

A multiplication on the twisted tensor product

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1 Introduction

Let G be a connected topological group. We define the right adjoint action $ad : G \times G \rightarrow G$ by $ad(g, h) = h^{-1}gh$. Then the cohomology $H^*(G; \mathbf{Z}/l)$ is regarded as a right $H^*(G; \mathbf{Z}/l)$ -comodule under the coaction induced by the adjoint action. The comodule is denoted by $H^*(G; \mathbf{Z}/l)_c$ below. In this note, the algebra structure of

$$E := \text{Cotor}_{H^*(G; \mathbf{Z}/l)}(H^*(G; \mathbf{Z}/l)_c, \mathbf{Z}/l)$$

is considered from the viewpoint of the a differential graded algebra structure of the twisted tensor product due to Brown [1]. The existence of the following three spectral sequences motivates the consideration of the algebra structure of E .

(1) Let $G(\mathbf{F}_q)$ be a finite Chevalley group of Lie type over the finite field $\mathbf{F}_q$ of q elements and l a prime number. By applying the Deligne spectral sequence in the case where the characteristic of $\mathbf{F}_q$ is prime to l , Tezuka [7] has constructed a spectral sequence converging to $H^*(BG(\mathbf{F}_q); \mathbf{Z}/l)$. In particular if $q - 1 \equiv 0$ modulo l , then the E_2 -term of the spectral sequence is isomorphic to E as an algebra for many cases.

(2) Let BLG be the classifying space of the loop group LG consisting of all continuous maps from the circle to G . Then there exists the

Eilenberg-Moore spectral sequence, whose E_2 -term is isomorphic to E as an algebra, converging to $H^*(BLG; \mathbf{Z}/l)$.

(3) Let X be a simply connected finite CW-complex. Following Milnor's description of universal bundles over a space, we can regard the loop space ΩX , which is the subspace of the free loop space LX consisting of based loops, as a topological group G . Therefore we have the Eilenberg-Moore spectral sequence converging to $H^*(LX; \mathbf{Z}/l)$ with $E_2 \cong E$ as an algebra.

One will know that it is important to clarify the algebra structure of E as the first step in computing those spectral sequences.

Let G be a connected complex Lie group with the same Lie type as that of a finite Chevalley group $G(\mathbf{F}_q)$. As for the cohomology algebras of $BG(\mathbf{F}_q)$ and BLG , Tezuka [15] has proposed a problem whether the cohomologies $H^*(BG(\mathbf{F}_q); \mathbf{Z}/l)$ and $H^*(BLG; \mathbf{Z}/l)$ are isomorphic as an algebra in the case where l is odd and divides $q - 1$ but does not divide q or $l = 2$ and 4 divides $q - 1$. As mentioned in [15], the answer is affirmative if the integral cohomology of G has no l -torsion. The main theorem in [6] and the explicit calculation of $H^*(BG(\mathbf{F}_q); \mathbf{Z}/l)$ due to Kleinerman [3] guarantee the result. To shed light on left part of the problem, we will consider the structure of E for the case where $H^*(G; \mathbf{Z})$ has l -torsion.

2 Results

Before stating our results, we recall a construction of the twisted tensor product due to Brown (see [1], [14] or [4]). Let A be a coalgebra over $\mathbf{Z}/l$ with coproduct ϕ_A and augmentation ε . Let L be a $\mathbf{Z}/lp$ -subspace of A , $\iota : L \rightarrow A$ the inclusion and $\theta : A \rightarrow L$ a map such that $\theta \circ \iota = id_L$. We define the map $\bar{\theta} : A \rightarrow sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota} : sL \rightarrow A$ by $\bar{\iota} = \iota \circ s^{-1}$, where $s : L \rightarrow sL$ is a suspension. Construct the tensor product $X = T(sL)$ and denote by ψ the product in $T(sL)$. The map $\bar{\theta}$ induces a map $A \rightarrow T(sL)$ which is again denoted by $\bar{\theta}$. Let I be the ideal of $T(sL)$ generated by $(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A) (\ker \bar{\theta})$. The twisted tensor product (W, d) with respect to $\bar{\theta}$ is defined as follows; we put

$W = A \otimes X/I = A \otimes \bar{X}$ and define the differential operator d_W by

$$d_W = 1 \otimes d_{\bar{X}} + (1 \otimes \psi) \circ (1 \otimes \bar{\theta} \otimes 1) \circ (\phi_A \otimes 1), \text{ where} \\ d_{\bar{X}} = -\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A \circ \bar{\iota}.$$

We may denote the twisted tensor product W with respect to $\bar{\theta} : A \rightarrow sL$ by $A \otimes_{\theta} \bar{X}$.

Let G be a compact, simply connected, simple exceptional Lie group. Then it is known [9] that a suitable choice of a subspace L of $H^*(G; \mathbf{Z}/l)$ makes the twisted tensor product into an injective resolution $0 \rightarrow \mathbf{Z}/l \rightarrow H^*(G; \mathbf{Z}/l) \otimes_{\theta} \bar{X}$ over the coalgebra A . Moreover the algebra structure of $\bar{X}$ induces that of the complex

$$(\mathbf{Z}/l \square_{H^*(G; \mathbf{Z}/l)} (H^*(G; \mathbf{Z}/l) \otimes_{\theta} \bar{X}), 1 \square d_W) \cong (\bar{X}, d_{\bar{X}})$$

Consequently we have

$$\text{Cotor}_{H^*(G; \mathbf{Z}/l)}(\mathbf{Z}/l, \mathbf{Z}/l) \cong H(\bar{X}, d_{\bar{X}}) \text{ as an algebra.}$$

In this note, we consider a multiplication m_W on the twisted tensor product $A \otimes_{\theta} \bar{X}$ for a Hopf algebra A , in the sense of Milnor and Moore [8], such that the differential d_W is derivative under the multiplication. In order to define a multiplication m_W explicitly, we will assume that the $\mathbf{Z}/l$ -subspace L of A satisfies the following condition.

(I) *There exist the set Q of indecomposable elements of A and a basis $\{x_i\}$ of L such that $\{x_i\} \subset Q \cup Q^2$, where $Q^2 = \{\alpha^2 | \alpha \in Q \cap \text{Prim } A\}$ and, as an algebra,*

$$A \cong \bigotimes_{x_s \in S} \mathbf{Z}/p[x_s]/(x_s^{p^{n_s}}) \otimes \bigotimes_{x_t \in T} \Lambda(x_t),$$

where $S \cup T = Q \cap \{x_i\}$ and $S \cap T = \emptyset$. Moreover, we also assume that

(II) $(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)(\ker \bar{\theta}) = \mathbf{Z}/l\{(\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi_A)(x_i x_j) | x_i, x_j \in \{x_i\}, i \neq j\}$,

(III) for any $a \in Q$, $\bar{\theta}(y a_i'') = 0$ for any $y \in \bar{A}$, where $\phi_A(a) = \sum_i a_i' \otimes a_i'' + a \otimes 1 + 1 \otimes a$ and that

(IV) for any x and $y \in \{x_i\}$, $\bar{\theta}(xy) \neq 0$ if and only if $x = y$ and

$$x^2 \in Q^2.$$

We mention here that the conditions (I), (II) (III) and (IV) hold in the cases $(PU(3), 3)$, $(F_4, 3)$, $(E_8, 3)$, (E_6, p) , (E_7, p) for $l = 2$ and 3 which have been studied by Kono, Mimura, Sambe and Shimada ([4],[5], [10], [11]).

The following is one of the our main theorem.

Theorem 2.1. *Let A be a Hopf algebra over $\mathbf{Z}/l$. For any elements $a \otimes \theta x$ and $b \otimes \theta y$ of $A \otimes_\theta \bar{X}$, define $m_W : A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X} \rightarrow A \otimes_\theta \bar{X}$ by*

$$m_W(a \otimes \theta x \otimes b \otimes \theta y) = a \otimes \theta x \cdot b \otimes \theta y = \sum_i (-1)^{|\theta x||b'_i|} ab'_i \otimes \theta(xb''_i)\theta y,$$

and

$$(\theta x_1 \cdots \theta x_s) \cdot a = (\theta x_1(\theta x_2(\cdots (\theta x_s \cdot a)) \cdots)),$$

where $\phi_A(b) = \sum_i b'_i \otimes b''_i$. If m_W is well-defined, then $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a differential graded algebra.

By comparing the differential algebra structure of the cobar resolution [13, 7.A, 1.2] of the left A -comodule $\mathbf{Z}/l$ and that of the twisted tensor product mentioned above, we can prove Theorem 1.

Theorem 2.2. *If $l = 2$ or 3 and the condition (I), (II), (III) and (IV) hold, then the multiplication m_W is well-defined.*

In the case where $A = H^*(E_8; \mathbf{Z}/5)$, explicit calculation for the differential d_W and the multiplication m_W on $A \otimes_\theta \bar{X}$ allow us to obtain the following theorem.

Theorem 2.3. *Let $A \otimes_\theta \bar{X}$ be the twisted tensor product of $H^*(E_8; \mathbf{Z}/5)$ constructed in [12]. Then $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a well-defined differential graded algebra.*

In the case where $A = H^*(E_8; \mathbf{Z}/2)$, indecomposable elements x on A can be chosen so that $\bar{\Delta}(x)$ is in $P \otimes P$, where P is the $\mathbf{Z}/2$ -subspace of A consisting of primitive elements. Thanks to this fact, we can easily verify that the multiplication m_W is well-defined.

Theorem 2.4. *Let $A \otimes_\theta \bar{X}$ be the twisted tensor product of $H^*(E_8; \mathbf{Z}/2)$ constructed in [9]. Then $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a well-defined differential graded algebra.*

In order to prove that the multiplication m_W induces the algebra structure on $\text{Cotor}_A(A, \mathbf{Z}/p)$, it suffices to prove

Proposition 2.5. *Let p be a prime number and $\mu : A \otimes A \rightarrow A$ the multiplication of A . Then the map $m_W : A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X} \rightarrow A \otimes_\theta \bar{X}$ is a μ -morphism if m_W is well-defined, that is, the following diagram is commutative:*

$$\begin{array}{ccc} A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X} & \xrightarrow{\psi_1} & (A \otimes A) \otimes A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X} \\ m_W \downarrow & & \downarrow \mu \otimes m_W \\ A \otimes_\theta \bar{X} & \xrightarrow[\psi_2]{} & A \otimes A \otimes_\theta \bar{X}, \end{array}$$

where ψ_1 and ψ_2 are the comodule structures of $A \otimes_\theta \bar{X} \otimes A \otimes_\theta \bar{X}$ and $A \otimes_\theta \bar{X}$ respectively.

Let A denote the mod l cohomology $H^*(G; \mathbf{Z}/p)$. Since $ad^* \otimes 1 : A \otimes \bar{X} \rightarrow A \square_A (A \otimes \bar{X})$ is the isomorphism with the inverse $1 \otimes \varepsilon \otimes 1$, we can define a differential on $A \otimes \bar{X}$ by the compositions

$$A \otimes \bar{X} \xrightarrow{ad^* \otimes 1} A \square_A (A \otimes \bar{X}) \xrightarrow{inc} A \otimes (A \otimes \bar{X}) \xrightarrow{1 \otimes d_W} A \otimes (A \otimes \bar{X}) \xrightarrow{1 \otimes \varepsilon \otimes 1} A \otimes \bar{X}.$$

A straightforward calculation for the differential $d : A \otimes \bar{X} \rightarrow A \otimes \bar{X}$ enables us to obtain the following explicit formula for d .

Lemma 2.6. *We write as $\Delta_A(x) = x \otimes 1 + 1 \otimes x + \sum_i x'_i \otimes x''_i$ for $x \in A$. If x'_i is primitive for any i , then*

$$dx = - \sum_i (-1)^{|x''_i|(|x'_i|+1)} x''_i \otimes \theta x'_i + \sum_i (-1)^{|x'_i|} x'_i \otimes \theta x''_i.$$

The multiplication m_W on the twisted tensor product $A \otimes_\theta \bar{X}$ induces a multiplication m on $A \otimes \bar{X}$ defined by

$$\begin{aligned} A \otimes \bar{X} \otimes A \otimes \bar{X} &\xrightarrow{ad^* \otimes 1 \otimes qd^* \otimes 1} A \square_A (A \otimes \bar{X}) \otimes A \square_A (A \otimes \bar{X}) \xrightarrow{inc} \\ A \otimes (A \otimes \bar{X}) \otimes A \otimes (A \otimes \bar{X}) &\longrightarrow A \otimes A \otimes (A \otimes \bar{X}) \otimes (A \otimes \bar{X}) \xrightarrow{m_A \otimes m_W} \\ &A \otimes (A \otimes \bar{X}) \xrightarrow{1 \otimes \varepsilon \otimes 1} A \otimes \bar{X}. \end{aligned}$$

We can obtain an explicit formula for the multiplication m on $A \otimes \bar{X}$.

Lemma 2.7. *We write as $\Delta_A(a) = a \otimes 1 + 1 \otimes a + \sum_i a'_i \otimes a''_i$ for $a \in A$. If a'_i is primitive for any i , then*

$$\begin{aligned} \theta x \cdot a &= (-1)^{|\theta x||a|} a \otimes \theta x - \sum_i (-1)^{|a''_i||a'_i|+|a''_i||\theta x|} a''_i \otimes \theta(xa'_i) \\ &\quad + \sum_i (-1)^{|a'_i||\theta x|} a'_i \otimes \theta(xa''_i). \end{aligned}$$

Thus we can obtain a differential graded algebra $(A \otimes \bar{X}, d, m)$. From the construction of this differential graded algebra, we have

Theorem 2.8. *For the case where $A = H^*(G; \mathbf{Z}/l)$, if the twisted tensor product $(A \otimes_\theta \bar{X}, d_W, m_W)$ is a well-defined differential graded algebra, then, as an algebra,*

$$\text{Cotor}_{H^*(G; \mathbf{Z}/l)}(H^*(G; \mathbf{Z}/l)_c, \mathbf{Z}/l) \cong H(A \otimes \bar{X}, d, m).$$

The proofs of theorems and propositions in this note will be given in a further article [7].

This note will be concluded with some examples of the differential graded algebras $A \square_A (A \otimes_\theta \bar{X})$ for computing the algebras $\text{Cotor}_A(A, \mathbf{Z}/l)$.

The case $(G, p) = (PU(3), 3)$.

$$W' = A \square_A (A \otimes_\theta \bar{X}) = \mathbf{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1, x_3) \otimes \mathbf{Z}/3\{a_2, a_3, c_5, b_4\}/I,$$

$$\begin{aligned} db_4 &= -a_2a_3, \quad dc_5 = a_3^2, \\ d(x_3) &= x_2 \otimes a_2 + x_1 \otimes a_3, \end{aligned}$$

$$a_3 \cdot x_3 = -x_3 \otimes a_3 + x_1 \otimes c_5.$$

Therefore, we have, as a $\text{Cotor}_{H^*(PU(3); \mathbf{Z}/3)}(\mathbf{Z}/3, \mathbf{Z}/3)$ -module,

$$\begin{aligned} &\text{Cotor}_{H^*(PU(3); \mathbf{Z}/3)}(H^*(PU(3); \mathbf{Z}/3), \mathbf{Z}/3) \cong \\ &\{\mathbf{Z}/3[x_2]/(x_2^3) \otimes \Lambda(x_1) \otimes \mathbf{Z}/3[y_2, y_3, y_7, y_8, y_{12}]/(y_2y_3, y_3^2, y_2y_7, y_7^2, \\ &\quad y_2y_8 + y_3y_7) \\ &\quad \oplus x_3 \cdot (x_1x_2^2, x_1y_7, x_1y_8 + x_2y_7, x_2^2y_2, y_3)\} / (x_2y_2 + x_1y_3). \end{aligned}$$

The case $(G, p) = (F_4, 3)$.

$$W' = A \square_A (A \otimes_{\theta} \bar{X}) = \mathbf{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}) \\ \otimes \mathbf{Z}/3\{a_4, a_8, a_9, b_{12}, b_{16}, c_{17}\}/I,$$

$$d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15), \\ d|_{\mathbf{Z}/3\{ \}}/I = \text{the ordinary differential on } \mathbf{Z}/3\{ \}/I,$$

$$a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15).$$

The case $(G, p) = (E_6, 3)$.

$$W' = A \square_A (A \otimes_{\theta} \bar{X}) = \\ \mathbf{Z}/3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}) \otimes \\ \mathbf{Z}/3\{a_4, a_8, a_9, a_{10}, b_{12}, b_{16}, b_{18}, c_{17}\}/I,$$

$$d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15, 17), \\ d|_{\mathbf{Z}/3\{ \}}/I = \text{the ordinary differential on } \mathbf{Z}/3\{ \}/I,$$

$$a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15, 17).$$

The case $(E_7, 3)$.

$$W' = A \square_A (A \otimes_{\theta} \bar{X}) = \mathbf{Z}/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}) \\ \otimes \mathbf{Z}/3\{a_4, a_8, a_9, a_{20}, b_{12}, b_{16}, b_{28}, c_{17}, e_{36}\}/I,$$

$$d(x_j) = x_8 \otimes a_{j-8+1} + x_{j-8} \otimes a_9 \quad (j = 11, 15, 27), \\ d(x_{35}) = x_8 \otimes b_{28} + x_{27} \otimes a_9 - x_8^2 \otimes a_{20} + x_{19} \otimes c_{17}, \\ d|_{\mathbf{Z}/3\{ \}}/I = \text{the ordinary differential on } \mathbf{Z}/3\{ \}/I,$$

$$a_9 \cdot x_j = -x_j \otimes a_9 + x_{j-8} \otimes c_{17} \quad (j = 11, 15, 27, 35).$$

The case $(E_8, 3)$.

$$W' = A \square_A (A \otimes_{\theta} \bar{X}) = \\ \mathbf{Z}/3[x_8, x_{20}]/(x_8^3, x_{20}^3) \otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47}) \\ \otimes \mathbf{Z}/3\{a_4, a_8, a_9, a_{20}, a_{21}, c_{17}, c_{41}, b_{16}, b_{40}, d_{28}, e_{36}, e_{48}\}/I,$$

$$d(x_{15}) = x_8 \otimes a_8 + x_7 \otimes a_9, \quad d(x_{39}) = x_{20} \otimes a_{20} + x_{19} \otimes a_{21}, \\ d(x_{27}) = x_8 \otimes a_{20} + x_{19} \otimes a_9 + x_{20} \otimes a_8 + x_7 \otimes a_{21},$$

$$d(x_{35}) = x_8 \otimes d_{28} + x_{27} \otimes a_9 - x_8^2 \otimes a_{20} + x_{19} \otimes c_{17} + x_{20} \otimes b_{16} \\ + x_{15} \otimes a_{21} + x_{20}x_8 \otimes a_8,$$

$$d(x_{47}) = x_8 \otimes b_{40} + x_{39} \otimes a_8 + x_{20} \otimes d_{28} + x_{27} \otimes a_{21} + x_7 \otimes c_{41} \\ - x_{20}^2 \otimes a_8 + x_{20}x_8 \otimes a_{20},$$

$d|_{\mathbf{Z}/3\{ \}}/I$ = the ordinary differential on $\mathbf{Z}/3\{ \}/I$,

$$a_9 \cdot x_{15} = -x_{15} \otimes a_9 + x_7 \otimes c_{17}, \quad a_{21} \cdot x_{39} = -x_{39} \otimes a_{21} + x_{19} \otimes c_{41},$$

$$a_9 \cdot x_{27} = -x_{27} \otimes a_9 + x_{19} \otimes c_{17}, \quad a_{21} \cdot x_{27} = -x_{27} \otimes a_{21} + x_7 \otimes c_{41},$$

$$a_9 \cdot x_{35} = -x_{35} \otimes a_9 + x_{27} \otimes c_{17}, \quad a_{21} \cdot x_{35} = -x_{35} \otimes a_{21} + x_{15} \otimes c_{41},$$

$$a_9 \cdot x_{47} = -x_{47} \otimes a_9 + x_{39} \otimes c_{17}, \quad a_{21} \cdot x_{47} = -x_{47} \otimes a_{21} + x_{27} \otimes c_{41}.$$

The differential operator d and the bracket $[,]$ are trivial on the generators if they are not indicated above.

References

- [1] E. H. Brown, Jr., Twisted tensor products, I, Ann. Math., **69**(1959), 223-246.
- [2] E. M. Friedlander, Computations of K-theories of finite fields, Topology, **15**(1976), 87-109.
- [3] S. Kleinerman, The cohomology of Chevalley groups of exceptional Lie type, Memoirs of AMS, **268**(1982).
- [4] A. Kono, M. Mimura and N. Shimada, Cohomology of classifying space of certain associative H -spaces, J. Math. Kyoto Univ., **15**(1975), 607-617.
- [5] A. Kono, M. Mimura and N. Shimada, On the cohomology mod 2 of the classifying space of the 1-connected exceptional Lie group E_7 , J. Pure and Applied Algebra **8**(1977), 755-776.
- [6] K. Kono and K. Kozima, The adjoint action of a Lie group on the space on loops, J. Math. Soc. Japan, **45**(1993), 495-510.
- [7] K. Kuribayashi, M. Mimura and M. Tezuka, On the cohomology of finite Chevalley groups, (In preparation).

- [8] J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. of Math.*, **81**(1965), 211-236.
- [9] M. Mimura, The characteristic classes for the exceptional Lie groups, *LMS Lecture Note Series*, **175**(1992), 103-130.
- [10] M. Mimura and Y. Sambe, On the cohomology mod p of the classifying spaces of the exceptional Lie groups, I, *J. Math. Kyoto Univ.*, **19**(1979), 553-581.
- [11] M. Mimura and Y. Sambe, On the cohomology mod p of the classifying spaces of the exceptional Lie groups, II, *J. Math. Kyoto Univ.*, **20**(1980), 327-349.
- [12] M. Mimura and Y. Sambe, Collapsing of Eilenberg-Moore spectral sequence mod 5 of the compact exceptional group E_8 , *J. Math. Kyoto Univ.*, **21**(1981), 203-230.
- [13] D.C.Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press,(1986).
- [14] N. Shimada and A. Iwai, On the cohomology of some Hopf algebras, *Nagoya Math. J.*, **30**(1967), 103-111.
- [15] M. Tezuka, On the cohomology of finite Chevalley groups and free loop spaces of classifying spaces, *Cohomology of Finite Groups and Related Topics*(Japanese), *KIMS Kokyuroku*, **1057**(1998), 54-55.