $j = 1, 2, \dots, m$. Thus we obtain

$$\begin{aligned} & \sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} \left(\sum_{j=1}^m d_j a_{n,j} \right) \\ &= \sum_{j=1}^m d_j \left(\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} a_{n,j} \right) \\ &\leq (1+\mu)\beta(1-\alpha) \left(1 - \sum_{i=1}^k c_i \right) \end{aligned}$$

which (in view of Theorem 1) implies, that $F(z) \in \Sigma_{p, c_k}^*(\alpha, \beta, \mu)$.

THEOREM 3. Let the functions $f_j(z)$ defined by (3.1) be in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ for each $j=1, 2, \dots, m$, then the function $h(z)$ defined by

$$h(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} b_n z^n \quad (b_n \geq 0) \quad (3.6)$$

is also in the same class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$, where

$$b_n = \frac{1}{m} \sum_{j=1}^m a_{n,j}. \quad (3.7)$$

PROOF. Since $f_j(z) \in \Sigma_{p, c_k}^*(\alpha, \beta, \mu)$, it follows from Theorem 1, that

$$\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} a_{n,j} \leq (1+\mu)\beta(1-\alpha) \left(1 - \sum_{i=1}^k c_i \right). \quad (3.8)$$

Hence

$$\begin{aligned}
& \sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} b_n \\
&= \sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} \left(\frac{1}{m} \sum_{j=1}^m a_{n,j} \right) \\
&= \frac{1}{m} \sum_{j=1}^m \left(\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} a_{n,j} \right) \\
&\leq (1+\mu)\beta(1-\alpha) \left(1 - \sum_{i=1}^k c_i \right)
\end{aligned} \tag{3.9}$$

and the result follows.

THEOREM 4. The class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ is closed under convex linear combination.

PROOF. Let the functions $f_j(z)$ ($j=1,2$) defined by (3.1) be in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \lambda f_1(z) + (1-\lambda)f_2(z) \quad (0 \leq \lambda \leq 1) \tag{3.10}$$

is also in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$.

Since

$$H(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} \{\lambda a_{n,1} + (1-\lambda)a_{n,2}\} z^n, \tag{3.11}$$

we observe that

$$\sum_{n=k+1}^{\infty} \left\{ (n+1) + \beta[\mu n + (1+\mu)\alpha - 1] \right\} \left\{ \lambda a_{n,1} + (1-\lambda)a_{n,2} \right\} \\ \leq (1+\mu)\beta(1-\alpha)(1 - \sum_{i=1}^k c_i), \quad (3.12)$$

with the aid of Theorem 1. Hence $H(z) \in \Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. This completes the proof of Theorem 4.

THEOREM 5. Let

$$f_k(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i \quad (3.13)$$

and

$$f_n(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i \\ + \frac{(1+\mu)\beta(1-\alpha)(1 - \sum_{i=1}^k c_i)}{(n+1) + \beta[\mu n + (1+\mu)\alpha - 1]} z^n \quad (n \geq k+1). \quad (3.14)$$

Then $f(z)$ is in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z) \quad (3.15)$$

where $\lambda_n \geq 0$ ($n \geq k$) and

$$\sum_{n=k}^{\infty} \lambda_n = 1. \quad (3.16)$$

PROOF. We suppose that $f(z)$ can be expressed in the form (3.15).

Then it follows from (3.13), (3.14) and (3.16) that

$$f(z) = \frac{1}{z} + \sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)c_i}{(i+1) + \beta[\mu i + (1+\mu)\alpha - 1]} z^i + \sum_{n=k+1}^{\infty} \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)\lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} z^n. \quad (3.17)$$

Note that

$$\begin{aligned} & \sum_{n=k+1}^{\infty} \left\{ (n+1)+\beta[\mu n+(1+\mu)\alpha-1] \right\} \left\{ \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)\lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} \right\} \\ &= (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i) \sum_{n=k+1}^{\infty} \lambda_n = (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)(1-\lambda_k) \\ &\leq (1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i), \end{aligned} \quad (3.18)$$

which implies that $f(z) \in \Sigma_{p, c_k}^*(\alpha, \beta, \mu)$.

For the converse, assume that the function $f(z)$ of the form (1.6) belongs to the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. Since $f(z)$ satisfies (2.5) for $n \geq k+1$, we may set

$$\lambda_n = \frac{\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}}{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)} a_n \leq 1 \quad (n \geq k+1) \quad (3.19)$$

and

$$\lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n. \quad (3.20)$$

Hence $f(z)$ has the representation (3.15). This evidently completes the proof of Theorem 5.

4. Radius of Convexity

THEOREM 6. Let the function $f(z)$ defined by (1.6) be in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$. Then $f(z)$ is meromorphically convex of order ρ ($0 \leq \rho < 1$) in the disc $0 < |z| < r = r(\alpha, \beta, \mu, c_i, \rho)$, where $r(\alpha, \beta, \mu, c_i, \rho)$ is the largest value for which

$$\sum_{i=1}^k \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} + \frac{n(n+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} r^{n+1} \leq 1-\rho, \quad (4.1)$$

for $n \geq k+1$. The result is sharp for the function $f(z)$ given by (2.4).

PROOF. It suffices to show that $\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1-\rho$ ($0 \leq \rho < 1$)

for $0 < |z| < r(\alpha, \beta, \mu, c_i, \rho)$. Note that

$$\begin{aligned} & \left| \frac{zf''(z)}{f'(z)} + 2 \right| \\ & \leq \frac{\sum_{i=1}^k \frac{i(i+1)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} + \sum_{n=k+1}^{\infty} \frac{n(n+1)a_n}{1 - \sum_{i=1}^k \frac{i(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} - \sum_{n=k+1}^{\infty} na_n} r^{n+1}}{1 - \sum_{i=1}^k \frac{i(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} - \sum_{n=k+1}^{\infty} na_n} r^{n+1} \\ & \leq 1-\rho \end{aligned} \quad (4.3)$$

for $0 < |z| \leq r$ if and only if

$$\sum_{i=1}^k \frac{(1+\mu)\beta(1-\alpha)i(i+2-\rho)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} + \sum_{n=k+1}^{\infty} \frac{n(n+2-\rho)a_n}{1 - \sum_{i=1}^k \frac{i(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} - \sum_{n=k+1}^{\infty} na_n} r^{n+1} \leq 1-\rho. \quad (4.4)$$

Since $f(z)$ is in the class $\Sigma_{p, c_k}^*(\alpha, \beta, \mu)$, from (2.5) we may take

$$a_n = \frac{(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i) \lambda_n}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} \quad (n \geq k+1), \quad (4.5)$$

where $\lambda_n \geq 0$ ($n \geq k+1$) and

$$\sum_{n=k+1}^{\infty} \lambda_n \leq 1. \quad (4.6)$$

For each fixed r , we choose the positive integer $n_0 = n_0(r)$ for which

$\frac{n(n+2-\rho)}{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]} r^{n+1}$ is maximal. Then it follows that

$$\sum_{n=k+1}^{\infty} n(n+2-\rho) a_n r^{n+1} \leq \frac{n_0(n_0+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)}{(n_0+1)+\beta[\mu n_0+(1+\mu)\alpha-1]} r^{n_0+1}. \quad (4.7)$$

Hence $f(z)$ is meromorphically convex of order ρ in $0 < |z| < r(\alpha, \beta, \mu, c_i, \rho)$ provided that

$$\begin{aligned} & \sum_{i=1}^k \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r^{i+1} \\ & + \frac{n_0(n_0+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)}{(n_0+1)+\beta[\mu n_0+(1+\mu)\alpha-1]} r^{n_0+1} \leq 1 - \rho. \end{aligned} \quad (4.8)$$

We find the value $r_0 = r_0(\alpha, \beta, \mu, c_i, \rho)$ and the corresponding integer $n_0(r_0)$ so that

$$\sum_{i=1}^k \frac{i(i+2-\rho)(1+\mu)\beta(1-\alpha)c_i}{(i+1)+\beta[\mu i+(1+\mu)\alpha-1]} r_o^{i+1} + \frac{n_o(n_o+2-\rho)(1+\mu)\beta(1-\alpha)(1-\sum_{i=1}^k c_i)}{(n_o+1)+\beta[\mu n_o+(1+\mu)\alpha-1]} r_o^{n_o+1} = 1-\rho. \quad (4.9)$$

Then this value r_o is the radius of meromorphically convex of order ρ for functions $f(z)$ belonging to the class $\Sigma_{p,c_k}^*(\alpha,\beta,\mu)$.

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