

Hankel-type operators on the spaces of analytic functions

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Abstract. We will study Hankel-type operators on the spaces of analytic functions on the open unit disk. These operators are a natural generalization of the classical Hankel operator on the Hilbert Hardy space. They are related to tight uniform algebras, the Dunford-Pettis property, and Bourgain algebras.

1 Introduction

Let X be a Banach space and Y a closed subspace of X . For an element g such that $gY \subset X$, we define the operator $S_g : Y \rightarrow X/Y$ by

$$S_g f = gf + Y$$

for all $f \in Y$. The norm is considered as the quotient norm, that is,

$$\|S_g f\| = \|gf + Y\| = \inf\{\|gf + h\| : h \in Y\}$$

for all $f \in Y$. The quotient norm is the distance from gf to Y : $d(gf, Y) = \inf\{\|gf + h\| : h \in Y\}$. This operator is called a Hankel-type operator and is a natural generalization of the classical Hankel operator on the Hilbert Hardy space. Recall that S_g is said to be (*weakly*) *compact* if S_g maps every bounded set into a relatively (weakly) compact one, and that S_g is said to be *completely continuous* if S_g maps every weakly convergent sequence into a norm convergent one. In general, every compact operator is completely continuous. But the converse is not always true. We define the following sets of symbols;

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$$Y_c = \{g : S_g \text{ is compact}\},$$

$$Y_{wc} = \{g : S_g \text{ is weakly compact}\},$$

$$Y_{cc} = \{g : S_g \text{ is completely continuous}\}.$$

The conditions for S_g to be compact, weakly compact and completely continuous have been investigated in various function spaces. The problem of whether all S_g are weakly compact on a uniform algebra is related to a tight algebra [3] and the problem of complete continuity appears in the Dunford-Pettis property. The latter introduced a notion of Bourgain algebras which have been actively researched in analytic and harmonic function spaces on the open unit disk ([1], [2] and [10]). Recently, Dudziak, Gamelin, and Gorkin [5] studied Hankel-type operators on analytic function spaces and Izuchi and the author [9] investigated Hankel-type operators on the space of bounded harmonic functions on the unit disk. See [8] and [13] as surveys for convenience.

We here consider Hankel-type operators on the spaces of analytic functions on the open unit disk, explicitly, the disk algebra, Hardy and Bergman spaces.

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\partial\mathbb{D}$ its boundary. Let $C(\partial\mathbb{D})$ and $C(\overline{\mathbb{D}})$ be the algebras of all continuous functions on $\partial\mathbb{D}$ and $\overline{\mathbb{D}}$ respectively. Let $A(\mathbb{D})$ be the disk algebra of all continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. Then $A(\mathbb{D})$ is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)|; z \in \overline{\mathbb{D}}\}.$$

For $1 \leq p \leq \infty$, let $L^p(\partial\mathbb{D})$ and $L^p(\mathbb{D})$ be the Lebesgue spaces on $\partial\mathbb{D}$ and $\mathbb{D}$ respectively. For $1 \leq p < \infty$, we denote by H^p the classical Hardy space that is the Banach space of all analytic function f on $\mathbb{D}$ for which

$$\|f\|_{H^p} = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty,$$

and denote by L_a^p the Bergman space consisting of all analytic function f on $\mathbb{D}$ for which

$$\|f\|_{L_a^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty$$

where dA is the normalized area measure on $\mathbb{D}$. Let H^∞ be the algebra of bounded analytic functions on $\mathbb{D}$. See [6], [7] and [15] for more information on the Hardy and Bergman spaces.

In the next section, we regard the disk algebra $A(\mathbb{D})$ as a closed subalgebra of $C(\partial\mathbb{D})$ or $C(\overline{\mathbb{D}})$ and H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$ or $L^\infty(\mathbb{D})$ respectively. In section 3, we will consider the case of Hardy space H^p : for $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator $S_g : H^p \rightarrow L^p(\partial\mathbb{D})/H^p$ by $S_g f = gf + H^p$ for $f \in H^p$. Trivially $S_g : H^p \rightarrow L^p(\partial\mathbb{D})/H^p$ is a bounded linear operator. When $p = 2$, let H_g be the classical Hankel operator on H^2 ; for $g \in L^\infty(\partial\mathbb{D})$, $H_g f = gf - P(gf)$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 . It is well known that H_g is compact if and only if $g \in H^\infty + C(\partial\mathbb{D})$. In section 3, Theorem 3.1 says that this equivalence holds on H^p for $1 < p < \infty$. When $p = 1$, Janson, Peetre and Semmes [11] studied the Hankel operator as form $H_b f = \overline{P}(bf)$ where f is analytic polynomial and $\overline{P}$ is the orthogonal projection of L^2 onto $\overline{H^2}$. So we will give the attention to Hankel operators on H^p from the another approach.

On the other hand, in the case of Bergman space, Leucking [12] characterized the compactness of Hankel operators on L_a^p , $1 < p < \infty$. So we will note them in section 4. In section 5 we add the result on the space of bounded harmonic functions and in the last section we pose some open questions.

2 The disk algebra and H^∞

(1) The disk algebra $A(\mathbb{D})$

At first we regard the disk algebra $A = A(\mathbb{D})|_{\partial\mathbb{D}}$ as a closed subalgebra of $C(\partial\mathbb{D})$. For $g \in C(\partial\mathbb{D})$, we define the linear operator $S_g : A \rightarrow C(\partial\mathbb{D})/A$ by

$$S_g f = gf + A \quad \text{for } f \in A.$$

Then we would characterize the sets A_c , A_{wc} and A_{cc} . Each set is a closed subalgebra of $C(\partial\mathbb{D})$.

Theorem 2.1. When we regard $A = A(\mathbb{D})|_{\partial\mathbb{D}}$ as a closed subalgebra of $C(\partial\mathbb{D})$, then

$$A_c = A_{wc} = A_{cc} = C(\partial\mathbb{D}).$$

Proof. It is trivial that $A \subset A_c \subset A_{cc} \subset C(\partial\mathbb{D})$ and that $A_c \subset A_{wc}$. Let $f_n \in A$ with $\|f_n\|_\infty \leq 1$. Then $|f_n(0)| \leq 1$. Thus there exists a subsequence (which we do not relabel) of $\{f_n(0)\}$ such that $f_n(0) \rightarrow c$ for some constant c . Then, since $\bar{z}(f_n - f_n(0)) \in A$,

$$\begin{aligned} \|\bar{z}f_n - \bar{z}c + A\|_\infty &\leq \|\bar{z}f_n - \bar{z}c - \bar{z}(f_n - f_n(0))\|_\infty \\ &= \|\bar{z}(c - f_n(0))\|_\infty \\ &= |c - f_n(0)| \rightarrow 0. \end{aligned}$$

So $S_{\bar{z}}$ is compact and $\bar{z} \in A_c$. Because A_c is a closed subalgebra of $C(\partial\mathbb{D})$,

$$A_c = A_{wc} = A_{cc} = C(\partial\mathbb{D}),$$

by the Stone-Weierstrass theorem. $\square$

We here estimate the norm and the essential norm of S_g . Recall that the essential norm of a bounded linear operator T from Y to X/Y is defined as

$$\|T\|_e = \inf\{\|T + K\| : K \text{ is compact operator from } Y \text{ to } X/Y\}.$$

Using the basic duality relation ([7: Chapter IV]), we have the following.

Theorem 2.2. For $g \in C(\partial\mathbb{D})$, then $\|S_g\| = d(g, A(\mathbb{D})|_{\partial\mathbb{D}})$ and $\|S_g\|_e = 0$.

Secondly we regard $A = A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$. For $g \in C(\overline{\mathbb{D}})$, we define the linear operator $S_g : A \rightarrow C(\overline{\mathbb{D}})/A$ by

$$S_g f = gf + A \quad \text{for } f \in A.$$

Then each set A_c , A_{wc} and A_{cc} is a closed subalgebra of $C(\overline{\mathbb{D}})$ and their equivalence was proved by Cole and Gamelin [3].

Theorem 2.3. When we regard $A(\mathbb{D})$ as a closed subalgebra of $C(\overline{\mathbb{D}})$, then

$$A_c = A_{wc} = A_{cc} = C(\overline{\mathbb{D}}).$$

In this case we also have the following using the basic duality relation.

Theorem 2.4. For $g \in C(\overline{\mathbb{D}})$, then $\|S_g\| = d(g, A(\mathbb{D}))$ and $\|S_g\|_e = 0$.

(2) H^∞

At first we regard H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$. For $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator $S_g : H^\infty \rightarrow L^\infty(\partial\mathbb{D})/H^\infty$ by

$$S_g f = gf + H^\infty \quad \text{for } f \in H^\infty.$$

Then using the fact that H^∞ has the Dunford-Pettis property, Cima, Janson and Yale [1] and Gorkin [8] showed the following.

Theorem 2.5. When we regard H^∞ as a closed subalgebra of $L^\infty(\partial\mathbb{D})$, then

$$H_c^\infty = H_{wc}^\infty = H_{cc}^\infty = H^\infty + C(\partial\mathbb{D}).$$

The estimation of norms is the following.

Theorem 2.6. For $g \in L^\infty(\partial\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and $\|S_g\|_e \leq d(g, H^\infty + C(\partial\mathbb{D}))$.

Secondly we regard H^∞ as a closed subalgebra of $L^\infty(\mathbb{D})$. For $g \in L^\infty(\mathbb{D})$, we define the linear operator $S_g : H^\infty \rightarrow L^\infty(\mathbb{D})/H^\infty$ by

$$S_g f = gf + H^\infty \quad \text{for } f \in H^\infty.$$

Then Cima, Stroethoff and Yale [2] obtained the following result.

Theorem 2.7. When we regard H^∞ as a closed subalgebra of $L^\infty(\mathbb{D})$, then

$$H_c^\infty = H_{wc}^\infty = H_{cc}^\infty = H^\infty + C(\overline{\mathbb{D}}) + V$$

where $V = \{g \in L^\infty(\mathbb{D}) : \|g\chi_{\mathbb{D} \setminus r\mathbb{D}}\| \rightarrow 0 \text{ as } r \rightarrow 1^-\}$.

Furthermore the estimation of norms is the following.

Theorem 2.8. For $g \in L^\infty(\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and $\|S_g\|_e \leq d(g, H^\infty + C(\overline{\mathbb{D}}) + V)$.

3 The case of Hardy spaces H^p for $1 < p < \infty$

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We here consider the case of Hardy spaces. Before starting our discussion, we recall results concerning the topology of Hardy spaces H^p .

Fact 1. ([4: Chap.20, Proposition 3.15]) If $1 < p < \infty$, $f \in H^p$, and f_n is a sequence in H^p , then the following are equivalent.

- (a) $\{f_n\}$ converges weakly to $f \in H^p$.
- (b) $\{f_n\}$ is bounded and $f_n \in H^p$ converges to f uniformly on every compact subset of $\mathbb{D}$.
- (c) $\{f_n\}$ is bounded and $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.
- (d) $\{f_n\}$ is bounded and $f_n^k(0)$ converges to $f^k(0)$ for all $k \geq 0$.

Fact 2. ([4: Chap.20, Proposition 3.16]) Put $S = \{f \in H^1 : \|f\|_{H^1} = 1\}$. Then S is weak*-compact and metrizable, but not weak compact.

For $f_n \in S$, the following are equivalent:

- (i) f_n converges to f in the weak*-topology in H^1 .
- (ii) $f_n(z)$ converges to $f(z)$ for all $z \in \mathbb{D}$.
- (iii) f_n converges to f uniformly on every compact subset of $\mathbb{D}$.

For $1 \leq p < \infty$ and $g \in L^\infty(\partial\mathbb{D})$, we define the linear operator $S_g : H^p \rightarrow L^p(\mathbb{D})/H^p$ by

$$S_g f = gf + H^p \quad \text{for } f \in H^p.$$

Fact 3. For $1 < p < \infty$ and $g \in L^\infty(\partial\mathbb{D})$, the following are equivalent:

- (i) $S_g : H^p \rightarrow L^p(\partial\mathbb{D})/H^p$ is compact (completely continuous).
- (ii) If $\{f_n\}$ is bounded in H^p and converges to 0 uniformly on every compact subset of $\mathbb{D}$, then $\|S_g f_n\| \rightarrow 0$.

For $1 < p < \infty$, H^p is reflexive. So all completely continuous operator on H^p is compact and every bounded operator on H^p is always weakly compact. Thus $H_c^p = H_{cc}^p$ and $H_{wc}^p = L^\infty(\partial\mathbb{D})$.

For $g \in L^\infty(\partial\mathbb{D})$, let H_g be the classical Hankel operator on H^2 defined by $H_g f = gf - P(gf)$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 . Hartman's theorem says that $H_g : H^2 \rightarrow L^2(\partial\mathbb{D})$ is compact if and only if $g \in H^\infty + C(\partial\mathbb{D})$. Then for $1 < p < \infty$, P is bounded from

$L^p(\partial\mathbb{D})$ onto H^p and we can easily see the equivalence of compactness of H_g and S_g . But the next result will give the characterization of Hankel operators on H^p from the another approach.

Theorem 3.1. *For $1 < p < \infty$, the following hold:*

$$H_c^p = H_{cc}^p = H^\infty + C(\partial\mathbb{D}) \quad \text{and} \quad H_{wc}^p = L^\infty(\partial\mathbb{D}).$$

Proof. First, we note that $H^\infty \subset H_c^p = H_{cc}^p \subset L^\infty(\partial\mathbb{D})$. Then $B := H_c^p = H_{cc}^p$ is a closed algebra and so a Douglas algebra.

Suppose that $H^\infty + C(\partial\mathbb{D}) \subsetneq B$. Thus there exists an interpolating Blaschke product $\psi \in H^\infty$ with $\bar{\psi} \in B$. That is, $S_{\bar{\psi}}$ is compact (completely continuous). Write $\psi(z) = e^{i\alpha} \prod_{n=1}^\infty b_n(z)$ where $b_n(z) = (z - z_n)/(1 - \bar{z}_n z)$. Put $f_k(z) = \prod_{n=k}^\infty b_n(z)$. Then $f_k \in H^p$, $\|f_k\|_{H^p} = 1$ and $f_k(z) \rightarrow 0$ for $z \in \mathbb{D}$ as $k \rightarrow \infty$. By Fact 1, $f_k(z) \rightarrow 0$ weakly in H^p . On the other hand, we have

$$\begin{aligned} \|S_{\bar{\psi}} f_k\| &= \|\bar{\psi} f_k + H^p\| \\ &= \|e^{i\alpha} b_1 b_2 \cdots b_{k-1} + H^p\| \\ &= \inf_{f \in H^p} \|1 + e^{i\alpha} b_1 b_2 \cdots b_{k-1} f\|_{H^p} \\ &\geq \inf_{f \in H^p} |1 + e^{i\alpha} (b_1 b_2 \cdots b_{k-1} f)(z)| (1 - |z|^2)^{1/p}, \end{aligned}$$

for $z \in \mathbb{D}$.

Put $z = z_1$, a zero of b_1 . So

$$\|S_{\bar{\psi}} f_k\| \geq (1 - |z_1|^2)^{1/q} > 0.$$

As $S_{\bar{\psi}}$ is completely continuous,

$$\|S_{\bar{\psi}} f_k\| \rightarrow 0.$$

This contradicts. So $B = H^\infty + C(\partial\mathbb{D})$. □

Furthermore the estimation of norms is the following.

Theorem 3.2. *For $1 \leq p < \infty$ and $g \in L^\infty(\mathbb{D})$, then $\|S_g\| = d(g, H^\infty)$ and for $1 < p < \infty$, $\|S_g\|_e \leq d(g, H^\infty + C(\partial\mathbb{D}))$.*

4 The case of Bergman spaces L_a^p for $1 < p < \infty$

For $g \in L^\infty(\mathbb{D})$, let H_g be the classical Hankel operator defined by $H_g f = gf - P(gf)$, where P is the Bergman projection from L^p onto L_a^p .

Then we can easily see the equivalence of compactness of H_g and S_g . On the other hand, Leucking [12] characterized the compactness of Hankel operators on L_a^p , $1 < p < \infty$. And so we have the following.

Theorem 4.1. *For $1 < p < \infty$, then $g \in (L_a^p)_c = (L_a^p)_{cc}$ if and only if g admits a decomposition $g = g_1 + g_2$ so that*

$$\lim_{|z| \rightarrow 1} \frac{1}{|D(z)|} \int_{D(z)} |g_1|^2 dA = 0$$

and

$$g_2 \in C^1(\mathbb{D}), \quad \lim_{|z| \rightarrow 1} (1 - |z|) \bar{\partial} g_2(z) = 0$$

where $D(z)$ is the Bergman disk with center z .

Moreover it holds that $(L_a^p)_{wc} = L^\infty(\mathbb{D})$.

5 The space of bounded harmonic functions

We here consider Hankel-type operators on the space of bounded harmonic functions. Let $h^\infty := h^\infty(\mathbb{D})$ be the set of all bounded harmonic functions on $\mathbb{D}$. It follows that h^∞ is a closed subspace of $L^\infty(\mathbb{D})$. We can define h_c^∞ , h_{wc}^∞ and h_{cc}^∞ as before. The Bourgain algebra h_{cc}^∞ is characterized by Izuchi, Stroethoff and Yale [10].

For a function $f \in L^\infty(\partial\mathbb{D})$, we denote by $\hat{f}$ the Poisson integral of f on $\mathbb{D}$, that is,

$$\hat{f}(z) = \int_0^{2\pi} f(e^{i\theta}) P_z(e^{i\theta}) d\theta / 2\pi,$$

where P_z is the Poisson kernel of $z \in \mathbb{D}$. Then $\hat{f} \in h^\infty$. For any nonempty subset B of $L^\infty(\partial\mathbb{D})$, we write $\hat{B} = \{\hat{f} : f \in B\}$. It is known that f in h^∞ has a boundary function f^* on $\partial\mathbb{D}$ and $\hat{f}^* = f$ on $\mathbb{D}$, so that $h^\infty = \widehat{L^\infty(\partial\mathbb{D})}$. Let $H^\infty(\partial\mathbb{D})$ be the space of boundary functions of

bounded analytic functions on $\mathbb{D}$. The algebra QC of bounded quasi-continuous functions on $\partial\mathbb{D}$ is given by

$$QC = (H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D})) \cap \overline{(H^\infty(\partial\mathbb{D}) + C(\partial\mathbb{D}))}.$$

Refer to [7] and [14] for more information.

The equality $h_{cc}^\infty = \widehat{QC} + V$ was given as Corollary 3 in [10], where V is the same set as in Theorem 2.7. Then Izuchi and the author [9] show the following result.

Theorem 5.1. $h_c^\infty = h_{cc}^\infty = \widehat{QC} + V$.

6 Problems

Problem 6.1. Estimate the essential norms of Hankel-type operators in cases of Hardy and Bergman spaces.

Problem 6.2. How about the case $p = 1$? That is, what are $H_c^1, H_{wc}^1, H_{cc}^1$?

Problem 6.3. Let h^∞ be the space of bounded harmonic functions on $\mathbb{D}$. Then characterize h_{wc}^∞ .

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