

$\mathbb{Q}$ -bases of the Néron-Severi groups of certain elliptic surfaces

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1 Introduction

P. Stiller computed the Mordell-Weil ranks and hence the Picard numbers of several families of elliptic surfaces by studying the action of certain automorphisms on the cohomology group ([Stiller 1987]). He considered five families of elliptic surfaces $\mathcal{E}_n^i$ ($1 \leq i \leq 5$, $n \in \mathbb{N}$) (Example 1, 2, 3, 4 and 5 in [Stiller 1987]). For each $\mathcal{E}_n^i$, he proved that there exists finite set Adm_i of natural numbers such that the Mordell-Weil rank $r_i(n)$ is given by

$$r_i(n) = \sum_{d|n, d \in \text{Adm}_i} \varphi(d),$$

where φ is the Euler function. However he did not give generators of the Néron-Severi groups of these surfaces. In [Kuroda], $\mathbb{Q}$ -bases or $\mathbb{Z}$ -bases of these groups are given explicitly. **In this poster, we explain briefly properties with respect to such bases, and we give a basis in the most complicated case Example 1 of these five examples:**

$$\begin{aligned} \mathcal{E}_n^1 : y^2 &= x^3 + t^n x + t^n \quad (n \in \mathbb{N}, t \in \mathbb{P}_{\mathbb{C}}^1), \\ \text{Adm}_1 &= \{1, 2, 3, 7, 8, 10, 12, 15, 18, 20, 42\}. \end{aligned}$$

2 The Néron-Severi group of an elliptic surface

Notations

$f : \mathcal{E} \rightarrow \mathbb{P}_{\mathbb{C}}^1$: an elliptic surface with a zero section
 $E/\mathbb{C}(t)$: the generic fiber of $f : \mathcal{E} \rightarrow \mathbb{P}_{\mathbb{C}}^1$
 $E(\mathbb{C}(t))$: the Mordell-Weil group of $E/\mathbb{C}(t)$
 (P) : the image in $\mathcal{E}$ of the section corresponding to $P \in E(\mathbb{C}(t))$
 ∞ : the image of zero section
 $\Sigma(\mathcal{E}) := \{t \in \mathbb{P}_{\mathbb{C}}^1 \mid \mathcal{E}_t := f^{-1}(t) \text{ is a singular fiber}\}$
 $F_{t,a}$ ($0 \leq a \leq m_t - 1$) : the irreducible components of the fiber $\mathcal{E}_t$
 $F_{t,0}$: the unique component of $\mathcal{E}_t$ intersecting with ∞
 $C_0 := \mathcal{E}_{t_0}$, $t_0 \in \mathbb{P}_{\mathbb{C}}^1 \setminus \Sigma(\mathcal{E})$: a general fiber

Properties with respect to $\mathbb{Q}$ -bases of $\text{NS}(\mathcal{E})$

Lemma 2.1. Let $P_1, \dots, P_r$ be rational points of E and let M be the intersection matrix of the associated divisors

$$C_0, \infty, D_1, \dots, D_r, F_{t,a} \quad (t \in \Sigma(\mathcal{E}), 1 \leq a \leq m_t - 1), \quad (1)$$

where $D_i = (P_i) - \infty$. Put

$$L_{t,\alpha} = (F_{t,\alpha,i} \cdot F_{t,\alpha,j})_{1 \leq i, j \leq m_t-1} \text{ and } N = ((D_i + \Phi_i) \cdot D_j)_{1 \leq i, j \leq r}, \quad (2)$$

where $\{t_1, \dots, t_s\} = \{t \in \Sigma(\mathcal{E}) \mid m_t \geq 2\}$ and

$$\begin{aligned} \Phi_i &:= \sum_{t \in \mathbb{P}_{\mathbb{C}}^1} \sum_{k=0}^{m_t-1} a_{t,k} F_{t,k} = \sum_{t \in \{t_1, \dots, t_s\}} \sum_{k=1}^{m_t-1} a_{t,k} F_{t,k}, \\ (a_{t,1}, \dots, a_{t,m_t-1}) &= -(D_i \cdot F_{t,1}, \dots, D_i \cdot F_{t,m_t-1}) L_t^{-1}. \end{aligned}$$

Then we have

$$\det(M) = -\det(N) \prod_{\alpha=1}^s \det(L_{t_\alpha}).$$

Remark 2.2. (i) $P_1, \dots, P_r$ form a $\mathbb{Z}$ -basis of $E(\mathbb{C}(t))_{\text{free}}$
 $\Rightarrow$ (1) form a $\mathbb{Z}$ -basis of $\text{NS}(\mathcal{E})_{\text{free}}$ ([Shioda 1972]).

(ii) In [Shioda 1990], $(P) - \infty + \Phi_P$ is denoted by D_P . Then the pairing $\langle P, Q \rangle := -D_P \cdot D_Q = -((P) - \infty + \Phi_P) \cdot ((Q) - \infty)$ is the height pairing.

Properties

The divisors $C_0, \infty, D_j, F_{t,a}$ form a $\mathbb{Q}$ (resp. $\mathbb{Z}$)-basis of $\text{NS}(\mathcal{E})$
 $\iff \det(M) \neq 0$
 $\iff \det(N) \neq 0$ ($\because \det(L_{t_\alpha}) \neq 0$ ($1 \leq \alpha \leq s$))
 (resp. $\iff |\det(M)| = \det(\text{NS}(\mathcal{E}))$)

3 $\mathbb{Q}$ -bases of $\text{NS}(\mathcal{E}_n^1)$

We give $r_1(n)$ rational points of the generic fiber $E_n^1/\mathbb{C}(t)$ of $\mathcal{E}_n^1$, and calculate the determinant of the intersection matrix M_n of the associated divisors (or $\det(N_n)$, where N_n is similar to (2)).

Definition 3.1. $\mathbb{C}(t)$ -rational points $P_{d,j}$ of E_d^1 ($d \in \text{Adm}_1$)

$$\begin{aligned} P_{1,1} &= (-1, \sqrt{-1}), & j &\in (\mathbb{Z}/d\mathbb{Z})^\times \\ P_{2,1} &= (\sqrt{-1}t, -t), & \zeta_d &= \exp(2\pi\sqrt{-1}/d) \\ P_{3,j} &= (-\zeta_d^j t, \sqrt{-1}\zeta_d^{2j} t^2), & [*] &: \text{the Gauss symbol} \\ P_{7,j} &= (-\zeta_d^{2j} t^2 - \zeta_d^{3j} t^3, \sqrt{-1}(\zeta_d^{3j} t^3 + \zeta_d^{4j} t^4 + \zeta_d^{5j} t^5)), \\ P_{10,j} &= (2^{\frac{2}{5}} \zeta_{10}^{4j} t^4, -\zeta_{10}^{5j} t^5 - 2^{\frac{1}{5}} \zeta_{10}^{7j} t^7), \\ P_{15,j} &= (-\zeta_{15}^{5j} t^5 - 3^{\frac{1}{5}} \zeta_{15}^{6j} t^6 - 3^{\frac{2}{5}} \zeta_{15}^{7j} t^7, \\ &\quad \sqrt{-1}(3^{\frac{3}{5}} \zeta_{15}^{8j} t^8 + 3^{\frac{4}{5}} \zeta_{15}^{9j} t^9 + 2\zeta_{15}^{10j} t^{10} + 3^{\frac{1}{5}} \zeta_{15}^{11j} t^{11})), \\ P_{d,j} &= \left(\sum_{k=0}^2 a_k(d, j) t^{2[\frac{d}{6}] + k}, \sum_{k=0}^3 b_k(d, j) t^{3[\frac{d}{6}] + k} \right) \quad (d = 8, 12), \\ P_{d,j} &= \left(\sum_{k=0}^2 a_k(d, j) (\zeta_d^j t)^{6+2k}, \sum_{k=0}^3 b_k(d, j) (\zeta_d^j t)^{9+2k} \right) \quad (d = 18, 20), \\ P_{42,j} &= \left(\sum_{k=0}^4 a_k(42, j) (\zeta_{42}^j t)^{14+2k}, \sum_{k=0}^6 b_k(42, j) (\zeta_{42}^j t)^{21+2k} \right). \end{aligned}$$

$a_k(d, j), b_k(d, j)$ satisfy the system given by comparing the coefficients of

$$y(P_{d,j})^2 = x(P_{d,j})^3 + t^d x(P_{d,j})^2 + t^d.$$

Remark 3.2. (i) $f : \mathcal{E}_n^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ has a global section and is not smooth.

$\rightsquigarrow \text{NS}(\mathcal{E}_n^1)$ is torsion free ([Shioda 1990]).

(ii) $\rho : E_n^1 \rightarrow E_d^1$; $(x, y, t) \mapsto (x, y, t^{\frac{n}{d}})$: surjective map ($d \mid n$)

$\rightsquigarrow E_d^1(\mathbb{C}(t)) \hookrightarrow E_n^1(\mathbb{C}(t))$; $P_{d,j} \mapsto \rho^*(P_{d,j})$

We use the same symbol $P_{d,j}$ for $\rho^*(P_{d,j})$.

(iii) $\mathcal{E}_n^1$: rational (i.e., $n = 1, 2, 3, 4, 6, 7, 8$ or 12)

$\Rightarrow E_n^1(\mathbb{C}(t))$ is generated by points of the form

$$x = t^{2[\frac{n}{6}]} \sum_{k=0}^2 a_k t^k, \quad y = t^{3[\frac{n}{6}]} \sum_{k=0}^3 b_k t^k \quad ([\text{Shioda 1990}]).$$

(iv) We can choose the coefficients $a_k(d, j), b_k(d, j)$ such that

$\det(N_n) \neq 0$, and $|\det(M_n)| = 1$ if $n = 1, 2, 3, 4, 6, 7, 8, 12$.

(In [Kuroda], these coefficients are given explicitly.)

Theorem 3.3.

- (i) $\text{NS}(\mathcal{E}_n^1)$ has a $\mathbb{Q}$ -basis $C_0, \infty, D_d, j, F_{t,a}$
 $(d \in \text{Adm}_1, d \mid n, j \in (\mathbb{Z}/d\mathbb{Z})^\times, t \in \Sigma(\mathcal{E}_n^1), 1 \leq a \leq m_t - 1)$.
 (ii) $\mathcal{E}_n$ is rational (i.e., $n = 1, 2, 3, 4, 6, 7, 8$ or 12)
 $\Rightarrow$ these divisors form a $\mathbb{Z}$ -basis of $\text{NS}(\mathcal{E}_n^1)$.

References

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