

Ph. D. Thesis

**Magnetized twisted orbifold models for
the origin of generation and chirality**

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Abstract

The Standard Model (SM) is completed by the discovery of the Higgs particle and the measurement of its mass. The SM has achieved great successes for explaining many experimental results, e.g., non-zero gauge boson masses, suppression of flavor changing neutral currents, and CP-violating processes. However, there are some puzzles which are not explained in the SM as follows: The origin of the generation, the chiral structure, the mass hierarchy of the matters, the gauge structure, and the gravitational interaction. In this thesis, we focus on former four puzzles and investigate models which can solve them.

One of the attractive ideas for such model-building is higher-dimensional field theory where the dimension of the space-time is more than four. We obtain four-dimensional (4D) effective field theory by performing dimensional reduction. The advantages of this scenario are the follows: First, the 4D scalar field arises from higher-dimensional vector field. Second, hierarchical Yukawa couplings may be obtained by overlap integrals of extra-dimensional wavefunctions.

In addition to higher-dimensional theory, toroidal compactification with constant magnetic flux is useful for solving some of the puzzles which are mentioned in the above. On this background, we can solve the zero-mode equation of Kaluza–Klein modes analytically. This analyticity helps us to perform overlap integrals and to compute the Yukawa couplings in the effective theory. Furthermore, we obtain the solution which is chiral and degenerate. Thanks to these properties of the solution, three generations of chiral fermions can be realized in the effective theory.

We also consider the twisted orbifolding. This is useful for realizing actual flavor structure as well as for projecting out exotic particles. We analyze the system on magnetized twisted orbifold $\mathbb{T}^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$) by using the operator formalism and investigate the number of independent zero-modes on the orbifold for each $\mathbb{Z}_N$ eigenvalues.

Based on these aspects, we propose the ten-dimensional (10D) super Yang–Mills (SYM) theory on magnetized twisted orbifolds as a framework for solving the puzzles in the SM. 10D SYM is a well-motivated higher-dimensional theory since this appears as low-energy limit of some types of superstring theories. By using the results obtained by the above investigation, we classify all the possible configurations of the parameters which realize three generations of chiral fermions in 4D effective theory. This classification is a first step for building realistic models with three generations, chiral fermions, and hierarchical Yukawa couplings.

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Chapter 1

Introduction

The Standard Model (SM) of particle physics [1–8] is completed by the discovery of the Higgs particle and the measurement of its mass at the Large Hadron Collider [9, 10]. The SM has reproduced many experimental results with high accuracy as follows:

- Gauge boson masses [2]:
Non-zero gauge boson masses in the SM are explained by the Higgs mechanism.
- g -factor of the electron:
Perturbative calculation in the SM reproduces its experimental value with the amazing accuracy of $O(10^{-12})$.
- Suppression of flavor changing neutral currents [5]:
Flavor changing neutral currents are suppressed at tree-level.
- Anomaly cancellation [6]:
Anomalies arising from chiral fermions is canceled among the quarks and the leptons.
- CP-violation [7]:
CP-violating processes such as $K_L \rightarrow \pi^+\pi^-$ are explained.

While the SM has achieved these great successes, there are following puzzles which are not explained in the SM:

- Why do three generations of the matters exist?
- Why do the left-handed and the right-handed matters interact differently?
- Why do the masses of the matters differ greatly among the generations?
- Why is the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ chosen?
- Why is not the gravitational interaction included in the SM?

Solving these puzzles is an important problem of particle physics. In this thesis, we focus on matter and Higgs sectors of the SM, and propose a desirable framework for solving former four puzzles. The virtue of this framework is that this can be obtained as the low-energy limit of the superstring

theories and solves the above puzzles by simple and calculable ways. We will explain this in the below.

One of attractive ideas for solving the puzzles is higher-dimensional field theory, which is the field theory defined on space-time whose dimension is more than four. Historically, higher-dimensional field theory was proposed by T. Kaluza [11] and O. Klein [12] for unifying the gravitational and the electromagnetic interactions. A. Einstein also has investigated higher-dimensional field theory in same direction [13, 14]. Nowadays the superstring theories are also a kind of higher-dimensional field theory where the space-time has ten dimensions. These are almost the only known candidates which can describe the gravitational interaction consistently at quantum level. In $(4 + d)$ -dimensional field theory (d is a positive integer), we obtain four-dimensional (4D) effective field theory through dimensional reduction. That is, 4D effective Lagrangian $\mathcal{L}_{4\text{D effective}}$ is obtained by integrating $(4 + d)$ -dimensional Lagrangian $\mathcal{L}_{4+d}$ over extra dimensions,

$$\mathcal{L}_{4\text{D effective}} = \int d^d y \mathcal{L}_{4+d},$$

where y is the coordinates of d -dimensional compact space. Higher-dimensional field theory has the following desirable features:

- The first feature is that infinite number of fields called Kaluza–Klein (KK) modes appear in effective theory. Every higher-dimensional field generates such KK modes by the compactification of extra dimensions. It is noted that the mode function (which is obtained by solving the equation of motion on the extra dimensions) and the mass of the KK modes depend on the compactification. For example, in five-dimensional field theory, which is the simplest higher-dimensional field theory, the mode function for circle compactification (with periodic boundary condition) is in the form of $e^{in\theta}$ where n ($\in \mathbb{Z}$) is the mode index and θ ($0 \leq \theta \leq 2\pi$) is the coordinate of the fifth dimension, while the mode function for line segment compactification (with Dirichlet boundary condition) is in the form of $\sin(2\pi n'y/L)$ where n' ($\in \mathbb{N}$) is the mode index, L is the length of the fifth dimension, and y ($0 \leq y \leq L$) is the coordinate of the fifth dimension.
- The second feature is that the extra-dimensional components of vector fields behave as scalars in 4D. For example, in 5D field theory, the fifth dimensional component of the vector field, A_5 , becomes a scalar field in 4D, while A_μ ($\mu = 0, 1, 2, 3$) behaves as a 4D vector field. The gauge-Higgs unification [15, 16] is a scenario based on this feature, where the extra-dimensional components of the gauge field are identified with the Higgs field. The advantage of this scenario is that the quadratic divergence of quantum correction to the Higgs mass is controlled by the higher-dimensional gauge symmetry [17, 18].
- The third feature is that 4D effective coupling constants are expressed as overlap integrals of mode functions. This feature is very helpful for solving the puzzle of the mass hierarchy of the matters as follows: 4D effective Yukawa couplings arise from higher-dimensional gauge vertex $\Psi \Gamma^m A_m \Psi$ ($m = 5, 6, \dots, 4 + d$), where Ψ is a fermion field, A_m are extra-dimensional components of the gauge field, and Γ^m are gamma matrices. Therefore, effective Yukawa couplings become hierarchical if the mode functions of Ψ and A_m have quasi-localized profile.

On the other hand, how the extra dimensions are compactified is essential for realizing chiral fermions in 4D effective field theory. For example, toroidal compactification in 6D field theory leads to non-chiral 4D effective theory. On the other hand, point interaction on S^1 in 5D field theory can realize chiral fermions [19–21]. Thus, it is phenomenologically important to find the compactification which leads to chiral fermions in 4D effective theory as required in the SM.

Toroidal compactification with constant magnetic flux is a prescription for deriving chiral 4D effective theory [22–26]. It is well known that the non-trivial gauge background, such as constant magnetic flux, in higher-dimensional field theory leads to chiral fermions in 4D effective theory [27–29]. In this thesis, we consider 2-torus with homogeneous magnetic field for extradimensional geometry. In order to define the theory on this geometry, we should determine appropriate boundary conditions of fields for the discrete translation along two basis of the torus, which we call “torus-shift” operation. The single-valuedness of fields on the torus leads to the quantization of the magnetic flux. On this background, we can solve the equation of motion for KK zero-mode analytically.¹ This opens a way to the calculation of the Yukawa couplings which are expressed as overlap integrals. The zero-mode solution, described by Jacobi’s theta function (the definition of this function is in Eq.(3.1.35)), has following properties:

- The zero-mode solution is degenerate:
The number of independent solution is determined by the magnetic flux and equal to the number of the generation. This property leads to generations of the matters in 4D effective theory.
- The zero-mode solution is chiral:
2D spinor has two components, but only one component becomes a physical solution. Whether a component becomes physical is determined by the sign of the magnetic flux. This property leads to chiral fermions in 4D effective theory.
- The zero-mode solution has Gaussian-like quasi-localized profile:
This leads to hierarchical Yukawa couplings as is mentioned in the above.

For these reasons, this scenario is highly desirable for solving the puzzles in the SM. However, since the magnetic flux is restricted to the certain values for realizing three generations of the matters, the flavor structure is limited to particular type. Therefore, we need further idea for resolving this disadvantage and obtaining realistic flavor structure.

Orbifolding is also helpful in actual model-buildings. Orbifold is defined by the identification of points on a manifold under certain discrete operation. For example, on 2D torus, shifted orbifold is defined by the identification under the discrete translation of $\mathbf{u}_1/N$ where $\mathbf{u}_1$ is one of the basis vectors of the torus (see Chapter 3), and twisted orbifold is defined by the identification under $2\pi/N$ -radians discrete rotation. In physics, orbifold was firstly proposed in stringy context [31,32], and has been considered in various model-buildings. For example, shifted orbifold with magnetic

¹ This is an eigenvalue problem and the equation of motion can be solved by using group theoretic method. See Ref. [30] and references therein.

flux was studied in Ref. [33]. In this thesis, as we mentioned, we consider twisted orbifold on 2-torus with magnetic flux. The advantages of twisted orbifolding are followings:

- Removing exotic particles.
In general, particles which are not in the SM matter content appear in model-buildings. By using orbifold, we can remove such particles called exotic particles.
- Deforming the profile of zero-mode solution.
Orbifolding deforms the profile of the mode functions (in particular, zero-mode functions) by changing the geometry of extra dimensions. This is useful for realizing actual flavor structure which is observed in experiments.

It is known that only $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifoldings are possible on 2D torus $\mathbb{T}^2$ [34]. The simplest case, magnetized $\mathbb{T}^2/\mathbb{Z}_2$ without the Wilson-line phase [35–37] and the Scherk–Schwarz phases [38, 39], was studied in Refs. [40, 41]. However, more complicated cases, namely $\mathbb{T}^2/\mathbb{Z}_3, \mathbb{T}^2/\mathbb{Z}_4, \mathbb{T}^2/\mathbb{Z}_6$ with magnetic flux, have been studied little. This is because the $\mathbb{Z}_N$ -twisted eigenfunctions consist of linear combinations of Jacobi’s theta function and have very complicated forms. Therefore, it was needed to rely almost entirely on numerical calculation for analyzing these orbifolds.

We investigated all the possible orbifolds, magnetized $\mathbb{T}^2/\mathbb{Z}_{2,3,4,6}$ with non-trivial Wilson-line phase and the Scherk–Schwarz phases, and resolved the difficulty of magnetized orbifolds by introducing a new method [42, 43]. In these references, we find that the Wilson-line phase and the Scherk–Schwarz phases have the following features:

1. The Wilson-line phase can be absorbed into the Scherk–Schwarz phases by the redefinition of the fields and vice versa (note that this can be performed in the case without orbifolding).
2. These phases are restricted to the particular values on magnetized orbifolds. Although such a restriction is found on $\mathbb{T}^2/\mathbb{Z}_N$ without magnetic flux, these phases should take non-zero values on some of magnetized orbifolds in order to make the system consistent.
3. The number of independent solutions depends not only on the magnetic flux and the $\mathbb{Z}_N$ eigenvalue but also on these phases. In other words, the number of generation in 4D effective theory is determined by the magnetic flux, the Wilson-line phase (or the Scherk–Schwarz phases), and the $\mathbb{Z}_N$ eigenvalue.

The method introduced in Ref. [43] is called the operator formalism analysis, which is dual method to the analysis by the wavefunction formalism. The advantage of this method is to provide exact and analytic expression of $\mathbb{Z}_N$ -twisted eigenstates. We investigated the number of generation on magnetized $\mathbb{T}^2/\mathbb{Z}_N$ by using this analytic expression.

Based on the results obtained in Refs. [42, 43], we start with the ten-dimensional (10D) super Yang–Mills (SYM) theory on magnetized twisted orbifolds and construct models for explaining the puzzles in the SM which are mentioned in the above. 10D SYM theory is a well-motivated higher-dimensional field theory, since this theory appears as low-energy limit of Type I, Type IIB and heterotic superstring theories. From phenomenological point of view, 10D SYM on magnetized twisted orbifolds has the following advantages:

- By introducing Abelian magnetic flux which is in a form of diagonal matrix in the gauge group space, the original gauge group is explicitly broken down into smaller gauge groups. Therefore, we can obtain the SM gauge group (up to $U(1)$ factors) by this mechanism.
- The SM particles arise from only two fields, the gauge field and the gaugino field in 10D. The SM fermions and the MSSM-like higgsinos arise from the off-diagonal parts of the gaugino, while the Higgs fields which are the supersymmetric partners of the higgsinos arise from extra-dimensional components of the gauge field.
- Since the off-diagonal parts of the gaugino possess “effective” magnetic flux through the gauge covariant derivative, three generations of the chiral fermions can be realized [44]. This is, of course, thanks to the introduction of the magnetic flux.
- Hierarchical Yukawa couplings and realistic flavor structure can be also realized [45–48]. The former originates from overlap integrals in higher-dimensional field theory, and the later is thanks to the twisted orbifold.

In this thesis, we assume $U(8)$ as the original gauge group of 10D SYM since this group includes Pati–Salam gauge group $U(4) \times U(2)_L \times U(2)_R$. In other words, we assume explicit gauge symmetry breaking pattern $U(8) \rightarrow U(4) \times U(2)_L \times U(2)_R$ by the magnetic flux. We assume such gauge background configuration, and classify all the possible combinations of the parameters which realize three generations of the matters by using the results obtained by the analysis in the above [49].

This thesis is organized as follows: In Chapter 2, we review the Standard Model briefly. Here, we see remaining puzzles in the SM and present some of approaches for explaining these puzzles. In Chapter 3, in order to investigate the aspects of the approaches mentioned in Chapter 2 in detail, we study six-dimensional $U(1)$ gauge theory on torus $\mathbb{T}^2$ or possible twisted orbifolds $\mathbb{T}^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$). Here, we solve zero-mode equations for KK zero-mode of fermion with non-trivial SS phases and the Wilson-line phase. We show that the zero-mode solution, which are described by Jacobi’s theta function, has desirable properties which are helpful for solving the puzzles in the SM, that is, degenerate, chiral, and quasi-localized profile. Moreover, we show that the SS phases and/or the Wilson line phase are absorbed into each other. In addition, we show that these phases are restricted to particular values on twisted orbifolds. In Chapter 4, we study the operator formalism analysis. This method resolves the difficulty of the analysis in Chapter 3. We obtain the exact and analytic expression of $\mathbb{Z}_N$ eigenstates given by the linear combination of the eigenstates on magnetized $\mathbb{T}^2$. By using the analytic expression, we investigate the number of independent eigenstates on twisted orbifolds which is equal to the number of generation. In Chapter 5, based on the aspects obtained in Chapter 3 and 4, we give a first step of model-building for solving the puzzles in the SM mentioned in Chapter 2. We first review 10D SYM on magnetized twisted orbifolds. Then, we classify possible combinations of the parameters which lead to three generations of chiral fermions in 4D effective theory. The brief result is presented. In Chapter 6, we summarize the research in this thesis and present future works. Appendix A is a summary on some mathematical formulas which are useful for calculations in Chapter 3 and 4. Appendix B is a summary on higher-dimensional gamma matrices which are used in Chapter 3 and

5. In Appendix C, we discuss KK higher modes on magnetized torus and orbifold. In Appendix D, the full tables of the classification in Chapter 5 are presented.

Chapter 2

The mysteries of the Standard Model

2.1 The Standard Model

The Standard Model (SM) is the established model in particle physics [8] by many experimental results. The SM is the theory symmetric under the gauge group $G_{\text{SM}} := \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_{\text{Y}}$, and contains the following matters,

		quantum numbers of G_{SM}
$\text{SU}(3)_{\text{C}}$ gauge bosons	G_{μ}^A	$(8, 1)_0$
$\text{SU}(2)_{\text{L}}$ gauge bosons	W_{μ}^a	$(1, 3)_0$
$\text{U}(1)_{\text{Y}}$ gauge boson	B_{μ}	$(1, 1)_0$
left-handed quarks	$q^i = \begin{pmatrix} u^i \\ d^i \end{pmatrix}_{\text{L}}$	$(3, 2)_{1/6}$
right-handed up-type quarks	u_{R}^i	$(3, 1)_{2/3}$
right-handed down-type quarks	d_{R}^i	$(3, 1)_{-1/3}$
left-handed leptons	$l^i = \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_{\text{L}}$	$(1, 2)_{-1/2}$
right-handed charged leptons	e_{R}^i	$(1, 1)_{-1}$
Higgs doublet	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(1, 2)_{1/2}$

Here, the indexes i runs over 1, 2, 3 for three generations, A runs from 1 to 8 for $\text{SU}(3)_{\text{C}}$, a runs from 1 to 3 for $\text{SU}(2)_{\text{L}}$.

The SM Lagrangian is given as the minimal symmetric form with given matter content,

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{Yukawa}} , \quad (2.1.1)$$

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} G^{A\mu\nu} G_{\mu\nu}^A - \frac{1}{4} W^{a\mu\nu} W_{\mu\nu}^a - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} , \quad (2.1.2)$$

$$\mathcal{L}_{\text{Higgs}} = \left| \left(\partial_{\mu} - i g_2 W_{\mu}^a \frac{\sigma^a}{2} - i g' B_{\mu} \frac{1}{2} \right) H \right|^2 + \mu^2 H^{\dagger} H - \frac{\lambda}{2} (H^{\dagger} H)^2 , \quad (2.1.3)$$

$$\begin{aligned}
\mathcal{L}_{\text{matter}} = & \bar{q}^i i \gamma^\mu \left[\partial_\mu - i g_3 G_\mu^A \left(\frac{1}{2} \lambda^A \right) - i g_2 W_\mu^a \left(\frac{1}{2} \sigma^a \right) - i g' B_\mu \left(\frac{1}{6} \right) \right] q^i \\
& + \bar{u}_R^i i \gamma^\mu \left[\partial_\mu - i g_3 G_\mu^A \left(\frac{1}{2} \lambda^A \right) - i g' B_\mu \left(\frac{2}{3} \right) \right] u_R^i \\
& + \bar{d}_R^i i \gamma^\mu \left[\partial_\mu - i g_3 G_\mu^A \left(\frac{1}{2} \lambda^A \right) - i g' B_\mu \left(-\frac{1}{3} \right) \right] d_R^i \\
& + \bar{l}^i i \gamma^\mu \left[\partial_\mu - i g_2 W_\mu^a \left(\frac{1}{2} \sigma^a \right) - i g' B_\mu \left(-\frac{1}{2} \right) \right] l^i \\
& + \bar{e}_R^i i \gamma^\mu \left[\partial_\mu - i g' B_\mu \left(-1 \right) \right] e_R^i,
\end{aligned} \tag{2.1.4}$$

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} = & - \left[y_d^{ij} (\bar{q}^i H) d_R^j + \text{h.c.} \right] - \left[y_u^{ij} (\bar{q}^i H^c) u_R^j + \text{h.c.} \right] \\
& - \left[y_e^i (\bar{l}^i H) e_R^i + \text{h.c.} \right],
\end{aligned} \tag{2.1.5}$$

where

$$G_{\mu\nu}^A := \partial_\mu G_\nu^A - \partial_\nu G_\mu^A + g_3 f^{ABC} G_\mu^B G_\nu^C, \tag{2.1.6}$$

$$W_{\mu\nu}^a := \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g_2 \epsilon^{abc} W_\mu^a W_\nu^b, \tag{2.1.7}$$

$$B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.1.8}$$

are gauge field strengths of $\text{SU}(3)_C$, $\text{SU}(2)_L$ and $\text{U}(1)_Y$ respectively, g_3, g_2, g' are corresponding gauge coupling constants, λ^A s are Gell-Mann matrices, σ^a s are Pauli matrices, and $H^c := i \sigma^2 H^*$.

The crucially important mechanism in the SM is the Higgs mechanism which provides the masses of the gauge bosons, the quarks, the leptons, and the Higgs particle. This mechanism is caused by the potential with so-called champagne-bottle profile as fig.2.1. Since the minima of the Higgs potential are not invariant under G_{SM} , $\text{SU}(2)_L \times \text{U}(1)_Y$ symmetry is spontaneously broken down into $\text{U}(1)_{\text{EM}}$.

The Higgs potential is rewritten as

$$V(H) = \frac{\lambda}{2} \left(H^\dagger H - \frac{\mu^2}{\lambda} \right)^2 - \frac{\mu^4}{2\lambda}, \tag{2.1.9}$$

so the Higgs field has non-zero vacuum expectation value (vev),

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \quad \frac{v}{\sqrt{2}} := \sqrt{\frac{\mu^2}{\lambda}}. \tag{2.1.10}$$

This vev produces the mass terms of the weak bosons through the covariant derivative term of the Higgs field as

$$\left| \left(\partial_\mu - i g_2 W_\mu^a \frac{\sigma^a}{2} - i g' B_\mu \frac{1}{2} \right) H \right|^2 = M_W^2 W^\mu W_\mu^\dagger + \frac{1}{2} M_Z^2 Z^\mu Z_\mu + \dots, \tag{2.1.11}$$

where

$$W_\mu := \frac{W_\mu^1 - i W_\mu^2}{\sqrt{2}}, \quad Z_\mu := \frac{g_2 W_\mu^3 - g' B_\mu}{\sqrt{g_2^2 + g'^2}} \tag{2.1.12}$$

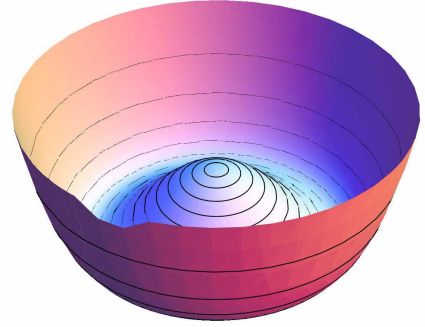


Figure 2.1: Higgs potential.

are the W boson field and the Z boson field, and

$$M_W^2 := \frac{g_2^2 v^2}{4}, \quad M_Z^2 := \frac{(g_2^2 + g'^2) v^2}{4} \quad (2.1.13)$$

are their mass-squared. The Higgs vev provides also the masses of the quarks and the leptons through the Yukawa couplings in the SM Lagrangian as

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= -[y_d^{ij}(\bar{q}^i H)d_R^j + \text{h.c.}] - [y_u^{ij}(\bar{q}^i H^c)u_R^j + \text{h.c.}] - [y_e^i(\bar{l}^i H)e_R^i + \text{h.c.}] \\ &= -(y_d^{ij}v\bar{d}_L^i d_R^j + \text{h.c.}) - (y_u^{ij}v\bar{u}_L^i u_R^j + \text{h.c.}) - (y_e^i v\bar{e}_L^i e_R^i + \text{h.c.}) + \dots \end{aligned} \quad (2.1.14)$$

After diagonalizing the quark sector, we find the masses of the quarks and the leptons

$$m_\alpha^i = y_\alpha^i v, \quad (\alpha = u, d, e). \quad (2.1.15)$$

The mass term of the physical Higgs is produced from the Higgs potential itself. To show this, let us expand the Higgs field H as

$$H(x) = \begin{pmatrix} 0 \\ (v + h(x))/\sqrt{2} \end{pmatrix}, \quad (2.1.16)$$

with the fluctuation around the vev $h(x)$. Substituting this into the Higgs potential, we obtain the Higgs mass term

$$V(H) = \frac{M_H^2}{2} h^2 + \dots, \quad M_H^2 = \lambda v^2. \quad (2.1.17)$$

The values of these masses are measured accurately in collider experiments [8]. The weak boson masses are measured as

$$M_W \sim 80 \text{ GeV}, \quad M_Z \sim 91 \text{ GeV}. \quad (2.1.18)$$

The photon mass is, of course, strictly zero,

$$M_A = 0 \text{ GeV}. \quad (2.1.19)$$

The masses of quarks and leptons measured are measured as follows,

	1st generation	2nd generation	3rd generation
up-type quarks	$m_u \sim 3 \text{ MeV}$	$m_c \sim 1.3 \times 10^3 \text{ MeV}$	$m_t \sim 1.7 \times 10^5 \text{ MeV}$
down-type quarks	$m_d \sim 5 \text{ MeV}$	$m_s \sim 100 \text{ MeV}$	$m_b \sim 4.2 \times 10^3 \text{ MeV}$
charged leptons	$m_e \sim 0.5 \text{ MeV}$	$m_\mu \sim 105 \text{ MeV}$	$m_\tau \sim 1.8 \times 10^3 \text{ MeV}$

(2.1.20)

The Higgs mass is measured recently in the LHC experiments [9, 10] as

$$M_H \sim 126 \text{ GeV}. \quad (2.1.21)$$

2.1.1 The glories and the mysteries of the SM

The SM has achieved great successes for explaining experimental results as follows:

- Gauge boson mass

In general, gauge boson should be massless since mass term of gauge field is forbidden by gauge symmetry. However, the gauge bosons in the SM have non-zero masses whose values are measured in experiments as in Eqs. (2.1.18)–(2.1.19). This apparently-paradoxical phenomenon is explained by the Higgs mechanism [2] as is mentioned in the above.

- g -factor of the electron

The g -factor of the electron which originates from the electron spin is measured precisely in experiments. The quantum correction to the g -factor is calculated in the SM and the result fits with the experimental value with absolute accuracy,

$$\left| \frac{g_e^{\text{exp}} - g_e^{\text{th}}}{g_e^{\text{exp}}} \right| = O(10^{-12}).$$

- Flavor changing neutral current

In 1960's, only three quarks (up, down, and strange) are discovered. Three flavors cause the flavor changing neutral current (FCNC), while the FCNC is rarely discovered in experiments. Glashow–Illioupoulos–Maiani [5] explained this scarcity by the mechanism where the FCNC is canceled at tree-level and predicted charm quark. The advantage of this mechanism, called GIM mechanism, is to explain not only the FCNC suppressing but also the branching $B(K_L \rightarrow \mu^+ \mu^-) \sim 10^{-9}$ (where K_L is long-lived kaon).

- CP violation

The rare decay process of the long-lived kaon $K_L \rightarrow \pi^+ \pi^-$ is forbidden if CP symmetry is preserved, but this process is observed in experiments. This means that CP symmetry is broken. Kobayashi and Maskawa [7] explained this CP-violation by introducing the matters greater than or equal to three generations. Nowadays, the three generations of the matters are discovered and Kobayashi–Maskawa mechanism is verified.

- Anomaly cancellation

In general, anomaly arises in chiral gauge theory which is gauge theory interacting with chiral fermion. Nevertheless, the SM, which is a chiral gauge theory, is anomaly free [6]. This is because of the reason as follows: The anomaly is canceled between the one arising from the quarks and the leptons in each generations.

Nevertheless, many mysteries are remaining.

- ▷ The origin of generation: why are there repetition of three same packages?

There are repetitions of the quarks and the leptons in the SM, and they have same quantum numbers.

$$\underbrace{\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix}}_{\text{1st generation}} \quad \underbrace{\begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix}}_{\text{2nd generation}} \quad \underbrace{\begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}}_{\text{3rd generation}}$$

Why are there repetition of same packages?

- ▷ Chirality: why do left- and right-handed matters interact differently?

The left-/right-handed quarks and leptons are in different representation of gauge group, that is, the left-handed quarks and leptons are $SU(2)_L$ doublet while the right-handed matters belong $SU(2)_L$ singlet representation.

$$\underbrace{\begin{pmatrix} u_L \\ d_L \end{pmatrix}}_{SU(2)_L \text{ doublet}} \iff \underbrace{u_R, d_R}_{SU(2)_L \text{ singlet}}$$

Why are there such difference in the left-handed and the right-handed matters?

- ▷ The mass hierarchy of the quarks and the leptons: why do masses of quarks and leptons differ among the generations?

There are large differences among the masses of the generations of quarks and leptons, e.g.,

$$\frac{m_t}{m_u} \sim 10^5. \quad (2.1.22)$$

What is the origin of these differences, called mass hierarchy?

- ▷ Gauge structure: why is the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ chosen?

There is no reason that the gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$. Moreover, the anomaly cancellation which is mentioned in the above is accidental; no anomaly exists if the gauge symmetry is described by particular type of the group.

- ▷ Gravity: why is not the gravitational interaction included?

The gravitational interaction is not included in the SM. This is in contrast to the other three interactions, the strong interaction, the weak interaction, and the electro magnetic interaction, which are included in the SM. In addition, it is so unnatural that typical energy scale of the gravitational interaction, called Planck scale $M_{\text{Pl}} \sim 10^{18}$ GeV, differs enormously from the weak scale $M_{\text{weak}} \sim 10^2$ GeV as follows:

$$\frac{M_{\text{Pl}}}{M_{\text{weak}}} = O(10^{16}). \quad (2.1.23)$$

It is weighty problem of particle physics to investigate the origin of these mysteries.

2.2 Approaches for solving the puzzles

Many approaches have been proposed for solving the puzzles in the SM. In the following, we present some approaches which are adopted in this thesis for solving the puzzles in the SM.

- Higher-dimensional field theory

One of these approaches is higher-dimensional field theory, which has space-time dimensions more than four. In a $(4 + d)$ -dimensional field theory on d -dimensional compactified space,

any fields are factorized, for example $f(x, y) = \chi(x) \times \psi(y)$, where x is the coordinates of 4D Minkowski space and y is the one of d -dimensional compact space. Yukawa couplings in 4D effective field theory are calculated by the overlap integral of a product of extra-dimensional wavefunctions as

$$Y_{ij} = g_{4+d} \int d^d y \, \psi_i(y) \psi_j(y) \phi(y), \quad (2.2.1)$$

where g_{4+d} is a $(4+d)$ -dimensional coupling constant, $\psi_i(y)$ are wavefunctions of i -th generation of the quarks and the leptons, $\phi(y)$ is a wavefunction of the Higgs. If the wavefunctions $\psi_i(y), \phi(y)$ have quasi-localized profile, such overlap integrals can produce large differences among the elements of the matrix Y_{ij} . Therefore, higher-dimensional field theory is desirable framework to explain the mass hierarchy and the mixing of the quarks and the leptons.

- Toroidal compactification with homogeneous magnetic field

In addition to higher-dimensional field theory, toroidal compactification with magnetic flux is also an attractive idea. We introduce the background potential $\langle A_m(y) \rangle$ ($m = 5, 6$) on 2-torus $\mathbb{T}^2$ which provides homogeneous magnetic field $\langle F_{56} \rangle = \partial_5 \langle A_6 \rangle - \partial_6 \langle A_5 \rangle$. On magnetized torus, fields should satisfy the appropriate boundary condition (see Eqs. (3.1.13) and (3.1.14)) and the magnetic flux $\int_{\mathbb{T}^2} d^2 y \langle F_{56} \rangle$ is quantized:

$$\int_{\mathbb{T}^2} d^2 y \langle F_{56} \rangle \propto M, \quad M \in \mathbb{Z}. \quad (2.2.2)$$

On the other hand, the equation of motion for 2D spinor field $\psi(y) = (\psi_+(y), \psi_-(y))^T$ becomes

$$i \tilde{\gamma}^m (\partial_m - i g \langle A_m(y) \rangle) \psi(y) = 0, \quad (2.2.3)$$

where $\tilde{\gamma}^m$ is the gamma matrix on 2D spinor and g is the gauge coupling constant. This equation can be solved analytically with the appropriate boundary conditions. The analytical solution helps us for computing Yukawa couplings in effective theory. In addition, as known in Ref. [45], the solution $\psi_+(y)$ ($\psi_-(y)$) is normalizable and becomes a physical solution only for $M > 0$ ($M < 0$). In other words, non-trivial gauge background leads to chiral fermions in 4D effective theory. Regardless of the chirality, the solution is $|M|$ -fold degenerated. This degeneracy naturally leads to generations, and Yukawa couplings in effective theory can be computed because of analytical solutions.

- Orbifolding

Orbifolding is also useful in actual model-building, not only for removing exotic particles which often appear in model-building but also for realizing realistic flavor structure. In particular, orbifolding is important in magnetized extra dimensions. This is because of the following reason: In magnetized torus models without orbifolding, three generations of fermion can be realized only when $|M| = 3$. This value of the magnetic flux limits the flavor structure to the one with $\Delta(27)$ flavor symmetry [50–52].¹ On the other hand, we can obtain three generations of fermion in various combinations of the parameters in magnetized orbifold models. Therefore, we may obtain realistic flavor structure.

¹More on string-originated flavor structure are in [53–55].

For example, in this direction, $U(1)$ gauge theory on twisted $\mathbb{T}^2/\mathbb{Z}_2$ with magnetic flux is analyzed in earlier study [40], where the twisted orbifold $\mathbb{T}^2/\mathbb{Z}_2$ is defined by identifying the points on 2D torus $\mathbb{T}^2$ under the π -radians rotation. Although orbifolding is useful, twisted orbifolds $\mathbb{Z}_{3,4,6}$ with magnetic flux were not studied well presumably because of the complexity of the system on magnetized $\mathbb{T}^2/\mathbb{Z}_{3,4,6}$. Recently, magnetized twisted orbifolds $\mathbb{T}^2/\mathbb{Z}_{3,4,6}$ as well as $\mathbb{T}^2/\mathbb{Z}_2$ are analyzed and the system on the orbifolds exactly solved [43]. In that work, the operator formalism analysis which is dual to wavefunction formalism works grossly.

Chapter 3

Magnetized Torus and Orbifolds

In this chapter, we study six-dimensional $U(1)$ gauge theory on the torus or possible twisted orbifolds to investigate the aspects of magnetized torus/orbifold model.

We first specify appropriate boundary conditions on the magnetized torus and find the quantization of the magnetic flux. We solve the equation of motion for KK zero-mode with non-trivial Wilson-line phase and Scherk–Schwarz phases and obtain explicit form of the zero-mode solution. Furthermore, we find that zero-mode solution is chiral and the number of independent solution is determined by magnetic flux. We also show that the Wilson-line phase is absorbed into the Scherk–Schwarz phases and vice versa by the redefinition of the fields.

Next, we consider twisted orbifolding. We specify the boundary condition for twist transformation. We investigate the allowed values of the Wilson-line phase and the Scherk–Schwarz phases. We also the way how the eigenfunction on the magnetized orbifold is constructed by zero-mode solutions on the magnetized torus.

In addition, we discuss KK higher modes at the end of this chapter.

3.1 $U(1)$ gauge theory on $\mathbb{M}^4 \times \mathbb{T}^2$

Let us consider $U(1)$ gauge theory with a chiral fermion on $\mathbb{M}^4 \times \mathbb{T}^2$ [42, 45]. The action is

$$\begin{aligned} S &= S_A + S_{\Psi_+}, \\ S_A &= \int_{\mathbb{M}^4} d^4x \int_{\mathbb{T}^2} dz d\bar{z} \left(-\frac{1}{4} F^{MN} F_{MN} \right), \\ S_{\Psi_+} &= \int_{\mathbb{M}^4} d^4x \int_{\mathbb{T}^2} dz d\bar{z} \bar{\Psi}_+ i \Gamma^M D_M \Psi_+, \end{aligned} \tag{3.1.1}$$

where $M, N = \mu(= 0, 1, 2, 3), z, \bar{z}$. The coordinates z and $\bar{z}$ are defined by

$$z := y^5 + i y^6, \quad \bar{z} = (z)^* = y^5 - i y^6, \tag{3.1.2}$$

where y^5, y^6 are the 2D orthogonal coordinates. The basis vectors of the torus are taken as $\mathbf{u}_1 = (1, 0)^T$ and $\mathbf{u}_2 = (\text{Re } \tau, \text{Im } \tau)^T$ in the orthogonal coordinates language, and therefore the complex coordinates are identified as

$$z \sim z + 1 \sim z + \tau, \tag{3.1.3}$$

on the torus. Here, the complex parameter τ ($\text{Im } \tau > 0$) is called moduli and determines the shape of the torus.

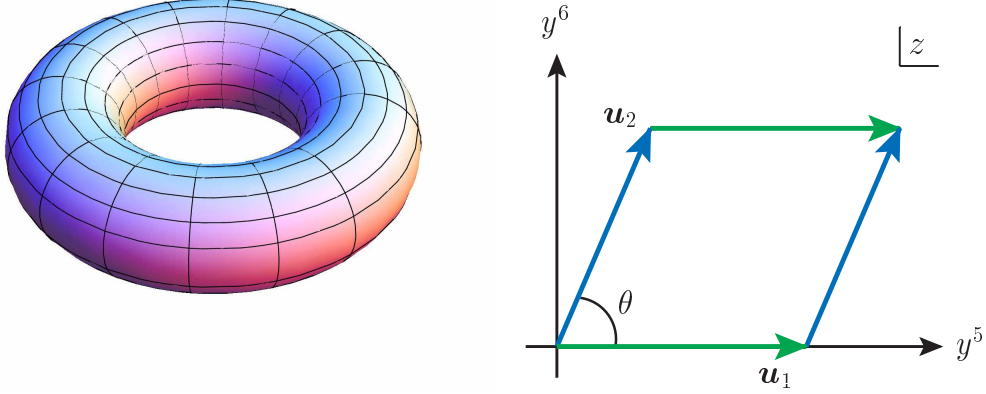


Figure 3.1: Two-dimensional torus. The left-hand side is an illustrative example of well-known donut-shaped torus. The right figure shows how a torus is constructed; by identifying two vectors of same color. The fundamental region of the torus is the parallelogram surrounded by the four vectors. The length of the vector $\mathbf{u}_2$ is equal to $|\tau|$. The angle between the basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, θ , is equal to $\text{Arg } \tau$.

Here, Ψ_+ is a 6D chiral fermion with positive sign chirality, Γ^M are the 6D gamma matrices, and

$$D_M = \partial_M - ie A_M \quad (3.1.4)$$

is the gauge covariant derivative. A_M is the gauge field of $U(1)$ gauge group and e is its coupling constant.

A 6D chiral spinor $\Psi_{\pm}(x, z)$ is decomposed as direct products of 4D chiral spinors $\chi_{R,L}$ and 2D Weyl spinors as

$$\begin{aligned} \Psi_{\pm}(x, z) &= \frac{\mathbf{1}_8 \pm \Gamma_7}{2} \Psi(x, z), \\ \Psi_+(x, z) &= \sum_n \left[\chi_{R,n}(x) \otimes \begin{pmatrix} \psi_{n,+}(z) \\ 0 \end{pmatrix} + \chi_{L,n}(x) \otimes \begin{pmatrix} 0 \\ \psi_{n,-}(z) \end{pmatrix} \right], \\ \Psi_-(x, z) &= \sum_n \left[\chi'_{L,n}(x) \otimes \begin{pmatrix} \psi_{n,+}(z) \\ 0 \end{pmatrix} + \chi'_{R,n}(x) \otimes \begin{pmatrix} 0 \\ \psi_{n,-}(z) \end{pmatrix} \right], \end{aligned} \quad (3.1.5)$$

where n is Kaluza–Klein modes index and Γ_7 is the matrix determining 6D chirality defined in Eq. (B.4). The mode functions $\psi_{n,\pm}(z)$ are the solutions of the eigenvalue equations

$$2 \langle D_{\bar{z}} \rangle \psi_{n,+}(z) = m_n \psi_{n,-}(z), \quad 2 \langle D_z \rangle \psi_{n,-}(z) = -m_n \psi_{n,+}(z), \quad (3.1.6)$$

where $\langle D_z \rangle$ and $\langle D_{\bar{z}} \rangle$ are the covariant derivative in the background gauge configurations $\langle A_z(z) \rangle$, $\langle A_{\bar{z}}(z) \rangle$,

$$\langle D_z \rangle = \partial - ie \langle A_z(z) \rangle, \quad \langle D_{\bar{z}} \rangle = \bar{\partial} - ie \langle A_{\bar{z}}(z) \rangle, \quad (3.1.7)$$

and $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$.

By integrating over the extra space, we obtain 4D effective action for fermions,

$$S_{\Psi_+} = \int_{\mathbb{M}^4} d^4x \left[\bar{\chi}_{R,0}(x) i \gamma^\mu D_\mu^{(0)} \chi_{R,0}(x) + \bar{\chi}_{L,0}(x) i \gamma^\mu D_\mu^{(0)} \chi_{L,0}(x) \right. \\ \left. + \sum_{m_n \neq 0} \{ \bar{\chi}_{R,n}(x) + \bar{\chi}_{L,n}(x) \} (i \gamma^\mu D_\mu^{(0)} - m_n) \{ \chi_{R,n}(x) + \chi_{L,n}(x) \} \right. \\ \left. + (\text{terms including higher-modes of gauge field}) \right], \quad (3.1.8)$$

where $D_\mu^{(0)} = \partial_\mu - i e A_\mu^{(0)}(x)$ is the covariant derivative in the zero-mode of the gauge field $A_\mu^{(0)}(x)$. The terms including higher-modes of gauge field provide Yukawa interaction.

In this paper, we are interested mainly in the zero-mode sector of the fermion, and omit the subscript $_0$ of the zero-mode functions, $\psi_\pm(z) := \psi_{0,\pm}(z)$.

3.1.1 Quantization of magnetic flux

The gauge field has a background configuration in its extra-dimensional components,

$$\langle A(z) \rangle := \langle A_z(z) \rangle dz + \langle A_{\bar{z}}(z) \rangle d\bar{z} = \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im} [(\bar{z} + \bar{a}) dz], \quad (3.1.9)$$

which provides the magnetic flux $b = \int_{\mathbb{T}^2} \langle F \rangle$ through the homogeneous magnetic field

$$\langle F \rangle = \frac{i b}{2 \operatorname{Im} \tau} dz \wedge d\bar{z}. \quad (3.1.10)$$

This configuration gives the following relations:

$$A(z+1) = A(z) + \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im} dz = A(z) + d\chi_1(z), \\ A(z+\tau) = A(z) + \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im}(\bar{\tau} dz) = A(z) + d\chi_\tau(z), \quad (3.1.11)$$

where

$$\chi_1(z+a) := \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im}(z+a), \quad \chi_\tau(z+a) := \frac{b}{2 \operatorname{Im} \tau} \operatorname{Im}[\bar{\tau}(z+a)]. \quad (3.1.12)$$

The relations (3.1.11) must become gauge transformations due to the invariance of the theory under the torus-shifts. Correspondingly, the matter field must satisfy the relations

$$\Psi(x, z+1) = U_1(z) \Psi(x, z), \quad \Psi(x, z+\tau) = U_\tau(z) \Psi(x, z), \quad (3.1.13)$$

where

$$U_1(z) := e^{i e \chi_1(z+a) + 2 \pi i \alpha_1}, \quad U_\tau(z) := e^{i e \chi_\tau(z+a) + 2 \pi i \alpha_\tau}, \quad (3.1.14)$$

and α_1, α_τ are real parameters called Scherk–Schwarz phases [38, 39].

It is remarkable that the magnetic flux should be quantized on $\mathbb{T}^2$. In order to show this, let us consider the contractible loop, e.g.

$$z \rightarrow z+1 \rightarrow z+1+\tau \rightarrow z+\tau \rightarrow z,$$

as in Fig. 3.2. For such loop, the functions $U_1(z)$ and $U_\tau(z)$ should satisfy the constraint

$$U_\tau^{-1}(z) U_1^{-1}(z + \tau) U_\tau(z + 1) U_1(z) = 1, \quad (3.1.15)$$

because of the single-valuedness of the field. This constraint condition lead the quantization condition of the magnetic flux,

$$\frac{eb}{2\pi} =: M \in \mathbb{Z}. \quad (3.1.16)$$

Note that, the condition (3.1.15) is equivalent to

$$U_\tau(z + 1) U_1(z) \Psi(x, z) = U_1(z + \tau) U_\tau(z) \Psi(x, z). \quad (3.1.17)$$

This equation means that the torus-shift operations along two directions $z \rightarrow z + 1$ and $z \rightarrow z + \tau$ are commutative.

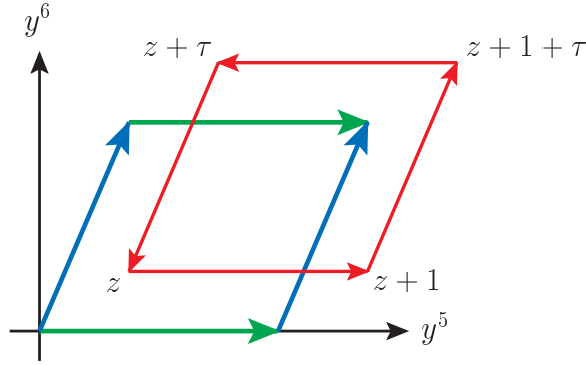


Figure 3.2: The contractible loop, $z \rightarrow z + 1 \rightarrow z + 1 + \tau \rightarrow z + \tau \rightarrow z$, is illustrated by red vectors. The blue and green vectors show the torus identifications.

3.1.2 Zero-mode eigenfunctions

From Eq. (3.1.6), the zero-mode functions $\psi_\pm(z)$ satisfy the eigenvalue equations,

$$\left(\bar{\partial} + \frac{\pi M}{2 \operatorname{Im} \tau} (z + a) \right) \psi_+^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = 0, \quad (3.1.18a)$$

$$\left(\partial - \frac{\pi M}{2 \operatorname{Im} \tau} (\bar{z} + \bar{a}) \right) \psi_-^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = 0. \quad (3.1.18b)$$

Note that, since the zero-mode functions actually depend on not only the coordinate z but also the Wilson line phase a , the Scherk–Schwarz phases α_1, α_τ , and the moduli τ , we represented these parameters explicitly as $\psi_\pm^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$. In addition, the solution for $\psi_-^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$ is obtained by the replacement $z \rightarrow \bar{z}, a \rightarrow \bar{a}$, and $\tau \rightarrow \bar{\tau}$ on the solution for $\psi_+(z)$, that is,

$$\psi_-^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = \psi_+^{(\alpha_1, \alpha_\tau)}(\bar{z}; \bar{a}, \bar{\tau}). \quad (3.1.19)$$

Therefore, we need to solve only the former equation,

$$\left(\bar{\partial} + \frac{\pi M}{2 \operatorname{Im} \tau} (z + a) \right) \psi_+^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = 0. \quad (3.1.20)$$

• Solving the equation of motion

In order to solve this equation satisfying the boundary conditions

$$\begin{aligned}\psi_+^{(\alpha_1, \alpha_\tau)}(z+p; a, \tau) &= U_p(z+a) \psi_+^{(\alpha_1, \alpha_\tau)}(z; a, \tau), \\ U_p(z+a) &= e^{i \frac{\pi M}{\text{Im} \tau} \text{Im}[\bar{p}(z+a)] + 2\pi i \alpha_p},\end{aligned}\tag{3.1.21}$$

where $p = 1, \tau$, we introduce the reduced wavefunction $\varphi_+(z)$ as

$$\psi_+^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = e^{i \frac{\pi M}{2 \text{Im} \tau} \text{Im}[(z+a)^2]} \varphi_+(z).\tag{3.1.22}$$

From this expression, we find that the reduced wavefunction satisfies the boundary conditions

$$\varphi_+(z+1) = e^{2\pi i \alpha_1} \varphi_+(z),\tag{3.1.23a}$$

$$\varphi_+(z+\tau) = e^{-\pi i M \text{Re}[2(z+a)+\tau] + 2\pi i \alpha_\tau} \varphi_+(z).\tag{3.1.23b}$$

These imply that the “real-axis” part of the reduced wavefunction satisfies the pseudo-periodic boundary condition. That is, if we decompose the complex coordinate z as

$$z = x + \tau y, \quad (x, y \in \mathbb{R}),\tag{3.1.24}$$

and correspondingly decompose the reduced wavefunction and the Wilson line phase as

$$\varphi_+(z) = X(x) \cdot Y(y), \quad a = a_1 + \tau a_\tau \quad (a_1, a_\tau \in \mathbb{R}),\tag{3.1.25}$$

the boundary condition of $X(x)$ is given by

$$X(x+1) = e^{2\pi i \alpha_1} X(x).\tag{3.1.26}$$

For this decomposition, the equation of motion is reduced to

$$\frac{\tau}{X(x)} \frac{dX(x)}{dx} - \frac{1}{Y(y)} \left(\frac{dY(y)}{dy} + 2\pi M(y+a_\tau) \text{Im} \tau \cdot Y(y) \right) = 0.\tag{3.1.27}$$

The solution $X(x)$ satisfying the condition (3.1.26) is found immediately as

$$X_r(x) = e^{2\pi i (r+\alpha_1)(x+a_1)}, \quad (r \in \mathbb{Z}).\tag{3.1.28}$$

On the other hand, the “moduli-axis” part $Y_r(y)$, $\varphi_+(z) = \sum_{r \in \mathbb{Z}} X_r(x) Y_r(y)$, should satisfy the differential equation

$$\frac{dY_r(y)}{dy} = [-2\pi M(y+a_\tau) \text{Im} \tau + 2\pi i (r+\alpha_1)\tau] Y_r(y),\tag{3.1.29}$$

and the boundary condition

$$Y_r(y+1) = e^{-2\pi i M(y+a_\tau+\frac{1}{2}) \text{Re} \tau + 2\pi i \alpha_\tau} Y_{r+M}(y).\tag{3.1.30}$$

The differential equation (3.1.29) is solved as

$$Y_r(y) = \mathcal{A}_r e^{-\pi M(y+a_\tau)^2 \text{Im} \tau + 2\pi i (r+\alpha_1)(y+a_\tau)\tau},\tag{3.1.31}$$

with a constant coefficient $\mathcal{A}_r$. The coefficient $\mathcal{A}_r$ is determined by considering the boundary condition (3.1.30) as

$$\mathcal{A}_r = \mathcal{N}_r \exp \left(2\pi i \frac{r + \alpha_1}{M} (-\alpha_\tau) + \pi i \frac{(r + \alpha_1)^2}{M} \tau \right), \quad (3.1.32)$$

where $\mathcal{N}_r$ is a constant coefficient satisfying the periodic condition $\mathcal{N}_{r+M} = \mathcal{N}_r$. This periodic condition implies that the solution of the Dirac equation is $|M|$ -fold degenerated. Denoting them as $\psi_{+,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$ ($j = 0, 1, \dots, |M|-1$) and splitting the summation index as $r = j + M \cdot \ell$ ($\ell \in \mathbb{Z}$), we obtain the solutions

$$\psi_{+,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = \mathcal{N}_j e^{i \frac{\pi M}{\text{Im} \tau} (z+a) \text{Im}(z+a)} \times \vartheta \left[\begin{matrix} (j + \alpha_1)/M \\ -\alpha_\tau \end{matrix} \right] (M(z+a), M\tau). \quad (3.1.33)$$

As we mentioned in the above, we can obtain the solution $\psi_{-,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$ immediately by replacing z, a, τ with these complex conjugation $\bar{z}, \bar{a}, \bar{\tau}$ in the above expression of $\psi_{+,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$,

$$\psi_{-,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau) = \mathcal{N}_j e^{i \frac{\pi M}{\text{Im} \tau} (\bar{z} + \bar{a}) \text{Im}(z+a)} \times \vartheta \left[\begin{matrix} (j + \alpha_1)/M \\ -\alpha_\tau \end{matrix} \right] (M(\bar{z} + \bar{a}), M\bar{\tau}). \quad (3.1.34)$$

Here, ϑ is Jacobi's theta function defined by

$$\vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu, \tau) := \sum_{\ell \in \mathbb{Z}} \exp \left[\pi i (a + \ell)^2 \tau + 2\pi i (a + \ell)(\nu + b) \right], \quad (3.1.35)$$

and has the following properties,

$$\vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu + m, \tau) = e^{2\pi i a m} \vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu, \tau), \quad (3.1.36)$$

$$\vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu + n\tau, \tau) = e^{-\pi i n^2 \tau - 2\pi i n(\nu + b)} \vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu, \tau), \quad (3.1.37)$$

$$\vartheta \left[\begin{matrix} a + m \\ b + n \end{matrix} \right] (\nu, \tau) = e^{2\pi i a n} \vartheta \left[\begin{matrix} a \\ b \end{matrix} \right] (\nu, \tau), \quad (3.1.38)$$

where n, m are integers.

In the following, we denote the solution $\psi_{\pm,j}^{(\alpha_1, \alpha_\tau)}(z; a, \tau)$ simply as $\psi_{\pm,j}(z)$.

• Normalization

We check the orthogonality of $\psi_{+,j}(z)$ and determine the normalization factor $\mathcal{N}_j$. To do this, we calculate the integration,

$$I_{j,k} := \int_{\mathbb{T}^2} d^2 z (\psi_{+,k}(z))^* \psi_{+,j}(z), \quad (3.1.39)$$

where the integral is

$$\int_{\mathbb{T}^2} d^2 z = \mathcal{A} \int_0^1 dx \int_0^1 dy, \quad (3.1.40)$$

and $\mathcal{A} = 1 \cdot \text{Im } \tau$ is torus area. Substituting the explicit form of $\psi_{+,j}(z)$, $I_{j,k}$ becomes

$$I_{j,k} = \mathcal{A} \mathcal{N}_j \mathcal{N}_k^* \sum_{\ell, \ell'} e^{\pi i \left[\left(\frac{j+\alpha_1}{M} + \ell \right)^2 M \tau - \left(\frac{k+\alpha_1}{M} + \ell' \right)^2 M \bar{\tau} \right]} \int_0^1 dx e^{2\pi i [(j+\alpha_1+M \cdot \ell) - (k+\alpha_1+M \cdot \ell')](x+a_1 - \frac{a_\tau}{M})} \\ \times \int_0^1 dy e^{-2\pi M (y+a_\tau)^2 \text{Im } \tau + 2\pi i [(j+\alpha_1+M \cdot \ell)\tau - (k+\alpha_1+M \cdot \ell')\bar{\tau}](y+a_\tau)} .$$

The integration with respect to x is performed immediately by using the formula

$$\int_0^1 dx e^{2\pi i (n-m)x} = \delta_{n,m} , \quad (\text{for } n, m \in \mathbb{Z}) , \quad (3.1.41)$$

and we obtain

$$I_{j,k} = \mathcal{A} \mathcal{N}_j \mathcal{N}_k^* \sum_{\ell, \ell'} e^{\pi i \left[\left(\frac{j+\alpha_1}{M} + \ell \right)^2 M \tau - \left(\frac{k+\alpha_1}{M} + \ell' \right)^2 M \bar{\tau} \right]} \delta_{(j+M \cdot \ell), (k+M \cdot \ell')} \\ \times \int_0^1 dy e^{-2\pi M (y+a_\tau)^2 \text{Im } \tau + 2\pi i [(j+\alpha_1+M \cdot \ell)\tau - (k+\alpha_1+M \cdot \ell')\bar{\tau}](y+a_\tau)} .$$

Furthermore, since the indices j and k run only from 0 to $|M| - 1$, the Kronecker's delta in the above is decomposed as

$$\delta_{(j+M \cdot \ell), (k+M \cdot \ell')} = \delta_{j,k} \delta_{\ell, \ell'} . \quad (3.1.42)$$

Therefore, $I_{j,k}$ becomes of a form

$$I_{j,k} = \mathcal{A} |\mathcal{N}_j|^2 \delta_{j,k} \sum_{\ell} \int_0^1 dy e^{-2\pi M \text{Im } \tau (y+a_\tau + \frac{j+\alpha_1}{M} + \ell)^2} . \quad (3.1.43)$$

Here, we confirm that $\{ \psi_{+,j}(z) \mid j = 0, 1, \dots, |M| - 1 \}$ is a orthogonal set. By shifting the integration variable as

$$y + a_\tau + \frac{j+\alpha_1}{M} + \ell \rightarrow y , \quad (3.1.44)$$

we obtain

$$I_{j,k} = \mathcal{A} |\mathcal{N}_j|^2 \delta_{j,k} \sum_{\ell} \int_{a_\tau + \frac{j+\alpha_1}{M} + \ell}^{1+a_\tau + \frac{j+\alpha_1}{M} + \ell} dy e^{-2\pi M \text{Im } \tau \cdot y^2} \\ = \mathcal{A} |\mathcal{N}_j|^2 \delta_{j,k} \int_{-\infty}^{+\infty} dy e^{-2\pi M \text{Im } \tau \cdot y^2} \\ = \mathcal{A} |\mathcal{N}_j|^2 \delta_{j,k} \cdot \sqrt{\frac{\pi}{2\pi M \text{Im } \tau}} . \quad (3.1.45)$$

Therefore the normalization constant is determined as

$$\mathcal{N}_j = \left(\frac{2M \text{Im } \tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} . \quad (3.1.46)$$

It should be noted that the Gaussian integral in the second last line of Eq.(3.1.45) can be performed only for $M > 0$. Thus the positive-chirality solutions $\psi_{+,j}(z)$ are not normalizable for

$M < 0$. Similar calculation holds for the negative-chirality solutions $\psi_{-,j}(z)$ by replacing τ with $\bar{\tau}$. Therefore, $\psi_{-,j}(z)$ are normalizable and become physical solutions only for $M < 0$.

Summarizing these, the zero-mode solutions are given by

$$\begin{aligned}\psi_{+,j}(z) &= \begin{cases} \Theta_M^{(j+\alpha_1, \alpha_\tau)}(z+a; \tau) & (\text{for } M > 0), \\ 0 & (\text{for } M < 0), \end{cases} \\ \psi_{-,j}(z) &= \begin{cases} 0 & (\text{for } M > 0), \\ \Theta_M^{(j+\alpha_1, \alpha_\tau)}(\bar{z}+\bar{a}; \bar{\tau}) & (\text{for } M < 0), \end{cases}\end{aligned}\quad (j = 0, 1, \dots, |M| - 1), \quad (3.1.47)$$

where the function $\Theta_M^{(j+\alpha_1, \alpha_\tau)}(z; \tau)$ is defined by

$$\Theta_M^{(j+\alpha_1, \alpha_\tau)}(z; \tau) := \mathcal{N}_M e^{\pi i M z \frac{\text{Im} z}{\text{Im} \tau}} \times \vartheta \left[\begin{matrix} (j + \alpha_1)/M \\ -\alpha_\tau \end{matrix} \right] (Mz, M\tau), \quad (3.1.48)$$

and

$$\mathcal{N}_M = \left(\frac{2|M| \text{Im} \tau}{\mathcal{A}^2} \right)^{\frac{1}{4}} \quad (3.1.49)$$

is the normalization constant.

• Remarks of the solution

We emphasize the following remarks:

1. The equation of motion can be solved analytically. The analytical solution is useful for computation of the Yukawa couplings in the effective theory.
2. Only $\psi_{+,j}(z)$ can be normalized for $M > 0$, while only $\psi_{-,j}(z)$ survive as physical solutions for $M < 0$,

$$\Psi_{+,0}(x, z) = \begin{cases} \sum_{j=0}^{M-1} \chi_{R,0,j}(x) \otimes \begin{pmatrix} \psi_{+,j}(z) \\ 0 \end{pmatrix} & (\text{for } M > 0), \\ \sum_{j=0}^{M-1} \chi_{L,0,j}(x) \otimes \begin{pmatrix} 0 \\ \psi_{-,j}(z) \end{pmatrix} & (\text{for } M < 0). \end{cases} \quad (3.1.50)$$

This property leads to the chiral structure in 4D effective theory.

3. The number of solutions is given by $|M|$ regardless of the sign of M . These $|M|$ -fold degenerated eigenfunctions are distinguished by the index $j = 0, 1, \dots, |M| - 1$. This leads to the generation structure in 4D effective theory, and the index j represents the generation.
4. In addition, the solution has quasi-localized profile. This may lead to hierarchical Yukawa couplings in 4D effective theory as is mentioned before. As an example, we show the wave-function profile for $M = 4$, $\alpha_1 = \alpha_\tau = 0$ and $\tau = i$ in Fig.3.3.

The aspects of the magnetized torus models, which are mentioned here, are very useful for solving the puzzles in the SM, the origin of the generations, the chirality, the mass hierarchy of the matters. However, the magnetic flux is restricted in $|M| = 3$ in order to realize three generations in the 4D

effective theory. For this value of the magnetic flux, only the flavor structure with $\Delta(27)$ flavor symmetry can be obtained. Therefore, it is not enough only by magnetized torus if one wants to obtain a rich flavor structure in addition to three generations of chiral fermions.

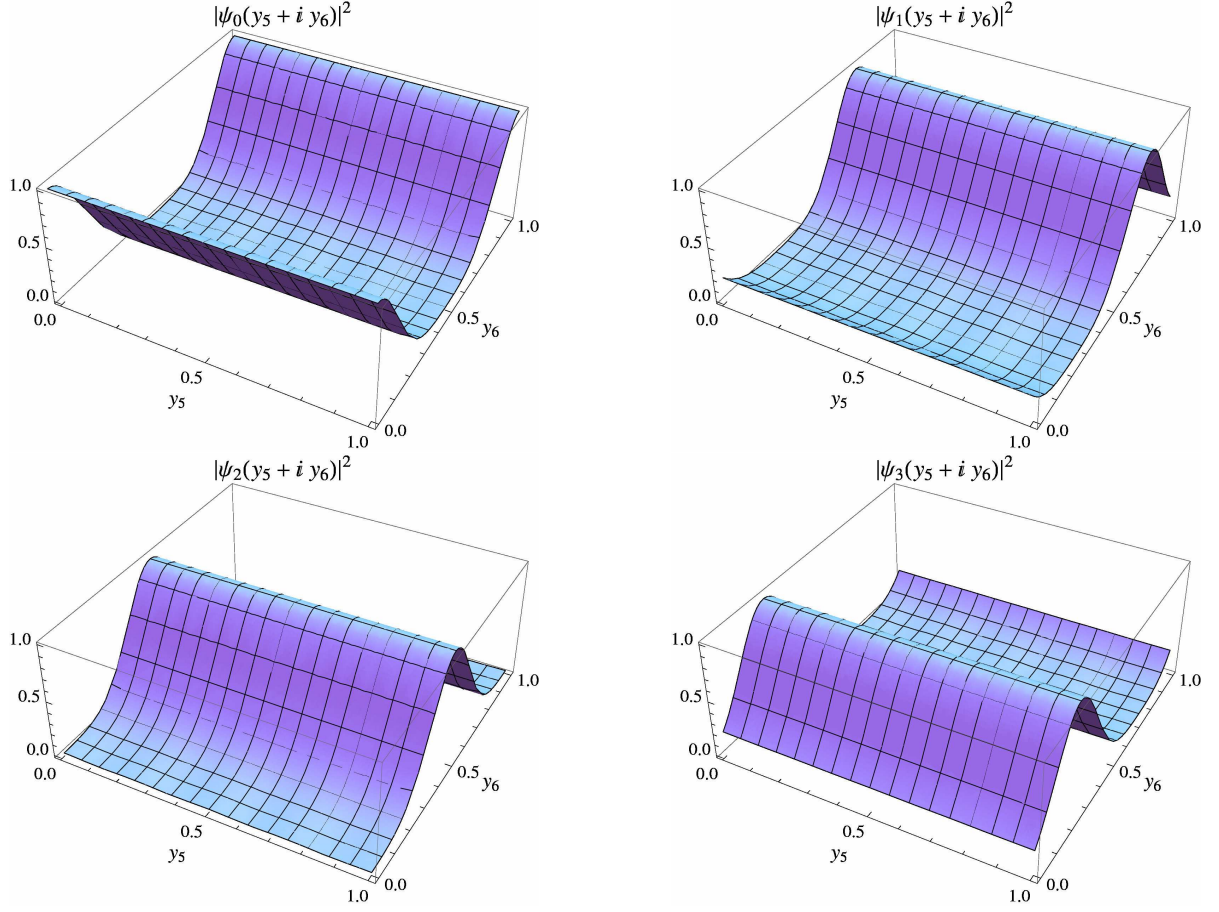


Figure 3.3: The probability densities $|\psi_j(z)|^2$ for $M = 4$, $\alpha_1 = \alpha_\tau = 0$ and $\tau = i$. These figures are depicted with the complex coordinate $z = y_5 + i y_6$ where y_5, y_6 ($0 \leq y_5, y_6 \leq 1$) are orthogonal coordinates. The upper-left figure is probability density for $j = 0$, the upper-right figure is for $j = 1$, the bottom-left figure is for $j = 2$, and the bottom-right is for $j = 3$. One can see that their profiles are Gaussian-like along y_6 -axis and the ridge moves as j increases.

3.1.3 Wilson-line phase VS Scherk–Schwarz phases

In this subsection, we will show that the Wilson-line phase and the Scherk–Schwarz phases relates through the redefinition of the field. Through this subsection, we use the differential form,

$$d = dz \frac{\partial}{\partial z} + d\bar{z} \frac{\partial}{\partial \bar{z}}, \quad A = A_z dz + A_{\bar{z}} d\bar{z}. \quad (3.1.51)$$

In addition, we denote the Wilson-line phase a explicitly in order to emphasize its role in the following discussion as $\psi(z; a)$ for the matter field and $A(z; a)$ for the gauge field.

Let us consider the redefinition of the field

$$\psi(z; a) = e^{ie \operatorname{Im}[i\bar{\gamma}(z+a')]} \psi'(z; a'), \quad (3.1.52)$$

where γ is a complex parameter and a' is the new Wilson-line phase. The covariant derivative becomes

$$\begin{aligned} & (d - ie \langle A(z; a) \rangle) \psi(z; a) \\ &= e^{ie \operatorname{Im}[i\bar{\gamma}(z+a')]} [d + ie \operatorname{Im}(i\bar{\gamma} dz) - ie \langle A(z; a) \rangle] \psi'(z; a'), \end{aligned} \quad (3.1.53)$$

by the redefinition. Therefore, we find that the new gauge field with redefined Wilson-line phase is given by

$$\langle A'(z; a') \rangle = \langle A(z; a) \rangle - \operatorname{Im}(i\bar{\gamma} dz). \quad (3.1.54)$$

This relation means

$$\begin{aligned} \frac{\pi M}{e \operatorname{Im} \tau} \operatorname{Im}(\bar{a}' dz) &= \frac{\pi M}{e \operatorname{Im} \tau} \operatorname{Im}(\bar{a} dz) - \operatorname{Im}(i\bar{\gamma} dz) \\ &= \frac{\pi M}{e \operatorname{Im} \tau} \operatorname{Im} \left[\left(\bar{a} - \frac{ie \operatorname{Im} \tau}{\pi M} \bar{\gamma} \right) dz \right]. \end{aligned} \quad (3.1.55)$$

Therefore, we find the relation between the new Wilson-line phase and the old one,

$$a' = a + \frac{ie \operatorname{Im} \tau}{\pi M} \gamma. \quad (3.1.56)$$

In the language of the redefined field, the boundary conditions (3.1.13) still hold with the redefinition of the parameters a, α_1, α_τ ,

$$\psi'(z + p; a') = e^{ie \chi_p(z+a') + 2\pi i \alpha'_p} \psi'(z; a'), \quad (p = 1, \tau), \quad (3.1.57)$$

where we used the shorthand notation $p = 1, \tau$. Our second work in this section is to find the relations of new Scherk–Schwarz phases α'_1, α'_τ and the old ones α_1, α_τ . From the redefinition (3.1.52), we find that

$$\psi(z + p; a) = e^{ie \operatorname{Im}(i\bar{\gamma} p)} e^{ie \operatorname{Im}[i\bar{\gamma}(z+a')]} \psi'(z + p; a'). \quad (3.1.58)$$

On the other hand, the boundary conditions for original field, Eq. (3.1.13), lead to

$$\psi(z + p; a) = e^{ie \chi_p(z+a) + 2\pi i \alpha_p} \cdot e^{ie \operatorname{Im}[i\bar{\gamma}(z+a')]} \psi'(z; a'). \quad (3.1.59)$$

Therefore, by comparing Eq. (3.1.58) with Eq. (3.1.59), we find the boundary conditions for redefined field as

$$\psi'(z + p; a') = e^{ie \chi_p(z+a) + 2\pi i \alpha_p - ie \operatorname{Im}(i\bar{\gamma} p)} \psi'(z; a'). \quad (3.1.60)$$

By using the relation (3.1.56), the exponential of the above becomes

$$\begin{aligned} & ie \chi_p(z + a) - ie \operatorname{Im}(i\bar{\gamma} p) + 2\pi i \alpha_p \\ &= i \frac{\pi M}{\operatorname{Im} \tau} \operatorname{Im} \left[\bar{p} \left(z + a' - \frac{ie \operatorname{Im} \tau}{\pi M} \gamma \right) \right] - ie \operatorname{Im}(i\bar{\gamma} p) + 2\pi i \alpha_p \\ &= ie \chi_p(z + a') + 2\pi i \left(\alpha_p - \frac{e}{\pi} \operatorname{Re}(\bar{p} \gamma) \right). \end{aligned} \quad (3.1.61)$$

Since this must equal to $i e \chi_p(z + a') + 2\pi i \alpha'_p$, we obtain the relations

$$\alpha'_p = \alpha_p - \frac{e}{\pi} \text{Re}(\bar{p} \gamma), \quad (p = 1, \tau). \quad (3.1.62)$$

Combining Eq.(3.1.56) and the above, we find that the Wilson-line phase can always be absorbed into Scherk–Schwarz phases and vice versa: By choosing the value of γ so that $a' = 0$, we obtain the new parameters

$$a' = 0, \quad \alpha'_1 = \alpha_1 + \text{Im} \frac{Ma}{\text{Im} \tau}, \quad \alpha'_\tau = \alpha_\tau + \text{Im} \left(\bar{\tau} \frac{Ma}{\text{Im} \tau} \right). \quad (3.1.63)$$

On the other hand, by choosing the value of γ so that $\alpha'_1 = \alpha'_\tau = 0$, we obtain

$$a' = a + \frac{\alpha_1 \tau - \alpha_\tau}{M}, \quad \alpha'_1 = 0, \quad \alpha'_\tau = 0. \quad (3.1.64)$$

3.2 Twisted Orbifolds

In this section, we study the U(1) gauge theory on the twisted orbifold $\mathbb{T}^2/\mathbb{Z}_N$. The purpose to consider the orbifold is as follows: As is mentioned at the end of previous section, toroidal compactification with magnetic flux is not enough for realizing the flavor structure which is observed in experiments. Twisted orbifolding with magnetic flux is useful for obtaining realistic flavor structure. Furthermore, twisted orbifold is useful also for removing exotic particles in actual model-buildings.

The twisted orbifolds $\mathbb{T}^2/\mathbb{Z}_N$ is defined by the symmetry under the discrete rotation [31, 32],

$$z \rightarrow \omega z, \quad \omega := e^{2\pi i/N}, \quad (3.2.1)$$

in addition to the torus-shifts $z \sim z + 1 \sim z + \tau$.

Note that, it is known that only $\mathbb{T}^2/\mathbb{Z}_2, \mathbb{T}^2/\mathbb{Z}_3, \mathbb{T}^2/\mathbb{Z}_4$ and $\mathbb{T}^2/\mathbb{Z}_6$ are allowed on $\mathbb{T}^2$. Furthermore, the moduli parameter τ has no limitation except $\text{Im} \tau > 0$ for $\mathbb{T}^2/\mathbb{Z}_2$, but τ must equal to ω for $N = 3, 4, 6$. These are, for example, in Ref. [34].

On $\mathbb{T}^2/\mathbb{Z}_N$, there are fixed points z_{fp} defined by ¹

$$z_{\text{fp}} = \omega z_{\text{fp}} + n_1 + n_\tau \tau, \quad (\exists n_1, n_\tau \in \mathbb{Z}). \quad (3.2.2)$$

We solve this equation with the torus identifications $z_{\text{fp}} \sim z_{\text{fp}} + n'_1 + n'_\tau \tau$ ($n'_1, n'_\tau \in \mathbb{Z}$) and find the fixed points for each $\mathbb{T}^2/\mathbb{Z}_N$ as

$\mathbb{T}^2/\mathbb{Z}_N$	fixed points
$\mathbb{T}^2/\mathbb{Z}_2$	$0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$
$\mathbb{T}^2/\mathbb{Z}_3$	$0, \frac{2+\tau}{3}, \frac{1+2\tau}{2}$
$\mathbb{T}^2/\mathbb{Z}_4$	$0, \frac{1+\tau}{2}$
$\mathbb{T}^2/\mathbb{Z}_6$	0

¹Fixed points are specified by the $\mathbb{Z}_N$ -twist ω and the torus-shifts $n_1 + n_\tau \tau$, and are defined as $(\omega, n_1 + n_\tau \tau)$ with the language of space group. See, for example, [37].

The fundamental region and the fixed points for each orbifold are depicted in Fig.3.4.

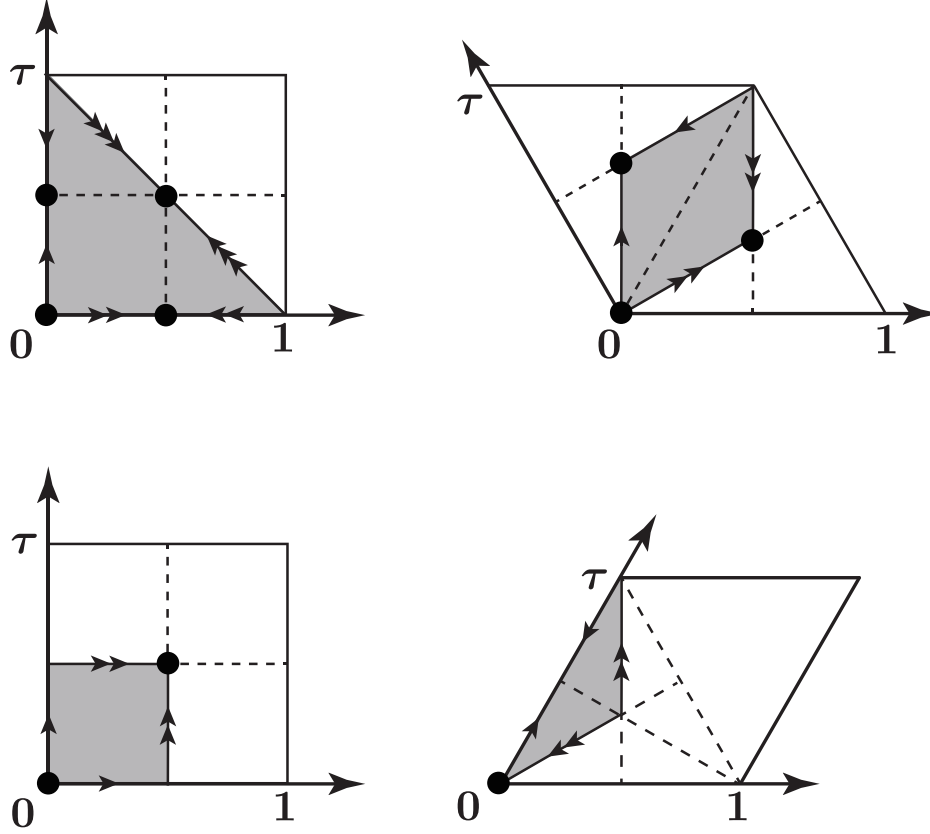


Figure 3.4: The fixed points and the fundamental region of the $\mathbb{T}^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$). The upper-left figure is for $\mathbb{T}^2/\mathbb{Z}_2$, the upper-right figure is for $\mathbb{T}^2/\mathbb{Z}_3$, the lower-left figure is for $\mathbb{T}^2/\mathbb{Z}_4$, and the lower-right figure is for $\mathbb{T}^2/\mathbb{Z}_6$. The black dots denote the fixed points, and the gray areas correspond to the fundamental region. Each of the lines with arrows means the identification by torus-shifts and $\mathbb{Z}_N$ -twist. The number of arrows on the line represents the pair of the identification, and the direction of arrows represents the direction of the identification. (Quoted from [42].)

3.2.1 The boundary conditions and allowed values of SS phases

The gauge fields transform under the twist as

$$\begin{aligned} A_\mu(x, \omega z) &= A_\mu(x, z), \\ A_z(x, \omega z) &= \bar{\omega} A_z(x, z), \quad A_{\bar{z}}(x, \omega z) = \omega A_{\bar{z}}(x, z), \end{aligned} \quad (3.2.3)$$

according to the general theory of coordinate transformations. The background configuration (3.1.9) satisfies these transformation properties if the Wilson-line phase satisfies $a = \omega a$. This condition means that the Wilson-line phase should be fixed points. For the torus-shifts, the boundary conditions mentioned in the last section holds,

$$\langle A(z+1) \rangle = \langle A(z) \rangle + d\chi_1(z), \quad \langle A(z+\tau) \rangle = \langle A(z) \rangle + d\chi_\tau(z), \quad (3.2.4)$$

with

$$\chi_1(z) = \frac{\pi M}{e \operatorname{Im} \tau} \operatorname{Im} z, \quad \chi_\tau(z) = \frac{\pi M}{e \operatorname{Im} \tau} \operatorname{Im}(\bar{\tau} z). \quad (3.2.5)$$

Here and hereafter, we discuss in the base where the Wilson-line phase vanishes, $a = 0$, without loss of generality.

The boundary conditions for the 6D spinor $\Psi_{\pm}(x, z)$ are determined so that the term $\Gamma^M D_M \Psi_{\pm}(x, z)$ to be covariant. This leads the following,

$$\Psi_{\pm}(x, \omega z) = \eta \mathcal{S} \Psi_{\pm}(x, z), \quad (3.2.6a)$$

$$\Psi_{\pm}(x, z + 1) = U_1(z) \Psi_{\pm}(x, z), \quad U_1(z) = e^{ie\chi_1(z) + 2\pi i\alpha_1}, \quad (3.2.6b)$$

$$\Psi_{\pm}(x, z + \tau) = U_{\tau}(z) \Psi_{\pm}(x, z), \quad U_{\tau}(z) = e^{ie\chi_{\tau}(z) + 2\pi i\alpha_{\tau}}, \quad (3.2.6c)$$

where $\mathcal{S} := -\frac{1}{4}(\Gamma^z \Gamma^{\bar{z}} + \omega \Gamma^{\bar{z}} \Gamma^z) = \text{diag}(\mathbb{I}_4, \omega \mathbb{I}_4)$ and η is the eigenvalue of the $\mathbb{Z}_N$ -twist transformation satisfying the consistency condition $\eta^N = 1$. The first condition reduces for extra-dimensional components $\psi_{\pm}(z)$ to

$$\begin{cases} \psi_+(\omega z) = \eta \psi_+(z), \\ \psi_-(\omega z) = (\omega \eta) \psi_-(z). \end{cases} \quad (3.2.7)$$

It is known that the Scherk–Schwarz phases are restricted in the particular values on the orbifold without magnetic flux, in contrast to the case of $\mathbb{T}^2$ where the Scherk–Schwarz phases can take the values arbitrarily [36, 37]. In the following, we show that there are similar constraints on the Scherk–Schwarz phases on the magnetized orbifold $\mathbb{T}^2/\mathbb{Z}_N$. In order to investigate the allowed values of the Scherk–Schwarz phases, we calculate $\Psi(\omega(z + 1))$ and $\Psi(\omega(z + \tau))$ in two different manners.

- $\mathbb{T}^2/\mathbb{Z}_2$ case: The first method of calculation is to use the boundary condition for $\mathbb{Z}_N$ -twist, Eq. (3.2.6a), before using the boundary condition for torus-shifts, Eqs. (3.2.6b) and (3.2.6c),

$$\Psi(\omega(z + 1)) = \eta \mathcal{S} \Psi(z + 1) = \eta \mathcal{S} \cdot U_1(z) \Psi(z), \quad (3.2.8a)$$

$$\Psi(\omega(z + \tau)) = \eta \mathcal{S} \Psi(z + \tau) = \eta \mathcal{S} \cdot U_{\tau}(z) \Psi(z), \quad (3.2.8b)$$

while the second method of calculation is to use the boundary conditions in the opposite order of the above, that is, to use firstly the boundary condition for the torus-shifts Eqs. (3.2.6b)–(3.2.6c) and secondary to use the boundary condition for the $\mathbb{Z}_N$ -twist Eq. (3.2.6a)

$$\Psi(\omega(z + 1)) = \Psi(\omega z - 1) = U_1^{\dagger}(\omega z) \Psi(\omega z) = U_1^{\dagger}(-z) \cdot \eta \mathcal{S} \Psi(z), \quad (3.2.9a)$$

$$\Psi(\omega(z + \tau)) = \Psi(\omega z - \tau) = U_{\tau}^{\dagger}(\omega z) \Psi(\omega z) = U_{\tau}^{\dagger}(-z) \cdot \eta \mathcal{S} \Psi(z), \quad (3.2.9b)$$

where we used $\omega = -1$ for $\mathbb{T}^2/\mathbb{Z}_2$. By comparing Eq. (3.2.8a) with Eq. (3.2.9a), we obtain the relation

$$U_1(z) = U_1^{\dagger}(-z). \quad (3.2.10)$$

On the other hand, we obtain the relation

$$U_{\tau}(z) = U_{\tau}^{\dagger}(-z), \quad (3.2.11)$$

by comparing Eq. (3.2.8b) with Eq. (3.2.9b). These lead to the following constraints by using the expression of $U_1(z)$ and $U_{\tau}(z)$ in Eqs. (3.2.6b) and (3.2.6c):

$$e^{2\pi i\alpha_1} = e^{-2\pi i\alpha_1}, \quad e^{2\pi i\alpha_{\tau}} = e^{-2\pi i\alpha_{\tau}}. \quad (3.2.12)$$

Finally, we obtain the allowed values of the Scherk–Schwarz phases on the orbifold $\mathbb{T}^2/\mathbb{Z}_2$ as

$$(\alpha_1, \alpha_\tau) = (0, 0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right). \quad (3.2.13)$$

- $\mathbb{T}^2/\mathbb{Z}_3$ case: The first method of calculation is

$$\Psi(\omega(z+1)) = \eta \mathcal{S} \Psi(z+1) = \eta \mathcal{S} \cdot U_1(z) \Psi(z), \quad (3.2.14a)$$

$$\Psi(\omega(z+\tau)) = \eta \mathcal{S} \Psi(z+\tau) = \eta \mathcal{S} \cdot U_\tau(z) \Psi(z), \quad (3.2.14b)$$

while the second method of calculation is

$$\Psi(\omega(z+1)) = \Psi(\omega z + \tau) = U_\tau(\omega z) \Psi(\omega z) = U_\tau(\omega z) \cdot \eta \mathcal{S} \Psi(z), \quad (3.2.15a)$$

$$\begin{aligned} \Psi(\omega(z+\tau)) &= \Psi(\omega z - 1 - \tau) = U_1^\dagger(\omega z - 1 - \tau) \Psi(\omega z - \tau) \\ &= U_1^\dagger(\omega z - 1 - \tau) \cdot U_\tau^\dagger(\omega z - \tau) \Psi(\omega z) \\ &= U_1^\dagger(\omega z - 1 - \tau) U_\tau^\dagger(\omega z - \tau) \cdot \eta \mathcal{S} \Psi(z), \end{aligned} \quad (3.2.15b)$$

where we used the relations $\omega\tau = -1 - \tau$ for $\mathbb{T}^2/\mathbb{Z}_3$. By comparing Eq. (3.2.14a) with Eq. (3.2.15a), we obtain the relation

$$U_1(z) = U_\tau(\omega z). \quad (3.2.16)$$

On the other hand, we obtain the relation

$$U_\tau(z) = U_1^\dagger(\omega z - 1 - \tau) U_\tau^\dagger(\omega z - \tau), \quad (3.2.17)$$

by comparing Eq. (3.2.14b) with Eq. (3.2.15b). These lead to the following constraints by using the expression of $U_1(z)$ and $U_\tau(z)$ in Eqs. (3.2.6b) and (3.2.6c):

$$e^{2\pi i \alpha_1} = e^{2\pi i \alpha_\tau}, \quad e^{2\pi i \alpha_\tau} = e^{-2\pi i (\alpha_1 + \alpha_\tau) + \pi i M} e^{-\frac{\pi i M}{1+m\tau} \text{Im}[(1+\omega+\bar{\tau})z]}. \quad (3.2.18)$$

Finally, we obtain the allowed values of the Scherk–Schwarz phases on the orbifold $\mathbb{T}^2/\mathbb{Z}_3$ as

$$(\alpha_1, \alpha_\tau) = \begin{cases} (0, 0), \left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{2}{3}\right) & (\text{for } M = \text{even}), \\ \left(\frac{1}{6}, \frac{1}{6}\right), \left(\frac{3}{6}, \frac{3}{6}\right), \left(\frac{5}{6}, \frac{5}{6}\right) & (\text{for } M = \text{odd}), \end{cases} \quad (3.2.19)$$

by using the relation

$$1 + \omega + \bar{\tau} = 1 + \omega + \omega^2 = 0. \quad (3.2.20)$$

- $\mathbb{T}^2/\mathbb{Z}_4$ case: The first method of calculation is

$$\Psi(\omega(z+1)) = \eta \mathcal{S} \Psi(z+1) = \eta \mathcal{S} \cdot U_1(z) \Psi(z), \quad (3.2.21a)$$

$$\Psi(\omega(z+\tau)) = \eta \mathcal{S} \Psi(z+\tau) = \eta \mathcal{S} \cdot U_\tau(z) \Psi(z), \quad (3.2.21b)$$

while the second method of calculation is

$$\Psi(\omega(z+1)) = \Psi(\omega z + \tau) = U_\tau(\omega z) \Psi(\omega z) = U_\tau(\omega z) \cdot \eta \mathcal{S} \Psi(z), \quad (3.2.22a)$$

$$\begin{aligned}\Psi(\omega(z + \tau)) &= \Psi(\omega z - 1) = U_1^\dagger(\omega z - 1) \Psi(\omega z) \\ &= U_1^\dagger(\omega z - 1) \cdot \eta \mathcal{S} \Psi(z),\end{aligned}\tag{3.2.22b}$$

where we used the relations $\omega \tau = -1$ for $\mathbb{T}^2/\mathbb{Z}_4$. By comparing Eq. (3.2.21a) with Eq. (3.2.22a), we obtain the relation

$$U_1(z) = U_\tau(\omega z).\tag{3.2.23}$$

On the other hand, we obtain the relation

$$U_\tau(z) = U_1^\dagger(\omega z - 1),\tag{3.2.24}$$

by comparing Eq. (3.2.21b) with Eq. (3.2.22b). These lead to the following constraints by using the expression of $U_1(z)$ and $U_\tau(z)$ in Eqs. (3.2.6b) and (3.2.6c):

$$e^{2\pi i \alpha_1} = e^{2\pi i \alpha_\tau}, \quad e^{2\pi i \alpha_\tau} = e^{-2\pi i \alpha_1}.\tag{3.2.25}$$

Finally, we obtain the allowed values of the Scherk–Schwarz phases on the orbifold $\mathbb{T}^2/\mathbb{Z}_4$ as

$$(\alpha_1, \alpha_\tau) = (0, 0), \left(\frac{1}{2}, \frac{1}{2}\right).\tag{3.2.26}$$

- $\mathbb{T}^2/\mathbb{Z}_6$ case: The first method of calculation is

$$\Psi(\omega(z + 1)) = \eta \mathcal{S} \Psi(z + 1) = \eta \mathcal{S} \cdot U_1(z) \Psi(z),\tag{3.2.27a}$$

$$\Psi(\omega(z + \tau)) = \eta \mathcal{S} \Psi(z + \tau) = \eta \mathcal{S} \cdot U_\tau(z) \Psi(z),\tag{3.2.27b}$$

while the second method of calculation is

$$\Psi(\omega(z + 1)) = \Psi(\omega z + \tau) = U_\tau(\omega z) \Psi(\omega z) = U_\tau(\omega z) \cdot \eta \mathcal{S} \Psi(z),\tag{3.2.28a}$$

$$\begin{aligned}\Psi(\omega(z + \tau)) &= \Psi(\omega z + 1 - \tau) = U_1(\omega z - \tau) \Psi(\omega z - \tau) \\ &= U_1(\omega z - \tau) \cdot U_\tau^\dagger(\omega z - \tau) \Psi(\omega z) \\ &= U_1(\omega z - \tau) U_\tau^\dagger(\omega z - \tau) \cdot \eta \mathcal{S} \Psi(z),\end{aligned}\tag{3.2.28b}$$

where we used the relations $\omega \tau = 1 - \tau$ for $\mathbb{T}^2/\mathbb{Z}_6$. By comparing Eq. (3.2.27a) with Eq. (3.2.28a), we obtain the relation

$$U_1(z) = U_\tau(\omega z).\tag{3.2.29}$$

On the other hand, we obtain the relation

$$U_\tau(z) = U_1(\omega z - \tau) U_\tau^\dagger(\omega z - \tau),\tag{3.2.30}$$

by comparing Eq. (3.2.27b) with Eq. (3.2.28b). These lead to the following constraints by using the expression of $U_1(z)$ and $U_\tau(z)$ in Eqs. (3.2.6b) and (3.2.6c):

$$e^{2\pi i \alpha_1} = e^{2\pi i \alpha_\tau}, \quad e^{2\pi i \alpha_\tau} = e^{2\pi i (\alpha_1 - \alpha_\tau) - \pi i M} e^{-\frac{\pi i M}{\text{Im}\tau} \text{Im}[(\omega - 1 + \bar{\tau})z]}.\tag{3.2.31}$$

Finally, we obtain the allowed values of the Scherk–Schwarz phases on the orbifold $\mathbb{T}^2/\mathbb{Z}_6$ as

$$(\alpha_1, \alpha_\tau) = \begin{cases} (0, 0) & (\text{for } M = \text{even}), \\ \left(\frac{1}{2}, \frac{1}{2}\right) & (\text{for } M = \text{odd}). \end{cases} \quad (3.2.32)$$

by using the relation

$$\omega - 1 + \bar{\tau} = \omega(1 + \omega^2 + \omega^4) = 0. \quad (3.2.33)$$

Summarizing the results, we obtain the allowed values of the Scherk–Schwarz phases as

orbifold	M	allowed values of (α_1, α_τ)	corresponding values of Ma
$\mathbb{T}^2/\mathbb{Z}_2$	-	$(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$	$0, \frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2}$
$\mathbb{T}^2/\mathbb{Z}_3$	even	$(0, 0), (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})$	$0, \frac{2+\tau}{3}, \frac{1+2\tau}{3}$
	odd	$(\frac{1}{6}, \frac{1}{6}), (\frac{3}{6}, \frac{3}{6}), (\frac{5}{6}, \frac{5}{6})$	$\frac{5+\tau}{6}, \frac{1+\tau}{2}, \frac{1+5\tau}{6}$
$\mathbb{T}^2/\mathbb{Z}_4$	-	$(0, 0), (\frac{1}{2}, \frac{1}{2})$	$0, \frac{1+\tau}{2}$
$\mathbb{T}^2/\mathbb{Z}_6$	even	$(0, 0)$	0
	odd	$(\frac{1}{2}, \frac{1}{2})$	$\frac{1+\tau}{2}$

Here, the corresponding values of the Wilson-line phase are obtained by the redefinition of the field discussed in the subsection 3.1.3 as

$$Ma' = \alpha_1\tau - \alpha_\tau, \quad \alpha'_1 = \alpha'_\tau = 0. \quad (3.2.34)$$

Note that the allowed values of the Wilson-line phase are equal to fixed points except for $\mathbb{T}^2/\mathbb{Z}_3$ and $\mathbb{T}^2/\mathbb{Z}_6$ with odd M . (See Refs. [35, 36].)

3.2.2 $\mathbb{Z}_N$ eigenfunctions

On twisted orbifold $\mathbb{T}^2/\mathbb{Z}_N$, the zero-mode part of the 6D fermion $\Psi_{+,0}(x, z)$ is decomposed in the same manner of the $\mathbb{T}^2$ case, Eq. (3.1.50), as follows

$$\Psi_{+,0}(x, z) = \begin{cases} \sum_{j=0}^{M-1} \chi_{R,0,j}(x) \otimes \begin{pmatrix} \psi_{+,j;\mathbb{Z}_N}(z) \\ 0 \end{pmatrix} & (\text{for } M > 0), \\ \sum_{j=0}^{M-1} \chi_{L,0,j}(x) \otimes \begin{pmatrix} 0 \\ \psi_{-,j;\mathbb{Z}_N}(z) \end{pmatrix} & (\text{for } M > 0), \end{cases} \quad (3.2.35)$$

except for replacement the zero-mode functions $\psi_{\pm,j}(z)$ by the eigenfunctions on $\mathbb{T}^2/\mathbb{Z}_N$, $\psi_{\pm,j;\mathbb{Z}_N,\eta}(z)$. Since the mode functions (including higher modes) on $\mathbb{T}^2/\mathbb{Z}_N$ must satisfy the boundary conditions (3.2.7),

$$\begin{cases} \psi_+(\omega z) = \eta \psi_+(z), \\ \psi_-(\omega z) = (\omega \eta) \psi_-(z), \end{cases} \quad (\eta^N = 1), \quad (3.2.36)$$

the eigenfunctions are sorted by the $\mathbb{Z}_N$ eigenvalues

$$\eta \in \{1, \omega, \omega^2, \dots, \omega^{N-1}\}. \quad (3.2.37)$$

The zero-mode wavefunctions $\psi_{\pm,j;\mathbb{Z}_N,\eta}(z)$ ($j = 0, 1, \dots, |M| - 1$), which satisfy the eigenvalue equation $\psi_{\pm,j;\mathbb{Z}_N,\eta}(\omega z) = \eta \psi_{\pm,j;\mathbb{Z}_N,\eta}(z)$, can be constructed as the following linear combinations,

$$\psi_{\pm,j;\mathbb{Z}_N,\eta}(z) = \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x \psi_{\pm,j}(\omega^x z). \quad (3.2.38)$$

Here, the $\mathbb{Z}_N$ -twisted functions $\psi_{\pm,j}(\omega^x z)$ ($x = 0, 1, \dots, N-1$) can be expanded by the orthonormal complete set $\{\psi_{\pm,k}(z) \mid k = 0, 1, \dots, |M| - 1\}$,

$$\psi_{\pm,j}(\omega^x z) = \sum_{k=0}^{M-1} C_{jk}^{(\omega^x)} \psi_{\pm,k}(z), \quad (x = 0, 1, \dots, N-1), \quad (3.2.39)$$

with the complex coefficient

$$C_{jk}^{(\omega^x)} = \int_{\mathbb{T}^2} d^2 z \psi_{\pm,j}(\omega^x z) (\psi_{\pm,k}(z))^*. \quad (3.2.40)$$

Using this coefficient, the $\mathbb{Z}_N$ eigenstates are rewritten in the form

$$\psi_{\pm,j;\mathbb{Z}_N,\eta}(z) = \sum_{k=0}^{M-1} \left(\frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x C_{jk}^{(\omega^x)} \right) \psi_{\pm,k}(z). \quad (3.2.41)$$

By integrating the zero-mode part of the 6D chiral action over the extra space, we obtain 4D effective action of chiral fermion,

$$\begin{aligned} & \int_{\mathbb{M}^4} d^4 x \int_{\mathbb{T}^2} d^2 z \bar{\Psi}_{\mathbb{Z}_N,+0}(x, z) i \Gamma^M D_M^{(0)} \Psi_{\mathbb{Z}_N,+0}(x, z) \\ &= \int_{\mathbb{M}^4} d^4 x \sum_{j,k=0}^{M-1} \bar{\chi}_{R,0,k}(x) [\mathcal{K}_{jk}^{(\mathbb{Z}_N;\eta)}] i \gamma^\mu D_\mu^{(0)} \chi_{R,0,j}(x). \end{aligned} \quad (3.2.42)$$

Here, the complex coefficient matrix $\mathcal{K}_{jk}^{(\mathbb{Z}_N;\eta)}$ is defined by

$$\begin{aligned} \mathcal{K}_{jk}^{(\mathbb{Z}_N;\eta)} &:= \int_{\mathbb{T}^2} d^2 z \psi_{\pm,j;\mathbb{Z}_N,\eta}(z) (\psi_{\pm,k;\mathbb{Z}_N,\eta}(z))^* \\ &= \frac{1}{N^2} \sum_{x,y=0}^{N-1} \sum_{\ell=0}^{M-1} \bar{\eta}^x \eta^y C_{j\ell}^{(\omega^x)} (C_{k\ell}^{(\omega^y)})^*, \end{aligned} \quad (3.2.43)$$

where we used the orthonormality

$$\int_{\mathbb{T}^2} d^2 z \psi_{\pm,j}(z) (\psi_{\pm,k}(z))^* = \delta_{j,k}. \quad (3.2.44)$$

We can obtain the number of physical states on $\mathbb{T}^2/\mathbb{Z}_N$ as its rank,

$$[\text{The number of physical states}] = \text{rank} \left[\mathcal{K}_{jk}^{(\mathbb{Z}_N;\eta)} \right], \quad (3.2.45)$$

and this number is just the number of generations. As an example, the profiles of $|\psi_{+,j;\mathbb{Z}_3,\eta}(z)|^2$ for $M = 2$ and $\alpha_1 = \alpha_\tau = 0$ are depicted in Fig. 3.5. One can see that the number of independent

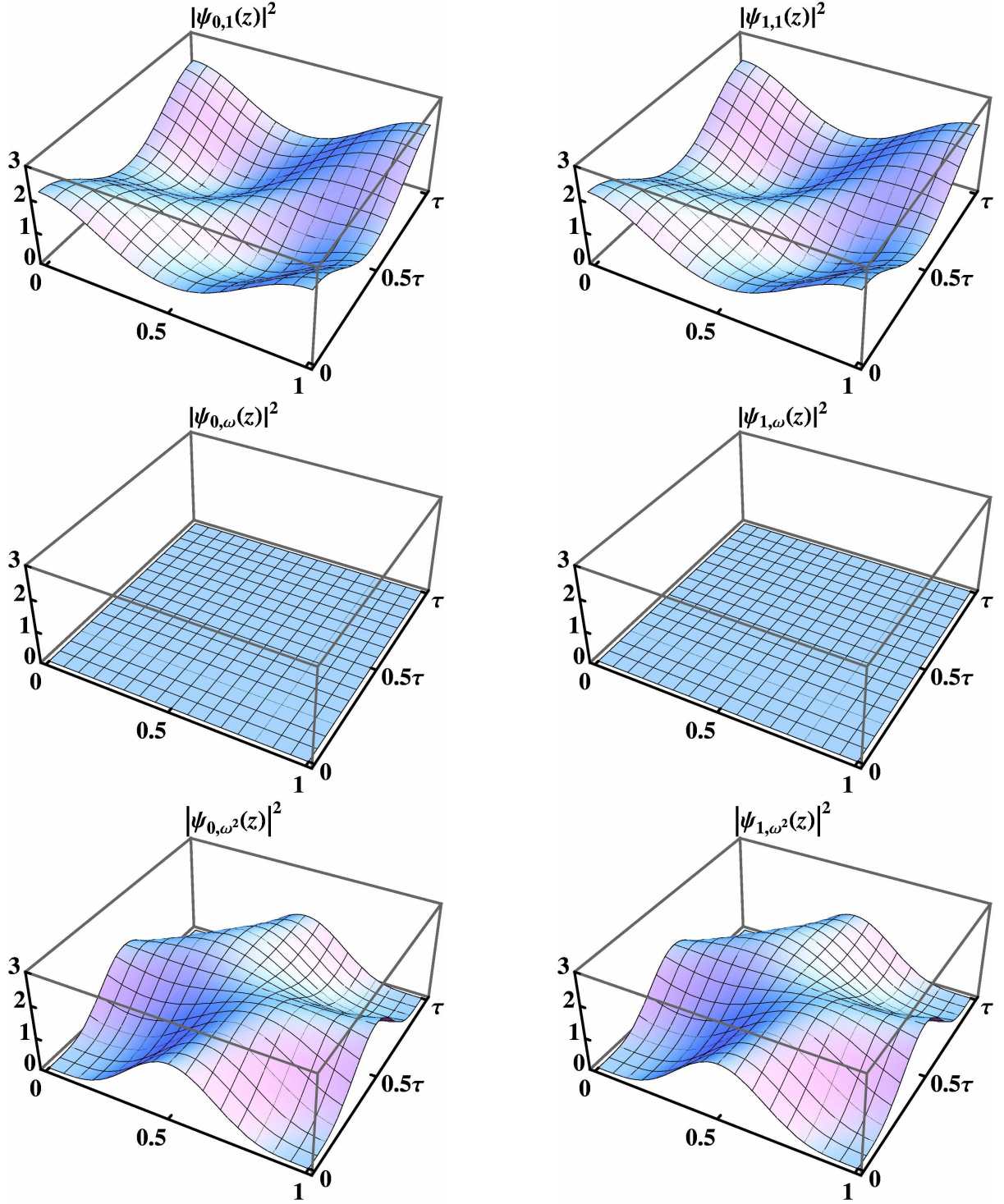


Figure 3.5: The probability densities $|\psi_{+,j;\mathbb{Z}_3,\eta}(z)|^2$ for $M = 2$. These figures are depicted with a complex coordinate $z = x + \tau y$ ($0 \leq x, y \leq 1$) for convenience of explanation. The top two figures are probability densities for $\eta = 1$, the middle two figures are for $\eta = \omega$, and the bottom two figures are for $\eta = \bar{\omega}$. Three figures on the left-hand side are the profiles with $j = 0$, and three figures on the right-hand side are for $j = 1$. The convention differs a little from the one in [42]. $\psi_{\mathbb{T}^2/\mathbb{Z}_{3+},0}^{(j,0)}(z)_\eta$ corresponds to $\psi_{+,j;\mathbb{Z}_N,\eta}(z)$ with $\alpha_1 = \alpha_\tau = 0$. (Quoted from Ref. [42].)

zero-modes is 0 for $\eta = \omega$ and 1 both for $\eta = 1, \bar{\omega}$, and this result is consistent with first row of Table 4.1.

In general, however, it is too difficult to evaluate the rank of the matrix $\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)}$. This is because of the reason as follows: In order to evaluate $\text{rank} \left[\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} \right]$, we should calculate the integral of the products of Jacobi's theta functions, as in Eq. (3.2.43). Such an integral is highly non-trivial and cannot be performed analytically. Furthermore, since we need to calculate M^2 components of the matrix $\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)}$, the evaluation becomes harder at an accelerated pace as the magnetic flux M becomes larger.

The operator formalism analysis discussed in Chapter 4 make this evaluation straightforward, because we can obtain the analytic form of the coefficients $C_{jk}^{(\omega^x)}$ with help of the operator formalism.

Chapter 4

Operator Formalism Analysis

In this chapter, we study the operator formalism analysis [43] which is very useful for analyzing magnetized orbifolds.

4.1 Quantum mechanics on $\mathbb{T}^2$ with magnetic flux

Let us consider the Schrödinger equation

$$\begin{aligned} H\psi(\mathbf{y}) &= E\psi(\mathbf{y}), \\ H &= (-i\nabla - e\mathbf{A}(\mathbf{y}))^2, \end{aligned} \tag{4.1.1}$$

where $\mathbf{y} = (y_1, y_2)^T$ is the 2D coordinate vector, e is the gauge coupling constant. $\mathbf{A}(\mathbf{y})$ is the gauge potential which is given by

$$\mathbf{A}(\mathbf{y}) = -\frac{1}{2}\Omega(\mathbf{y} + \mathbf{a}), \tag{4.1.2}$$

where $\mathbf{a} = (a_1, a_2)^T$ is the Wilson-line phase. This is represented in the components as

$$A_i(\mathbf{y}) = -\frac{1}{2} \sum_{j=1}^2 \Omega_{ij}(y_j + a_j). \tag{4.1.3}$$

The field strength B_{ij} is defined as follows and is equal to the anti-symmetric part of Ω ,

$$B_{ij} := \partial_i A_j - \partial_j A_i = \frac{1}{2}(\Omega_{ij} - \Omega_{ji}). \tag{4.1.4}$$

Since the field strength is gauge invariant, only the anti-symmetric part of Ω is physical, while the symmetric part of Ω is unphysical and can be gauged away.

On the two-dimensional torus $\mathbb{T}^2$, the coordinate vector is identified as

$$\mathbf{y} \sim \mathbf{y} + \sum_{a=1}^2 n_a \mathbf{u}_a \quad (n_a \in \mathbb{Z}), \tag{4.1.5}$$

where $\mathbf{u}_a$ ($a = 1, 2$) are the basis vectors of the torus. We choose $\mathbf{u}_1 = (1, 0)^T$ and $\mathbf{u}_2 = (\text{Re } \tau, \text{Im } \tau)^T$ without loss of generality. For the torus-shifts $\mathbf{y} \rightarrow \mathbf{y} + \mathbf{u}_a$, the gauge potential (4.1.3) changes as

$$\mathbf{A}(\mathbf{y} + \mathbf{u}_a) = \mathbf{A}(\mathbf{y}) - \frac{1}{2}\Omega \mathbf{u}_a. \tag{4.1.6}$$

This must become a gauge transformation because of the consistency of the system. Correspondingly, the torus-shifted matter field $\psi(\mathbf{y} + \mathbf{u}_a)$ should relate with the original field $\psi(\mathbf{y})$ through the gauge transformation. Therefore, we find the following boundary condition on the gauge fields

$$\mathbf{A}(\mathbf{y} + \mathbf{u}_a) = \mathbf{A}(\mathbf{y}) + \nabla \chi_a(\mathbf{y}), \quad (4.1.7)$$

$$\psi(\mathbf{y} + \mathbf{u}_a) = e^{ie\chi_a(\mathbf{y}) + 2\pi i\alpha_a} \psi(\mathbf{y}), \quad (4.1.8)$$

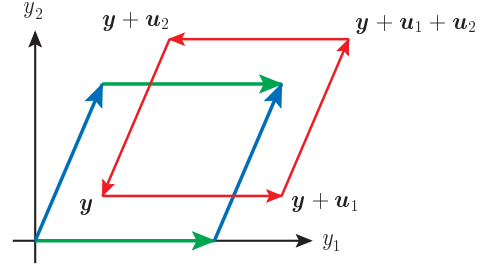
where

$$\chi_a(\mathbf{y}) = -\frac{1}{2} \mathbf{u}_a^T \Omega^T (\mathbf{y} + \mathbf{a}), \quad (4.1.9)$$

and α_a ($a = 1, 2$) are real parameters called the Scherk–Schwarz phases. Furthermore, the single-valuedness of the wavefunctions requires the quantization of the magnetic flux by considering contractible loops, e.g.,

$$\mathbf{y} \rightarrow \mathbf{y} + \mathbf{u}_1 \rightarrow \mathbf{y} + \mathbf{u}_1 + \mathbf{u}_2 \rightarrow \mathbf{y} + \mathbf{u}_2 \rightarrow \mathbf{y},$$

as in Fig. 4.1,



$$e \mathbf{u}_1^T B \mathbf{u}_2 = 2\pi M, \quad (M \in \mathbb{Z}), \quad (4.1.10)$$

where $B := \frac{1}{2}(\Omega - \Omega^T)$ is the magnetic field (field strength) defined in Eq. (4.1.4).

Figure 4.1: The contractible loop, $\mathbf{y} \rightarrow \mathbf{y} + \mathbf{u}_1 \rightarrow \mathbf{y} + \mathbf{u}_1 + \mathbf{u}_2 \rightarrow \mathbf{y} + \mathbf{u}_2 \rightarrow \mathbf{y}$, is illustrated by red vectors. The blue and green vectors show the torus identifications.

Next, we move on to the operator formalism, by introducing the coordinate operator $\hat{\mathbf{y}}$ and momentum operator $\hat{\mathbf{p}}$ satisfying the canonical commutation relations,

$$[\hat{y}_i, \hat{p}_j] = i\delta_{i,j}, \quad [\hat{y}_i, \hat{y}_j] = [\hat{p}_i, \hat{p}_j] = 0. \quad (4.1.11)$$

The wavefunction $\psi(\mathbf{y})$ is represented in the operator formalism as $\langle \mathbf{y} | \psi \rangle$, and the Hamiltonian and the Schrödinger equation becomes

$$\hat{H} = \left(\hat{\mathbf{p}} + \frac{e}{2} \Omega (\hat{\mathbf{y}} + \mathbf{a}) \right)^2 =: (\hat{\mathbf{p}}')^2, \quad \hat{H} |\psi\rangle = E |\psi\rangle. \quad (4.1.12)$$

The constraint conditions (4.1.8) are rewritten as

$$\langle \mathbf{y} | e^{+i\mathbf{u}_a^T \hat{\mathbf{p}}} | \psi \rangle = \langle \mathbf{y} | e^{-i\frac{e}{2}\mathbf{u}_a^T \Omega^T (\hat{\mathbf{y}} + \mathbf{a}) + 2\pi i\alpha_a} | \psi \rangle. \quad (4.1.13)$$

By using the formula (A.3), the exponentials of the above equation becomes

$$\begin{aligned} e^{+i\mathbf{u}_a^T \Omega^T (\hat{\mathbf{y}} + \mathbf{a})} e^{i\mathbf{u}_a^T \hat{\mathbf{p}}} &= e^{i\mathbf{u}_a^T \left(\hat{\mathbf{p}} + \frac{e}{2} \Omega^T (\hat{\mathbf{y}} + \mathbf{a}) \right)} \exp \left\{ \frac{1}{2} \left[i \frac{e}{2} u_{ai} \Omega_{ji} (\hat{y}_j + a_j), i u_{ak} \hat{p}_k \right] \right\} \\ &= e^{i\mathbf{u}_a^T \left(\hat{\mathbf{p}} + \frac{e}{2} \Omega^T (\hat{\mathbf{y}} + \mathbf{a}) \right)} e^{-i\frac{e}{4}\mathbf{u}_a^T \frac{\Omega + \Omega^T}{2} \mathbf{u}_a}. \end{aligned} \quad (4.1.14)$$

Therefore, we find the boundary conditions for the torus-shifts as follows:

$$e^{i\hat{T}_a} |\psi\rangle = e^{2\pi i\alpha_a} |\psi\rangle, \quad \hat{T}_a := \mathbf{u}_a^T \left(\hat{\mathbf{p}} + \frac{e}{2} \Omega^T (\hat{\mathbf{y}} + \mathbf{a}) \right). \quad (4.1.15)$$

Here and hereafter, we choose the gauge ¹

$$\frac{\Omega + \Omega^T}{2} = 0. \quad (4.1.16)$$

The operators $\hat{\mathbf{p}}'$ and $\hat{T}_a$ satisfy the following commutation relations:

$$[\hat{p}'_i, \hat{p}'_j] = i e B_{ij}, \quad (4.1.17)$$

$$[\hat{T}_a, \hat{T}_b] = -2\pi i M \epsilon_{ab}, \quad (4.1.18)$$

$$[\hat{p}'_i, \hat{T}_a] = 0, \quad (4.1.19)$$

where $\epsilon_{ab} = -\epsilon_{ba}$ is the anti-symmetric tensor with $\epsilon_{12} = +1$.

For further analysis of this system, we define new operators,

$$\hat{Y} := -\frac{\sqrt{2}}{\omega} \hat{p}'_2, \quad \hat{P} := \sqrt{2} \hat{p}'_1, \quad (4.1.20)$$

$$\hat{\mathcal{Y}} := \frac{\hat{T}_1 - 2\pi\alpha_1}{2\pi M}, \quad \hat{\mathcal{P}} := -\hat{T}_2 + 2\pi\alpha_2, \quad (4.1.21)$$

where $\omega := 2e B_{12}$. We find immediately that these operators satisfy the canonical commutation relations

$$[\hat{Y}, \hat{P}] = i, \quad (4.1.22a)$$

$$[\hat{\mathcal{Y}}, \hat{\mathcal{P}}] = i, \quad (4.1.22b)$$

$$[\hat{Y}, \hat{\mathcal{Y}}] = [\hat{Y}, \hat{\mathcal{P}}] = [\hat{P}, \hat{\mathcal{Y}}] = [\hat{P}, \hat{\mathcal{P}}] = 0. \quad (4.1.22c)$$

These relations tell us that the transformation $\{\hat{y}_i, \hat{p}_i \mid i = 1, 2\} \rightarrow \{\hat{Y}, \hat{P}, \hat{\mathcal{Y}}, \hat{\mathcal{P}}\}$ is canonical. With these new operators, the Hamiltonian (4.1.1) can be rewritten in the form of harmonic oscillator,

$$\hat{H} = \frac{1}{2} \hat{P}^2 + \frac{\omega^2}{2} \hat{Y}^2. \quad (4.1.23)$$

On the other hand, the constraint conditions (4.1.15) become

$$e^{2\pi i M \hat{\mathcal{Y}}} |\psi\rangle = |\psi\rangle, \quad (4.1.24a)$$

$$e^{-i \hat{\mathcal{P}}} |\psi\rangle = |\psi\rangle. \quad (4.1.24b)$$

The meanings of these constraint conditions can be shown clearly by taking an eigenstate of $\hat{\mathcal{Y}}$, which obeys the relation $\hat{\mathcal{Y}} |\mathcal{Y}\rangle = \mathcal{Y} |\mathcal{Y}\rangle$. The first condition reduces

$$e^{2\pi i M \mathcal{Y}} |\mathcal{Y}\rangle = |\mathcal{Y}\rangle, \quad (4.1.25)$$

and the second becomes

$$|\mathcal{Y} - 1\rangle = |\mathcal{Y}\rangle, \quad (4.1.26)$$

with help of the relation (4.1.22b). These imply that the eigenvalue $\mathcal{Y}$ is quantized as

$$\mathcal{Y} = \frac{j}{M}, \quad (j = 0, 1, 2, \dots, |M| - 1), \quad (4.1.27)$$

¹Since the symmetric part of Ω is gauge dependent as is mentioned in above, this gauge fixing is justified.

in every energy level. Finally, we find that

$$\hat{H} \left| n, \frac{j}{M} \right\rangle = E_n \left| n, \frac{j}{M} \right\rangle, \quad E_n = |\omega| \left(n + \frac{1}{2} \right), \quad (4.1.28)$$

$$\left\langle n, \frac{j}{M} \left| n', \frac{k}{M} \right. \right\rangle = \delta_{n,n'} \delta_{j,k}, \quad (4.1.29)$$

where $n, n' = 0, 1, \dots$, and $j, k = 0, 1, \dots, |M| - 1$.

4.2 Operator formalism on twisted orbifolds

In this section, we investigate the operator formalism on twisted orbifolds $\mathbb{T}^2/\mathbb{Z}_N$.

The twisted orbifold $\mathbb{T}^2/\mathbb{Z}_N$ is defined by the discrete symmetry under the rotation of $2\pi/N$ -radians described as

$$\mathbb{Z}_N : \mathbf{y} \rightarrow R_{\mathbb{Z}_N} \mathbf{y}, \quad R_{\mathbb{Z}_N} = \begin{pmatrix} \cos(2\pi/N) & -\sin(2\pi/N) \\ \sin(2\pi/N) & \cos(2\pi/N) \end{pmatrix}. \quad (4.2.1)$$

It is known that only $\mathbb{T}^2/\mathbb{Z}_N$ with $N = 2, 3, 4, 6$ are allowed on $\mathbb{T}^2$ as is mentioned in Section 3.2. The $\mathbb{Z}_N$ transformation property of the coordinate operator and the momentum operator is given by

$$\mathbb{Z}_N : \begin{cases} \hat{\mathbf{y}} \rightarrow \hat{U}_{\mathbb{Z}_N} \hat{\mathbf{y}} \hat{U}_{\mathbb{Z}_N}^{-1} = R_{\mathbb{Z}_N} \hat{\mathbf{y}}, \\ \hat{\mathbf{p}} \rightarrow \hat{U}_{\mathbb{Z}_N} \hat{\mathbf{p}} \hat{U}_{\mathbb{Z}_N}^{-1} = R_{\mathbb{Z}_N} \hat{\mathbf{p}}, \end{cases} \quad (4.2.2)$$

where the unitary operator $\hat{U}_{\mathbb{Z}_N}$ describes the $2\pi/N$ -rotation. The Hamiltonian transforms under the $\mathbb{Z}_N$ rotation as

$$\begin{aligned} \hat{H} \rightarrow \hat{U}_{\mathbb{Z}_N} \hat{H} \hat{U}_{\mathbb{Z}_N}^{-1} &= \left(R_{\mathbb{Z}_N} \hat{\mathbf{p}} + \frac{e}{2} \Omega (R_{\mathbb{Z}_N} \hat{\mathbf{y}} + \mathbf{a}) \right)^2 \\ &= \left(\hat{\mathbf{p}} + \frac{e}{2} (R_{\mathbb{Z}_N}^T \Omega R_{\mathbb{Z}_N}) (\hat{\mathbf{y}} + R_{\mathbb{Z}_N}^T \mathbf{a}) \right)^2. \end{aligned} \quad (4.2.3)$$

Thus, the discrete symmetry requires that

$$R_{\mathbb{Z}_N}^T \Omega R_{\mathbb{Z}_N} = \Omega, \quad (4.2.4)$$

and

$$R_{\mathbb{Z}_N}^T \mathbf{a} = \mathbf{a}. \quad (4.2.5)$$

The former condition can be rewritten as

$$[R_{\mathbb{Z}_N}, \Omega] = 0, \quad (4.2.6)$$

by using the property $R_{\mathbb{Z}_N} R_{\mathbb{Z}_N}^T = \mathbb{I}_2$, and is satisfied by the gauge condition (4.1.16), while the later condition means that the Wilson-line phase $\mathbf{a}$ must be a fixed point up to the torus-shifts,

$$R_{\mathbb{Z}_N}^T \mathbf{a} = \mathbf{a} + \sum_{a=1}^2 n_a \mathbf{u}_a, \quad (n_a \in \mathbb{Z}). \quad (4.2.7)$$

This constraint corresponds to Eq. (3.2.2) in the wavefunction analysis.

Torus-shift operators $\hat{T}_a$ ($a = 1, 2$) transform as follows:

$$\hat{T}_a \rightarrow \hat{U}_{\mathbb{Z}_N} \hat{T}_a \hat{U}_{\mathbb{Z}_N}^{-1} = (R_{\mathbb{Z}_N}^T \mathbf{u}_a)^T \left(\hat{\mathbf{p}} + \frac{e}{2} \Omega^T (\hat{\mathbf{y}} + \mathbf{a}) \right). \quad (4.2.8)$$

In order to find the transformation properties of $\hat{T}_a$, we need the expressions of the vector $R_{\mathbb{Z}_N}^T \mathbf{u}_a$,

$$\begin{aligned} \mathbb{Z}_2 : \quad & R_{\mathbb{Z}_2}^T \mathbf{u}_1 = -\mathbf{u}_1, & R_{\mathbb{Z}_2}^T \mathbf{u}_2 = -\mathbf{u}_2, \\ \mathbb{Z}_3 : \quad & R_{\mathbb{Z}_3}^T \mathbf{u}_1 = -\mathbf{u}_1 - \mathbf{u}_2, & R_{\mathbb{Z}_3}^T \mathbf{u}_2 = \mathbf{u}_1, \\ \mathbb{Z}_4 : \quad & R_{\mathbb{Z}_4}^T \mathbf{u}_1 = -\mathbf{u}_2, & R_{\mathbb{Z}_4}^T \mathbf{u}_2 = \mathbf{u}_1, \\ \mathbb{Z}_6 : \quad & R_{\mathbb{Z}_6}^T \mathbf{u}_1 = \mathbf{u}_1 - \mathbf{u}_2, & R_{\mathbb{Z}_6}^T \mathbf{u}_2 = \mathbf{u}_1. \end{aligned} \quad (4.2.9)$$

These lead to the transformation properties of torus-shift operators as follows:

$$\begin{aligned} \mathbb{Z}_2 : \quad & \hat{U}_{\mathbb{Z}_2} \hat{T}_1 \hat{U}_{\mathbb{Z}_2}^{-1} = -\hat{T}_1, & \hat{U}_{\mathbb{Z}_2} \hat{T}_2 \hat{U}_{\mathbb{Z}_2}^{-1} = -\hat{T}_2, \\ \mathbb{Z}_3 : \quad & \hat{U}_{\mathbb{Z}_3} \hat{T}_1 \hat{U}_{\mathbb{Z}_3}^{-1} = -\hat{T}_1 - \hat{T}_2, & \hat{U}_{\mathbb{Z}_3} \hat{T}_2 \hat{U}_{\mathbb{Z}_3}^{-1} = \hat{T}_1, \\ \mathbb{Z}_4 : \quad & \hat{U}_{\mathbb{Z}_4} \hat{T}_1 \hat{U}_{\mathbb{Z}_4}^{-1} = -\hat{T}_2, & \hat{U}_{\mathbb{Z}_4} \hat{T}_2 \hat{U}_{\mathbb{Z}_4}^{-1} = \hat{T}_1, \\ \mathbb{Z}_6 : \quad & \hat{U}_{\mathbb{Z}_6} \hat{T}_1 \hat{U}_{\mathbb{Z}_6}^{-1} = \hat{T}_1 - \hat{T}_2, & \hat{U}_{\mathbb{Z}_6} \hat{T}_2 \hat{U}_{\mathbb{Z}_6}^{-1} = \hat{T}_1. \end{aligned} \quad (4.2.10)$$

In the following analysis, we need the expression of the $\mathbb{Z}_N$ -twist transformation properties of the operators $\hat{\mathcal{Y}}, \hat{\mathcal{P}}$,

$$\mathbb{Z}_2 : \begin{cases} \hat{U}_{\mathbb{Z}_2} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_2}^{-1} = \hat{U}_{\mathbb{Z}_2}^{-1} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_2} = -\hat{\mathcal{Y}} - \frac{2\alpha_1}{M}, \\ \hat{U}_{\mathbb{Z}_2} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_2}^{-1} = \hat{U}_{\mathbb{Z}_2}^{-1} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_2} = -\hat{\mathcal{P}} + 4\pi\alpha_2, \end{cases} \quad (4.2.11)$$

$$\mathbb{Z}_3 : \begin{cases} \hat{U}_{\mathbb{Z}_3} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_3}^{-1} = -\hat{\mathcal{Y}} + \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{2\alpha_1 + \alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_3} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_3}^{-1} = -2\pi M \hat{\mathcal{Y}} - 2\pi(\alpha_1 - \alpha_2), \\ \hat{U}_{\mathbb{Z}_3}^{-1} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_3} = -\frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha_1 - \alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_3}^{-1} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_3} = 2\pi M \hat{\mathcal{Y}} - \hat{\mathcal{P}} + 2\pi(\alpha_1 + 2\alpha_2), \end{cases} \quad (4.2.12)$$

$$\mathbb{Z}_4 : \begin{cases} \hat{U}_{\mathbb{Z}_4} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_4}^{-1} = \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha_1 + \alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_4} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_4}^{-1} = -2\pi M \hat{\mathcal{Y}} - 2\pi(\alpha_1 - \alpha_2), \\ \hat{U}_{\mathbb{Z}_4}^{-1} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_4} = -\frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha_1 - \alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_4}^{-1} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_4} = 2\pi M \hat{\mathcal{Y}} + 2\pi(\alpha_1 + \alpha_2), \end{cases} \quad (4.2.13)$$

$$\mathbb{Z}_6 : \begin{cases} \hat{U}_{\mathbb{Z}_6} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_6}^{-1} = \hat{\mathcal{Y}} + \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_6} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_6}^{-1} = -2\pi M \hat{\mathcal{Y}} - 2\pi(\alpha_1 - \alpha_2), \\ \hat{U}_{\mathbb{Z}_6}^{-1} \hat{\mathcal{Y}} \hat{U}_{\mathbb{Z}_6} = -\frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha_1 - \alpha_2}{M}, \\ \hat{U}_{\mathbb{Z}_6}^{-1} \hat{\mathcal{P}} \hat{U}_{\mathbb{Z}_6} = 2\pi M \hat{\mathcal{Y}} + \hat{\mathcal{P}} + 2\pi\alpha_1. \end{cases} \quad (4.2.14)$$

In order to analyze the physical states on the $\mathbb{T}^2/\mathbb{Z}_N$, we calculate in the below along the following steps:

1. In order to investigate the allowed values of Scherk–Schwarz phases by requiring that the constraint conditions (4.1.24) hold for the $\mathbb{Z}_N$ -twisted state $\hat{U}_{\mathbb{Z}_N}|\psi\rangle$, i.e.,

$$e^{2\pi i M \hat{Y}}(\hat{U}_{\mathbb{Z}_N}|\psi\rangle) = (\hat{U}_{\mathbb{Z}_N}|\psi\rangle), \quad e^{-i\hat{P}}(\hat{U}_{\mathbb{Z}_N}|\psi\rangle) = (\hat{U}_{\mathbb{Z}_N}|\psi\rangle). \quad (4.2.15)$$

2. In order to calculate the analytic formula of the $\mathbb{Z}_N$ -twisted states $(\hat{U}_{\mathbb{Z}_N})^x|j/M\rangle$ ($x = 0, 1, \dots, N-1$),

$$(\hat{U}_{\mathbb{Z}_N})^x|j/M\rangle = \sum_{k=0}^{M-1} D_{jk}^{(\omega^x)}|k/M\rangle. \quad (4.2.16)$$

3. In order to construct the physical states on $\mathbb{T}^2/\mathbb{Z}_N$,

$$|j/M\rangle_{\mathbb{Z}_N, \eta} := \hat{Z}_N^{(\eta)}|j/M\rangle. \quad (4.2.17)$$

Here, the operator

$$\hat{Z}_N^{(\eta)} := \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x (\hat{U}_{\mathbb{Z}_N})^x, \quad (4.2.18)$$

is the projection operator on the $\mathbb{Z}_N$ eigenstate with the eigenvalue η and satisfies the relations

$$\hat{U}_{\mathbb{Z}_N} \hat{Z}_N^{(\eta)} = \eta \hat{Z}_N^{(\eta)}, \quad (\hat{Z}_N^{(\eta)})^2 = \hat{Z}_N^{(\eta)}, \quad (4.2.19)$$

because of the $\mathbb{Z}_N$ consistency, $(\hat{U}_{\mathbb{Z}_N})^N = \mathbf{1}, \eta^N = 1$.

4. In order to investigate the number of independent physical states, we evaluate the rank of the matrix $M_{jk}^{(\mathbb{Z}_N; \eta)}$ defined by

$$|j/M\rangle_{\mathbb{Z}_N, \eta} = \sum_{k=0}^{M-1} M_{jk}^{(\mathbb{Z}_N; \eta)}|k/M\rangle. \quad (4.2.20)$$

This matrix can be written as

$$M_{jk}^{(\mathbb{Z}_N; \eta)} = \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x D_{jk}^{(\omega^x)}. \quad (4.2.21)$$

As mentioned at the beginning of this chapter, the purpose of the operator formalism is to find the analytic expression of $C_{jk}^{(\omega^x)}$ in Eq. (3.2.40). For this purpose, we investigate the correspondence between the operator formalism analysis and the wavefunction analysis, and find the relations

$$C_{jk}^{(\omega^x)} = D_{jk}^{(\omega^x)}, \quad \mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} = M_{jk}^{(\mathbb{Z}_N; \eta)}, \quad (4.2.22)$$

in the next subsection.

In the following, we denote the $\mathbb{Z}_N$ -twist operator $\hat{U}_{\mathbb{Z}_N}$ as $\hat{U}$ simply in each case of $\mathbb{T}^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$).

4.2.1 Correspondence to the wavefunction analysis

In this subsection, we show the correspondence between the operator formalism analysis and the wavefunction analysis.

Firstly, we examine the meaning of the eigenvalue j in the states $|j/M\rangle$, i.e.,

$$\langle z | j/M \rangle = \psi_j(z), \quad (4.2.23)$$

where $|z\rangle$ is an eigenstate of the operator $\hat{z}$,

$$\hat{z} |z\rangle = z |z\rangle, \quad \hat{z} := \hat{y}_1 + i \hat{y}_2 = (1, i) \hat{\mathbf{y}}. \quad (4.2.24)$$

Through this subsection, we denote the operator $\tilde{O}$ as a differential operator acting on functions of the coordinates, which is the coordinate representation of the quantum mechanical operator $\hat{O}$ acting on states,

$$\langle z | \hat{O} = \tilde{O} \langle z |. \quad (4.2.25)$$

Since the operator $e^{2\pi i \hat{\mathcal{Y}}}$ gives the eigenvalue $e^{2\pi i \frac{j}{M}}$ by acting on the state $|j/M\rangle$, what we need to show is

$$e^{2\pi i \hat{\mathcal{Y}}} \psi_j(z) = e^{2\pi i \frac{j}{M}} \psi_j(z). \quad (4.2.26)$$

By replacing the operators as

$$\hat{\mathbf{y}} \rightarrow \mathbf{y}, \quad \hat{\mathbf{p}} \rightarrow -i \nabla, \quad (4.2.27)$$

we obtain the torus-shift operator $\tilde{T}_1$ in coordinate representation as follows:

$$\begin{aligned} \tilde{T}_1 &= (1, 0) \left[-i \begin{pmatrix} \partial/\partial y_1 \\ \partial/\partial y_2 \end{pmatrix} + \frac{e}{2} \cdot \frac{2\pi M}{e \operatorname{Im} \tau} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 + a_1 \\ y_2 + a_2 \end{pmatrix} \right] \\ &= -i(\partial + \bar{\partial}) - \frac{\pi M}{\operatorname{Im} \tau} \operatorname{Im}(z + a). \end{aligned} \quad (4.2.28)$$

This representation leads to

$$\begin{aligned} e^{i \tilde{T}_1 \frac{1}{M}} &= \exp \left(\frac{1}{M} (\partial + \bar{\partial}) - \frac{\pi i}{\operatorname{Im} \tau} \operatorname{Im}(z + a) \right) \\ &= e^{-\frac{\pi i}{\operatorname{Im} \tau} \operatorname{Im}(z + a)} e^{\frac{1}{M} \partial} e^{\frac{1}{M} \bar{\partial}}, \end{aligned} \quad (4.2.29)$$

with help of the formula (A.4). By acting the differential operator $e^{i \tilde{T}_1 \frac{1}{M}}$ on the wavefunction $\psi_j(z)$, we obtain

$$\begin{aligned} e^{i \tilde{T}_1 \frac{1}{M}} \psi_j(z) &= e^{-\frac{\pi i}{\operatorname{Im} \tau} \operatorname{Im}(z + a)} e^{\frac{1}{M} \partial} e^{\frac{1}{M} \bar{\partial}} \left\{ -\frac{\pi M}{2 \operatorname{Im} \tau} (z + a) \right\} \psi^{(j)}(z) \\ &= e^{-\frac{\pi i}{\operatorname{Im} \tau} \operatorname{Im}(z + a)} e^{-\frac{\pi i}{2 \operatorname{Im} \tau} (z + \frac{1}{M} + a)} \\ &\quad \times \mathcal{N} e^{\frac{\pi i M}{\operatorname{Im} \tau} (z + \frac{1}{M} + a) \{ \operatorname{Im}(z + a) + \frac{1}{2i} \}} \vartheta \left[\begin{matrix} (j + \alpha_1)/M \\ -\alpha_\tau \end{matrix} \right] (M(z + \frac{1}{M} + a), M\tau) \\ &= e^{-\frac{\pi i}{\operatorname{Im} \tau} \operatorname{Im}(z + a)} e^{-\frac{\pi i}{2 \operatorname{Im} \tau} (z + \frac{1}{M} + a)} \times \mathcal{N} e^{\frac{\pi i M}{\operatorname{Im} \tau} \frac{1}{M} \operatorname{Im}(z + a)} e^{\frac{\pi i M}{\operatorname{Im} \tau} (z + \frac{1}{M} + a) \frac{1}{2i}} \\ &\quad \times e^{\frac{\pi i M}{\operatorname{Im} \tau} (z + a) \operatorname{Im}(z + a)} \times e^{2\pi i \frac{j + \alpha_1}{M}} \cdot \vartheta \left[\begin{matrix} (j + \alpha_1)/M \\ -\alpha_\tau \end{matrix} \right] (M(z + a), M\tau) \end{aligned}$$

$$= e^{2\pi i \frac{j+\alpha_1}{M}} \times \psi_j(z), \quad (4.2.30)$$

where we used the equation of motion for $\psi^{(j)}(z)$, the formula

$$e^{\frac{1}{M}\partial} f(z, \bar{z}) = f\left(z + \frac{1}{M}, \bar{z}\right), \quad (4.2.31)$$

and the property of Jacobi's theta function (3.1.36). By the definition of the operator $\hat{\mathcal{Y}}$, Eq. (4.1.21), we find the relation

$$e^{2\pi i \hat{\mathcal{Y}}} \psi^{(j)}(z) = e^{-2\pi i \frac{\alpha_1}{M}} e^{i\tilde{T}_1 \frac{1}{M}} \psi^{(j)}(z) = e^{2\pi i \frac{j}{M}} \psi^{(j)}(z). \quad (4.2.32)$$

On the other hand, by replacing the operator

$$\tilde{T}_2 = -i(\tau\partial + \bar{\tau}\bar{\partial}) + \frac{\pi M}{\text{Im } \tau} \frac{\bar{\tau}(z+a) - \tau(\bar{z}+\bar{a})}{2i}, \quad (4.2.33)$$

and operating

$$e^{-i\tilde{\mathcal{P}} \frac{1}{M}} = e^{-\frac{i}{M}(-\frac{\pi M}{\text{Im } \tau} \text{Im}[\bar{\tau}(z+a)] - \tau\partial - \bar{\tau}\bar{\partial} + 2\pi\alpha_2)} \quad (4.2.34)$$

on the wavefunction $\psi_j(z)$, we obtain

$$e^{-i\tilde{\mathcal{P}} \frac{1}{M}} \psi_j(z) = \psi_{j+1}(z). \quad (4.2.35)$$

By comparing Eqs. (4.2.32) and (4.2.35) with the relations in the operator formalism,

$$e^{2\pi i \hat{\mathcal{Y}}} |j/M\rangle = e^{2\pi i \frac{j}{M}} |j/M\rangle, \quad e^{-i\hat{\mathcal{P}} \frac{1}{M}} |j/M\rangle = |(j+1)/M\rangle, \quad (4.2.36)$$

we find that the correspondence (4.2.23) is justified. In other words, the index j of $|j/M\rangle$ is same as the index j of the zero-mode function $\psi_j(z)$.

Next, we find the representation of the coefficients $C_{jk}^{(\omega^x)}$ in the operator formalism. We find that the state $|\omega z\rangle$ which is an eigenstate the coordinate operator $\hat{z}$ with the eigenvalue ωz is given by

$$|\omega z\rangle = \hat{U}_{\mathbb{Z}_N}^{-1} |z\rangle. \quad (4.2.37)$$

By using this relation, we find that the $\mathbb{Z}_N$ -twisted wavefunctions are represented in the operator formalism as

$$\psi_j(\omega^x z) = \langle \omega^x z | j/M \rangle = \langle z | (\hat{U}_{\mathbb{Z}_N})^x | j/M \rangle, \quad (4.2.38)$$

$$\psi_{\pm, j; \mathbb{Z}_N, \eta}(z) = \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x \langle z | (\hat{U}_{\mathbb{Z}_N})^x | j/M \rangle = \langle z | \hat{Z}_N^{(\eta)} | j/M \rangle. \quad (4.2.39)$$

Therefore, the coefficients $C_{jk}^{(\omega^x)}$ are represented as follows:

$$\begin{aligned} C_{jk}^{(\omega^x)} &:= \int_{\mathbb{T}^2} d^2 z \psi_j(\omega^x z) (\psi_k(z))^* \\ &= \int_{\mathbb{T}^2} d^2 z \langle \omega^x z | j/M \rangle \langle k/M | z \rangle \end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{T}^2} d^2 z \left\langle z \left| (\hat{U}_{\mathbb{Z}_N})^x \right| j/M \right\rangle \left\langle k/M \right| z \right\rangle \\
&= \left\langle k/M \left| (\hat{U}_{\mathbb{Z}_N})^x \right| j/M \right\rangle \\
&= D_{jk}^{(\omega^x)},
\end{aligned} \tag{4.2.40}$$

where we used the completeness relation

$$\int_{\mathbb{T}^2} dz |z\rangle \langle z| = \mathbf{1}. \tag{4.2.41}$$

Similar calculation holds for $\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)}$:

$$\begin{aligned}
\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} &:= \int_{\mathbb{T}^2} d^2 z \psi_{\pm, j; \mathbb{Z}_N, \eta}(z) (\psi_{\pm, k; \mathbb{Z}_N, \eta}(z))^* \\
&= \int_{\mathbb{T}^2} d^2 z \left\langle z \left| \hat{Z}_N^{(\eta)} \right| j/M \right\rangle \left\langle k/M \left| \hat{Z}_N^{(\eta)} \right| z \right\rangle \\
&= \left\langle k/M \left| \hat{Z}_N^{(\eta)} \cdot \hat{Z}_N^{(\eta)} \right| j/M \right\rangle \\
&= \left\langle k/M \left| \hat{Z}_N^{(\eta)} \right| j/M \right\rangle \\
&= M_{jk}^{(\mathbb{Z}_N; \eta)},
\end{aligned} \tag{4.2.42}$$

where we used the property of the projection operator $(\hat{Z}_N^{(\eta)})^2 = \hat{Z}_N^{(\eta)}$. Therefore, we find that

$$(\text{The number of physical states}) = \text{rank} \left[\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} \right] = \text{rank} \left[M_{jk}^{(\mathbb{Z}_N; \eta)} \right]. \tag{4.2.43}$$

Since we don't know the analytic expression of $\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)}$, we should calculate the integral in Eq. (3.2.43) whose integrand is the product of highly complicated linear combinations of Jacobi's theta. This troubles us about the investigation of the number of physical states. On the other hand, the calculation of $\text{rank} \left[M_{jk}^{(\mathbb{Z}_N; \eta)} \right]$ can be performed more smoothly than the calculation of $\text{rank} \left[\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} \right]$ since the analytic expression of $M_{jk}^{(\mathbb{Z}_N; \eta)}$ can be obtain as in the below. This is the biggest advantage of the operator formalism.

4.2.2 $\mathbb{T}^2/\mathbb{Z}_2$

In this subsection, we discuss the $\mathbb{T}^2/\mathbb{Z}_2$ orbifold.

• Allowed values of the Scherk–Schwarz phases on $\mathbb{T}^2/\mathbb{Z}_2$

In order to investigate allowed values of the Scherk–Schwarz phases, we operate $\hat{U}$ on the both sides of the constraint conditions (4.1.24). By using the $\mathbb{Z}_2$ transformation properties of the operators $\hat{\mathcal{Y}}$ and $\hat{\mathcal{P}}$, Eq. (4.2.11), we find that

$$\begin{aligned}
e^{-2\pi i M \hat{\mathcal{Y}}} \cdot e^{-2\pi i \cdot 2\alpha_1} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle, \\
e^{i\hat{\mathcal{P}}} \cdot e^{2\pi i \cdot 2\alpha_2} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle.
\end{aligned} \tag{4.2.44}$$

From the requirements (4.2.15), we find the following constraints,

$$e^{-2\pi i \cdot 2\alpha_1} = 1, \quad e^{2\pi i \cdot 2\alpha_2} = 1, \quad (4.2.45)$$

These lead to allowed values of the Scherk–Schwarz phases as follows:

$$(\alpha_1, \alpha_2) = (0, 0), \left(\frac{1}{2}, 0\right), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right). \quad (4.2.46)$$

• **$\mathbb{Z}_2$ -rotated states**

Next, we find the matrix elements $D_{jk}^{(\omega^x)}$ defined in Eq. (4.2.16). To do this, we first operate $e^{2\pi i \hat{\mathcal{Y}}}$ on the state $\hat{U} |j/M\rangle$. By using the formula (A.1) and the $\mathbb{Z}_2$ transformation properties (4.2.11), we find that

$$\begin{aligned} e^{2\pi i \hat{\mathcal{Y}}}(\hat{U} |j/M\rangle) &= \hat{U} e^{2\pi i (\hat{U}^{-1} \hat{\mathcal{Y}} \hat{U})} |j/M\rangle \\ &= \hat{U} e^{2\pi i (-\hat{\mathcal{Y}} - \frac{2\alpha_1}{M})} |j/M\rangle \\ &= e^{2\pi i \frac{-j-2\alpha_1}{M}} (\hat{U} |j/M\rangle). \end{aligned} \quad (4.2.47)$$

This eigenvalue equation implies that the $\mathbb{Z}_2$ -twisted state $\hat{U} |j/M\rangle$ is proportional to the state $|(-j-2\alpha_1)/M\rangle$, i.e.,

$$\hat{U} |j/M\rangle = e^{2\pi i \eta_j} |(-j-2\alpha_1)/M\rangle, \quad (4.2.48)$$

where $\eta_j \in [0, 1)$ is a real parameter. We operate the twist operator $\hat{U}$ on the both sides of the above equation and obtain

$$\hat{U}^2 |j/M\rangle = e^{2\pi i (\eta_j + \eta_{-j-2\alpha_1})} |j/M\rangle. \quad (4.2.49)$$

Since the twist operator $\hat{U}$ must satisfy the $\mathbb{Z}_2$ consistency condition, $\hat{U}^2 = \mathbf{1}$, we find that

$$\eta_j + \eta_{-j-2\alpha_1} = n, \quad (n \in \mathbb{Z}). \quad (4.2.50)$$

On the other hand, $\mathbb{Z}_2$ -twisted state $\hat{U} |j/M\rangle$ is also obtained as follows:

$$\begin{aligned} \hat{U} |j/M\rangle &= \hat{U} (e^{-i \hat{\mathcal{P}} \frac{j}{M}} |0/M\rangle) \\ &= e^{-i (\hat{U} \hat{\mathcal{P}} \hat{U}^{-1}) \frac{j}{M}} \hat{U} |0/M\rangle \\ &= e^{-i (-\hat{\mathcal{P}} + 4\pi\alpha_2) \frac{j}{M}} \cdot e^{2\pi i \eta_0} |-2\alpha_1/M\rangle \\ &= e^{2\pi i (\eta_0 - \frac{j}{M} \cdot 2\alpha_2)} |(-j/M) - 2\alpha_1/M\rangle. \end{aligned} \quad (4.2.51)$$

By comparing this with Eq. (4.2.48), we find that

$$\eta_j = \eta_0 - \frac{j}{M} \cdot 2\alpha_2 + n, \quad (n \in \mathbb{Z}). \quad (4.2.52)$$

Substituting this into Eq. (4.2.50), we find that

$$\eta_0 + \eta_{-j-2\alpha_1} = 2\eta_0 + \frac{2\alpha_1}{M} \cdot 2\alpha_2 = n, \quad (n \in \mathbb{Z}). \quad (4.2.53)$$

Therefore, the phase factor of twisted state $\hat{U} |0/M\rangle$ is obtained as

$$\eta_0 = -\frac{\alpha_1}{M} \cdot 2\alpha_2 + \frac{n}{2} \quad (n = 0, 1). \quad (4.2.54)$$

Here, we can put $n = 0$ without loss of generality since n gives unphysical phase. Substituting this into Eq. (4.2.52), the phase factor η_j becomes

$$\eta_j = -\frac{j + \alpha_1}{M} \cdot 2\alpha_2. \quad (4.2.55)$$

Thus, we find the $\mathbb{Z}_2$ -twisted states as follows:

$$\hat{U} |j/M\rangle = e^{-2\pi i \frac{j+\alpha_1}{M} \cdot 2\alpha_2} |(-j - 2\alpha_1)/M\rangle. \quad (4.2.56)$$

This gives the expression of the matrix element

$$D_{jk}^{(\omega)} = e^{-2\pi i \frac{j+\alpha_1}{M} \cdot 2\alpha_2} \delta_{k, -j-2\alpha_1}. \quad (4.2.57)$$

Here, the index of the Kronecker's delta is understood as mod $|M|$.

• **The number of physical solutions on $\mathbb{T}^2/\mathbb{Z}_2$**

By investigating the rank of the matrix

$$M_{jk}^{(\mathbb{Z}_2; \eta)} = \frac{1}{2} \left(\delta_{j,k} + \eta D_{jk}^{(\omega)} \right), \quad \eta \in \{+1, -1\}, \quad (4.2.58)$$

we obtain the number of physical states. On the $\mathbb{T}^2/\mathbb{Z}_2$, the expression of the matrix $M^{(\mathbb{Z}_2; \eta)}$ consists roughly in two Kronecker's delta. Because of this simplicity, we can evaluate analytically the rank of the matrix $M^{(\mathbb{Z}_2; \eta)}$. This is in contrast to the other orbifolds $\mathbb{T}^2/\mathbb{Z}_{3,4,6}$.

1. $(\alpha_1, \alpha_2) = (0, 0)$ **case**

In this pattern, the explicit form of $M_{jk}^{(\mathbb{Z}_2; \eta)}$ is

$$M_{jk}^{(\mathbb{Z}_2; \eta=\pm)} = \frac{1}{2} (\delta_{j,k} \pm \delta_{k, |M|-j}). \quad (4.2.59)$$

The $\mathbb{Z}_2$ eigenstates in this pattern are given by the following linear combination,

$$|j/M; \pm\rangle_{\mathbb{Z}_2} := \hat{\mathbb{Z}}_2^{\eta=\pm} |j/M\rangle = \frac{1}{2} \left(|j/M\rangle \pm |(|M| - j)/M\rangle \right), \quad (4.2.60)$$

where the sign $\pm$ means the $\mathbb{Z}_2$ eigenvalue $\eta = \pm 1$. It is found immediately that the above states have the property

$$|(|M| - j)/M; \pm\rangle_{\mathbb{Z}_2} = \pm |j/M; \pm\rangle_{\mathbb{Z}_2}, \quad (4.2.61)$$

because of the periodicity

$$|(j + M)/M\rangle = |j/M\rangle. \quad (4.2.62)$$

This implies that the independent states are

$$|j/M; \pm\rangle_{\mathbb{Z}_2} \quad \left(j \leq \frac{|M|}{2}\right). \quad (4.2.63)$$

In addition, for $j = 0$ and $j = |M|/2$ with $\mathbb{Z}_2$ eigenvalue $\eta = -1$ (the state for $j = |M|/2$ exists only when M is even), $\mathbb{Z}_2$ eigenstate vanishes,

$$|0/M; -\rangle_{\mathbb{Z}_2} = 0, \quad |(|M|/2)/M; -\rangle = 0. \quad (4.2.64)$$

These properties tell us the number of physical states as follows: For $M = \text{even}$, the independent $\mathbb{Z}_2$ eigenstates with $\eta = +1$ are

$$|j/M; +\rangle_{\mathbb{Z}_2} \quad \left(j = 0, \dots, \frac{|M|}{2}\right), \quad (4.2.65)$$

and the independent eigenstates with $\eta = -1$ are

$$|j/M; -\rangle_{\mathbb{Z}_2} \quad \left(j = 1, \dots, \frac{|M|}{2} - 1\right). \quad (4.2.66)$$

Therefore, the number of independent states are given by

$$\text{rank} \left[M_{jk}^{(\mathbb{Z}_2; \eta=+)} \right] = \frac{|M|}{2} + 1, \quad \text{rank} \left[M_{jk}^{(\mathbb{Z}_2; \eta=-)} \right] = \frac{|M|}{2} - 1. \quad (4.2.67)$$

On the other hand, for $M = \text{odd}$, since the independent states with $\eta = +1$ are

$$|j/M; +\rangle_{\mathbb{Z}_2} \quad \left(j = 0, \dots, \frac{|M| - 1}{2}\right), \quad (4.2.68)$$

and the independent eigenstates with $\eta = -1$ are

$$|j/M; -\rangle_{\mathbb{Z}_2} \quad \left(j = 1, \dots, \frac{|M| - 1}{2}\right), \quad (4.2.69)$$

the number of independent states are

$$\text{rank} \left[M_{jk}^{(\mathbb{Z}_2; \eta=+)} \right] = \frac{|M| - 1}{2} + 1, \quad \text{rank} \left[M_{jk}^{(\mathbb{Z}_2; \eta=-)} \right] = \frac{|M| - 1}{2}. \quad (4.2.70)$$

These can be expressed together as

$$\begin{aligned} (\# \text{ of physical states for } \eta = +1) &= \left\lfloor \frac{|M|}{2} \right\rfloor + 1, \\ (\# \text{ of physical states for } \eta = -1) &= \left\lfloor \frac{|M| - 1}{2} \right\rfloor, \end{aligned} \quad (4.2.71)$$

where $\lfloor x \rfloor$ is the floor function. This becomes in the form of a table as follows:

$ M $		0	1	2	3	4	5	6	7	8	9	10
η	+1	1	1	2	2	3	3	4	4	5	5	6
	-1	0	0	0	1	1	2	2	3	3	4	4

This case has been already studied. The above table obtained by the operator formalism analysis reproduces the same result as the earlier study, the Table 1 in [41].

2. $(\alpha_1, \alpha_2) = (\frac{1}{2}, 0)$ **case**

In this pattern, the explicit form of $M_{jk}^{(\mathbb{Z}_2; \eta)}$ is

$$M_{jk}^{(\mathbb{Z}_2; \eta=\pm)} = \frac{1}{2} (\delta_{j,k} \pm \delta_{k, |M| - j - 1}) . \quad (4.2.72)$$

The $\mathbb{Z}_2$ eigenstates in this pattern are given by

$$|j/M; \pm\rangle_{\mathbb{Z}_2} := \widehat{\mathbb{Z}}_2^{\eta=\pm} |j/M\rangle = \frac{1}{2} \left(|j/M\rangle \pm |(|M| - 1 - j)/M\rangle \right) . \quad (4.2.73)$$

This has the following properties:

$$\begin{aligned} |(|M| - 1 - j)/M; \pm\rangle &= \pm |j/M; \pm\rangle , \\ |0/M; -\rangle &= 0 , \quad \left| \left(\frac{|M|-1}{2} \right) / M; -\right\rangle = 0 . \end{aligned} \quad (4.2.74)$$

We count the number of independent states in the similar way to the $(\alpha_1, \alpha_2) = (0, 0)$ case as follows:

$$\begin{aligned} (\# \text{ of physical states for } \eta = +1) &= \left\lfloor \frac{|M| + 1}{2} \right\rfloor , \\ (\# \text{ of physical states for } \eta = -1) &= \left\lfloor \frac{|M|}{2} \right\rfloor . \end{aligned} \quad (4.2.75)$$

3. $(\alpha_1, \alpha_2) = (0, \frac{1}{2})$ **case**

In this pattern, the explicit form of $M_{jk}^{(\mathbb{Z}_2; \eta)}$ is

$$M_{jk}^{(\mathbb{Z}_2; \eta=\pm)} = \frac{1}{2} (\delta_{j,k} \pm e^{-2\pi i \frac{j}{M}} \delta_{k, |M| - j}) . \quad (4.2.76)$$

The $\mathbb{Z}_2$ eigenstates in this pattern are given by

$$|j/M; \pm\rangle_{\mathbb{Z}_2} := \widehat{\mathbb{Z}}_2^{\eta=\pm} |j/M\rangle = \frac{1}{2} \left(|j/M\rangle \pm e^{-2\pi i \frac{j}{M}} |(|M| - j)/M\rangle \right) . \quad (4.2.77)$$

This has the following properties:

$$\begin{aligned} |(|M| - j)/M; \pm\rangle &= \pm e^{2\pi i \frac{j}{M}} |j/M; \pm\rangle , \\ |0/M; -\rangle &= 0 , \quad \left| \left(\frac{|M|}{2} \right) / M; +\right\rangle = 0 . \end{aligned} \quad (4.2.78)$$

We count the number of independent states in the similar way to the $(\alpha_1, \alpha_2) = (0, 0)$ case as follows:

$$\begin{aligned} (\# \text{ of physical states for } \eta = +1) &= \left\lfloor \frac{|M| + 1}{2} \right\rfloor , \\ (\# \text{ of physical states for } \eta = -1) &= \left\lfloor \frac{|M|}{2} \right\rfloor . \end{aligned} \quad (4.2.79)$$

4. $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ case

In this pattern, the explicit form of $M_{jk}^{(\mathbb{Z}_2; \eta)}$ is

$$M_{jk}^{(\mathbb{Z}_2; \eta=\pm)} = \frac{1}{2} (\delta_{j,k} \pm e^{-2\pi i \frac{j+\frac{1}{2}}{M}} \delta_{k, |M|-j-1}) . \quad (4.2.80)$$

The $\mathbb{Z}_2$ eigenstates in this pattern are given by

$$|j/M; \pm\rangle_{\mathbb{Z}_2} := \hat{\mathbb{Z}}_2^{\eta=\pm} |j/M\rangle = \frac{1}{2} \left(|j/M\rangle \pm e^{-2\pi i \frac{j+\frac{1}{2}}{M}} |(|M|-1-j)/M\rangle \right) . \quad (4.2.81)$$

This has the following properties:

$$\begin{aligned} |(|M|-1-j)/M; \pm\rangle &= \pm e^{2\pi i \frac{j+\frac{1}{2}}{M}} |j/M; \pm\rangle , \\ \left| \left(\frac{|M|-1}{2} \right) / M; + \right\rangle &= 0 . \end{aligned} \quad (4.2.82)$$

We count the number of independent states in the similar way to the $(\alpha_1, \alpha_2) = (0, 0)$ case as follows:

$$\begin{aligned} (\# \text{ of physical states for } \eta = +1) &= \left\lfloor \frac{|M|}{2} \right\rfloor , \\ (\# \text{ of physical states for } \eta = -1) &= \left\lfloor \frac{|M|+1}{2} \right\rfloor . \end{aligned} \quad (4.2.83)$$

4.2.3 $\mathbb{T}^2/\mathbb{Z}_3$

In this subsection, we discuss the $\mathbb{T}^2/\mathbb{Z}_3$ orbifold. Although we can analyze this orbifold in the similar manner to the $\mathbb{T}^2/\mathbb{Z}_2$ case, it is more complicated than the $\mathbb{T}^2/\mathbb{Z}_2$ case because of the mixing of the operators $\hat{T}_1$ and $\hat{T}_2$ under the $\mathbb{Z}_3$ transformation.

• **Allowed values of the Scherk–Schwarz phases on $\mathbb{T}^2/\mathbb{Z}_3$**

First, we investigate the allowed values of the Scherk–Schwarz phases on the $\mathbb{T}^2/\mathbb{Z}_3$. Similarly to the case of $\mathbb{T}^2/\mathbb{Z}_2$, we operate $\hat{U}$ on the both sides of the constraint conditions (4.1.24). By using formulae (A.1) and (A.4), we find that the left-hand sides of Eq. (4.1.24) become as follows:

$$\begin{aligned} \hat{U}(e^{2\pi i M \hat{\mathcal{Y}}} |\psi\rangle) &= \hat{U} e^{2\pi i M \hat{\mathcal{Y}}} \hat{U}^{-1} \hat{U} |\psi\rangle \\ &= e^{2\pi i M (\hat{U} \hat{\mathcal{Y}} \hat{U}^{-1})} \hat{U} |\psi\rangle \\ &= e^{2\pi i M (-\hat{\mathcal{Y}} + \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{2\alpha_1 + \alpha_2}{M})} \hat{U} |\psi\rangle \\ &= e^{-2\pi i (2\alpha_1 + \alpha_2)} \cdot e^{-2\pi i M \hat{\mathcal{Y}}} e^{i\hat{\mathcal{P}}} e^{-\frac{1}{2}[-2\pi i M \hat{\mathcal{Y}}, \pi M \hat{\mathcal{P}}]} \hat{U} |\psi\rangle \\ &= e^{-2\pi i (2\alpha_1 + \alpha_2 + \frac{M}{2})} e^{-2\pi i M \hat{\mathcal{Y}}} e^{i\hat{\mathcal{P}}} \hat{U} |\psi\rangle , \end{aligned} \quad (4.2.84)$$

$$\begin{aligned} \hat{U}(e^{-i\hat{\mathcal{P}}} |\psi\rangle) &= \hat{U} e^{-i\hat{\mathcal{P}}} \hat{U}^{-1} \hat{U} |\psi\rangle \\ &= e^{-i(\hat{U} \hat{\mathcal{P}} \hat{U}^{-1})} \hat{U} |\psi\rangle \\ &= e^{-i[-2\pi M \hat{\mathcal{Y}} - 2\pi(\alpha_1 - \alpha_2)]} \hat{U} |\psi\rangle . \end{aligned} \quad (4.2.85)$$

On the other hand, the right-hand sides of Eq. (4.1.24) become $\hat{U} |\psi\rangle$. Therefore, we find that

$$e^{-2\pi i(2\alpha_1 + \alpha_2 + \frac{M}{2})} e^{-2\pi i M \hat{\mathcal{Y}}} e^{i\hat{\mathcal{P}}} (\hat{U} |\psi\rangle) = \hat{U} |\psi\rangle, \quad (4.2.86a)$$

$$e^{2\pi i(\alpha_1 - \alpha_2)} e^{2\pi i M \hat{\mathcal{Y}}} (\hat{U} |\psi\rangle) = \hat{U} |\psi\rangle. \quad (4.2.86b)$$

By multiplying $e^{2\pi i(2\alpha_1 + \alpha_2 + \frac{M}{2})} e^{+2\pi i M \hat{\mathcal{Y}}}$ on the both sides of Eq. (4.2.86a) and using Eq. (4.2.86b), we find that

$$e^{i\hat{\mathcal{P}}} (\hat{U} |\psi\rangle) = e^{2\pi i(2\alpha_1 + \alpha_2 + \frac{M}{2})} \cdot e^{-2\pi i(\alpha_1 - \alpha_2)} (\hat{U} |\psi\rangle). \quad (4.2.87)$$

By comparing Eqs. (4.2.87) and (4.2.86b) with the requirement (4.2.15), we find the following conditions:

$$\alpha_1 + 2\alpha_2 + \frac{M}{2} = n, \quad \alpha_1 - \alpha_2 = n', \quad (n, n' \in \mathbb{Z}). \quad (4.2.88)$$

Therefore, the allowed values of the Scherk–Schwarz phases are follows:

$$\alpha := \alpha_1 = \alpha_2 = \begin{cases} 0, \frac{1}{3}, \frac{2}{3} & (\text{for } M = \text{even}), \\ \frac{1}{6}, \frac{3}{6}, \frac{5}{6} & (\text{for } M = \text{odd}). \end{cases} \quad (4.2.89)$$

• $\mathbb{Z}_3$ -rotated states

Next, we find the matrix elements $D_{jk}^{(\omega^x)}$ defined in Eq. (4.2.16). For later convenience, we note the $\mathbb{Z}_3$ transformation properties of the operators $\hat{\mathcal{Y}}, \hat{\mathcal{P}}$ in Eq. (4.2.12) for the case $\alpha = \alpha_1 = \alpha_2$,

$$\hat{U} \hat{\mathcal{Y}} \hat{U} = -\hat{\mathcal{Y}} + \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha}{M}, \quad (4.2.90a)$$

$$\hat{U} \hat{\mathcal{P}} \hat{U} = -2\pi M \hat{\mathcal{Y}}, \quad (4.2.90b)$$

$$\hat{U}^{-1} \hat{\mathcal{Y}} \hat{U} = -\frac{\hat{\mathcal{P}}}{2\pi M}, \quad (4.2.90c)$$

$$\hat{U}^{-1} \hat{\mathcal{P}} \hat{U} = 2\pi M \hat{\mathcal{Y}} - \hat{\mathcal{P}} + 6\pi\alpha. \quad (4.2.90d)$$

By operating $e^{2\pi i \hat{\mathcal{Y}}}$ on the state $\hat{U} |j/M\rangle$ and using the formula (A.1), we get the relation

$$\begin{aligned} e^{2\pi i \hat{\mathcal{Y}}} (\hat{U} |j/M\rangle) &= \hat{U} e^{2\pi i(\hat{U}^{-1} \hat{\mathcal{Y}} \hat{U})} |j/M\rangle \\ &= \hat{U} e^{-i\hat{\mathcal{P}} \frac{1}{M}} |j/M\rangle \\ &= \hat{U} |(j+1)/M\rangle. \end{aligned} \quad (4.2.91)$$

This implies that the state $\sum_j \hat{U} |j/M\rangle$ is an eigenstate of the operator $e^{2\pi i \hat{\mathcal{Y}}}$ with the eigenvalue $e^{2\pi i \frac{0}{M}}$, i.e.,

$$\sum_{j=0}^{M-1} \hat{U} |j/M\rangle = \mathcal{A} e^{\pi i \eta} |0/M\rangle, \quad (4.2.92)$$

where $\mathcal{A}, e^{\pi i \eta}$ are the magnitude and phase parts of the proportional coefficient, respectively.

Operating $e^{-i\hat{\mathcal{P}} \frac{k}{M}}$ on the both sides of the above equation, the left-hand side becomes

$$e^{-i\hat{\mathcal{P}} \frac{k}{M}} \sum_{j=0}^{M-1} \hat{U} |j/M\rangle = \hat{U} \sum_{j=0}^{M-1} e^{-i(\hat{U}^{-1} \hat{\mathcal{P}} \hat{U}) \frac{k}{M}} |j/M\rangle$$

$$\begin{aligned}
&= \hat{U} \sum_{j=0}^{M-1} e^{-2\pi i \hat{Y}k + i \hat{P} \frac{k}{M} - 6\pi i \alpha \frac{k}{M}} |j/M\rangle \\
&= \hat{U} \sum_{j=0}^{M-1} e^{-\pi i \frac{k \cdot 6\alpha}{M}} \cdot e^{i \hat{P} \frac{k}{M}} e^{-2\pi i \hat{Y}k} e^{-\frac{1}{2}[i \hat{P} \frac{k}{M}, -2\pi i \hat{Y}k]} |j/M\rangle \\
&= e^{-\pi i \frac{k \cdot 6\alpha}{M} + \pi i \frac{k^2}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |(j-k)/M\rangle \\
&= e^{-\pi i \frac{k \cdot 6\alpha}{M} + \pi i \frac{k^2}{M}} \sum_{j=-k}^{M-1-k} e^{-2\pi i \frac{(j+k) \cdot k}{M}} \hat{U} |j/M\rangle \\
&= e^{-\pi i \frac{k(k+6\alpha)}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |j/M\rangle, \tag{4.2.93}
\end{aligned}$$

with help of the formulae (A.1), (A.4), and the transformation properties (4.2.90). On the other hand, right-hand side becomes $\mathcal{A} e^{\pi i \eta} |k/M\rangle$. Therefore, we find that

$$e^{-\pi i \frac{k(k+6\alpha)}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |j/M\rangle = \mathcal{A} e^{\pi i \eta} |k/M\rangle. \tag{4.2.94}$$

Multiplying $e^{+\pi i \frac{k(k+6\alpha)}{M}}$ on the both sides, summing over $k = 0, 1, \dots, |M|-1$, and using the formula (A.6), we find that the twisted state of $|0/M\rangle$ is given by the following linear combination,

$$\hat{U} |0/M\rangle = \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{\pi i \frac{k(k+6\alpha)}{M}} |k/M\rangle. \tag{4.2.95}$$

We can obtain the expression for the state $\hat{U} |j/M\rangle$ by using the translation property, $|j/M\rangle = e^{-i \hat{P} \frac{j}{M}} |0/M\rangle$, and the $\mathbb{Z}_3$ -twist transformation property, Eq. (4.2.90), as follows:

$$\begin{aligned}
\hat{U} |j/M\rangle &= \hat{U} (e^{-i \hat{P} \frac{j}{M}} |0/M\rangle) \\
&= e^{-i(\hat{U} \hat{P} \hat{U}^{-1}) \frac{j}{M}} \hat{U} |0/M\rangle \\
&= e^{2\pi i j \hat{Y}} \cdot \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{\pi i \frac{k(k+6\alpha)}{M}} |k/M\rangle \\
&= \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{\pi i \frac{k(k+2j+6\alpha)}{M}} |k/M\rangle. \tag{4.2.96}
\end{aligned}$$

Finally, we determine the magnitude factor $\mathcal{A}$ and the phase factor $e^{\pi i \eta}$ so that $\hat{U}^3$ is the identity operator, $\hat{U}^3 = \mathbf{1}$. This is done as follows: First we operate $\hat{U}^3$ on the state $|j/M\rangle$. By using the expression in Eq. (4.2.96) and the formulae (A.8), (A.6) in Appendix A, we find that

$$\hat{U}^3 |j/M\rangle = \sum_{k=0}^{M-1} \frac{\mathcal{A}^3}{M^3} e^{3\pi i \eta} \sum_{m=0}^{M-1} \sum_{l=0}^{M-1} e^{\pi i \frac{m(m+2j+6\alpha) + l(l+2m+6\alpha) + k(k+2m+6\alpha)}{M}} |k/M\rangle$$

$$= \frac{\mathcal{A}^3}{M^3} e^{3\pi i \theta} \cdot M^{3/2} e^{-\pi i (\frac{9\alpha^2}{M} - \frac{1}{4})} |j/M\rangle. \quad (4.2.97)$$

Therefore, the factors $\mathcal{A}$ and η are determined by the $\mathbb{Z}_3$ consistency, $\hat{U}^3 = \mathbf{1}$, as follows:

$$\mathcal{A} = \sqrt{M}, \quad \eta = \frac{3\alpha^2}{M} - \frac{1}{12}. \quad (4.2.98)$$

As the result, we find the following expressions of the matrix elements:

$$D_{jk}^{(\omega)} = \frac{1}{\sqrt{M}} e^{\pi i (\frac{3\alpha^2}{M} - \frac{1}{12})} e^{\pi i \frac{k(k+2j+6\alpha)}{M}}, \quad (4.2.99)$$

$$D_{jk}^{(\omega^2)} = \frac{1}{\sqrt{M}} e^{-\pi i (\frac{3\alpha^2}{M} - \frac{1}{12})} e^{-\pi i \frac{j(j+2k+6\alpha)}{M}}. \quad (4.2.100)$$

• **The number of physical solutions on $\mathbb{T}^2/\mathbb{Z}_3$**

By investigating the rank of the matrix

$$M_{jk}^{(\mathbb{Z}_3; \eta)} = \frac{1}{3} \sum_{x=0}^2 \bar{\eta}^x D_{jk}^{(\omega^x)}, \quad \eta \in \{1, \omega, \bar{\omega}\}, \quad (4.2.101)$$

we obtain Table 4.1–4.4 of the number of physical solutions.

$ M $		2	4	6	8	10	12	14	16	18	20	22
η	1	1	1	3	3	3	5	5	5	7	7	7
	ω	0	2	2	2	4	4	4	6	6	6	8
	$\bar{\omega}$	1	1	1	3	3	3	5	5	5	7	7

Table 4.1: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_3$ eigenvalue η for $M = \text{even}$ and $(\alpha_1, \alpha_2) = (0, 0)$ on $\mathbb{T}^2/\mathbb{Z}_3$.

$ M $		2	4	6	8	10	12	14	16	18	20	22
η	1	1	2	2	3	4	4	5	6	6	7	8
	ω	1	1	2	3	3	4	5	5	6	7	7
	$\bar{\omega}$	0	1	2	2	3	4	4	5	6	6	7

Table 4.2: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_3$ eigenvalue η for $M = \text{even}$ and $(\alpha_1, \alpha_2) = (\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{2}{3})$ on $\mathbb{T}^2/\mathbb{Z}_3$.

$ M $		1	3	5	7	9	11	13	15	17	19	21
η	1	1	1	2	3	3	4	5	5	6	7	7
	ω	0	1	2	2	3	4	4	5	6	6	7
	$\bar{\omega}$	0	1	1	2	3	3	4	5	5	6	7

Table 4.3: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_3$ eigenvalue η for $M = \text{odd}$ and $(\alpha_1, \alpha_2) = (\frac{1}{6}, \frac{1}{6}), (\frac{5}{6}, \frac{5}{6})$ on $\mathbb{T}^2/\mathbb{Z}_3$.

$ M $		1	3	5	7	9	11	13	15	17	19	21
η	1	0	2	2	2	4	4	4	6	6	6	8
	ω	1	1	1	3	3	3	5	5	5	7	7
	$\bar{\omega}$	0	0	2	2	2	4	4	4	6	6	6

Table 4.4: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_3$ eigenvalue η for $M = \text{odd}$ and $(\alpha_1, \alpha_2) = (\frac{3}{6}, \frac{3}{6})$ on $\mathbb{T}^2/\mathbb{Z}_3$.

4.2.4 $\mathbb{T}^2/\mathbb{Z}_4$

In this subsection, we discuss the $\mathbb{T}^2/\mathbb{Z}_4$ orbifold. The situation is similar to the $\mathbb{T}^2/\mathbb{Z}_3$ case.

• Allowed values of the Scherk–Schwarz phases on $\mathbb{T}^2/\mathbb{Z}_4$

First, we investigate the allowed values of the Scherk–Schwarz phases on $\mathbb{T}^2/\mathbb{Z}_4$. By acting the $\mathbb{Z}_4$ -twist operator $\hat{U}$ on the both sides of the constraint conditions (4.1.24) and using formula (A.1), we find that

$$\begin{aligned} e^{i\hat{\mathcal{P}}} e^{-2\pi i(\alpha_1 + \alpha_2)} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle, \\ e^{2\pi i(\alpha_1 - \alpha_2)} e^{2\pi i M \hat{\mathcal{Y}}} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle. \end{aligned} \quad (4.2.102)$$

By comparing the above with the requirement (4.2.15), we find the following conditions:

$$\alpha_1 + \alpha_2 = n, \quad \alpha_1 - \alpha_2 = n', \quad (n, n' \in \mathbb{Z}), \quad (4.2.103)$$

Therefore, the allowed values of Scherk–Schwarz phases are follows:

$$\alpha := \alpha_1 = \alpha_2 = 0, \frac{1}{2}. \quad (4.2.104)$$

• $\mathbb{Z}_4$ -rotated states

Next, we find the matrix elements $D_{jk}^{(\omega^x)}$ defined in Eq. (4.2.16). For later convenience, we note the $\mathbb{Z}_4$ transformation properties of the operators $\hat{\mathcal{Y}}, \hat{\mathcal{P}}$ in Eq. (4.2.12) for the case $\alpha = \alpha_1 = \alpha_2$,

$$\hat{U} \hat{\mathcal{Y}} \hat{U} = \frac{\hat{\mathcal{P}}}{2\pi M}, \quad (4.2.105a)$$

$$\hat{U} \hat{\mathcal{P}} \hat{U} = -2\pi M \hat{\mathcal{Y}}, \quad (4.2.105b)$$

$$\hat{U}^{-1} \hat{\mathcal{Y}} \hat{U} = -\frac{\hat{\mathcal{P}}}{2\pi M}, \quad (4.2.105c)$$

$$\hat{U}^{-1} \hat{\mathcal{P}} \hat{U} = 2\pi M \hat{\mathcal{Y}} + 4\pi\alpha. \quad (4.2.105d)$$

By the calculation same as the $\mathbb{Z}_3$ case, we find that

$$\sum_{j=0}^{M-1} \hat{U} |j/M\rangle = \mathcal{A} e^{\pi i \eta} |0/M\rangle. \quad (4.2.106)$$

By operating $e^{-i\hat{P}\frac{k}{M}}$ on the both sides of the above equation, the left-hand side becomes

$$\begin{aligned} e^{-i\hat{P}\frac{k}{M}} \sum_{j=0}^{M-1} \hat{U} |j/M\rangle &= \hat{U} \sum_{j=0}^{M-1} e^{-i(\hat{U}^{-1}\hat{P}\hat{U})\frac{k}{M}} |j/M\rangle \\ &= \hat{U} \sum_{j=0}^{M-1} e^{-2\pi i \hat{Y}k - 4\pi i \alpha \frac{k}{M}} |j/M\rangle \\ &= e^{-2\pi i \frac{k \cdot 2\alpha}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{k \cdot j}{M}} \hat{U} |j/M\rangle, \end{aligned} \quad (4.2.107)$$

with help of the formulae (A.1) and the transformation properties (4.2.105). On the other hand, the right-hand side becomes $\mathcal{A}e^{\pi i \eta} |k/M\rangle$. Therefore, we find that

$$e^{-2\pi i \frac{k \cdot 2\alpha}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{k \cdot j}{M}} \hat{U} |j/M\rangle = \mathcal{A}e^{\pi i \eta} |k/M\rangle \quad (4.2.108)$$

By multiplying $e^{+2\pi i \frac{k \cdot 2\alpha}{M}}$ on the both sides, summing over $k = 0, 1, \dots, |M| - 1$, and using the formula (A.6), we find that the twisted state of $|0/M\rangle$ is given by the following linear combination,

$$\hat{U} |0/M\rangle = \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{2\pi i \frac{k \cdot 2\alpha}{M}} |k/M\rangle. \quad (4.2.109)$$

We can obtain the expression for the state $\hat{U} |j/M\rangle$ by using the translation property, $|j/M\rangle = e^{-i\hat{P}\frac{j}{M}} |0/M\rangle$, and the $\mathbb{Z}_4$ -twist transformation property, Eq. (4.2.105), as follows:

$$\begin{aligned} \hat{U} |j/M\rangle &= \hat{U} (e^{-i\hat{P}\frac{j}{M}} |0/M\rangle) \\ &= e^{-i(\hat{U}\hat{P}\hat{U}^{-1})\frac{j}{M}} \hat{U} |0/M\rangle \\ &= e^{2\pi i j \hat{Y}} \cdot \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{2\pi i \frac{k \cdot 2\alpha}{M}} |k/M\rangle \\ &= \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{2\pi i \frac{k \cdot (j+2\alpha)}{M}} |k/M\rangle. \end{aligned} \quad (4.2.110)$$

Finally, we determine the magnitude factor $\mathcal{A}$ and the phase factor $e^{\pi i \eta}$ so that $\hat{U}^4$ is the identity operator, $\hat{U}^4 = \mathbf{1}$. This is done as follows: First we operate $\hat{U}^4$ on the state $|j/M\rangle$. By using the expression in Eq. (4.2.110) and the formula (A.6) in Appendix A, we find that

$$\begin{aligned} \hat{U}^4 |j/M\rangle &= \sum_{k=0}^{M-1} \frac{\mathcal{A}^4}{M^4} e^{4\pi i \eta} \sum_{l_1=0}^{M-1} \sum_{l_2=0}^{M-1} \sum_{l_3=0}^{M-1} e^{2\pi i \frac{l_1(j+2\alpha) + l_2(l_1+2\alpha) + l_3(l_2+2\alpha) + k(l_3+2\alpha)}{M}} |k/M\rangle \\ &= \frac{\mathcal{A}^4}{M^4} e^{4\pi i \eta} \cdot M^2 e^{-2\pi i \frac{4\alpha^2}{M}} |j/M\rangle \end{aligned} \quad (4.2.111)$$

where we used the formulae (A.8) and (A.6). Therefore, the factors $\mathcal{A}$ and η are determined by the $\mathbb{Z}_4$ consistency, $\hat{U}^4 = \mathbf{1}$, as follows:

$$\mathcal{A} = \sqrt{M}, \quad \eta = \frac{2\alpha^2}{M}. \quad (4.2.112)$$

As the result, we find the following expressions of the matrix elements:

$$D_{jk}^{(\omega)} = \frac{e^{2\pi i \alpha^2/M}}{\sqrt{M}} e^{2\pi i \frac{k \cdot (j+2\alpha)}{M}}, \quad (4.2.113)$$

$$D_{jk}^{(\omega^2)} = e^{4\pi i \alpha^2/M} e^{-2\pi i \frac{(j+2\alpha) \cdot 2\alpha}{M}} \delta_{k, -j-2\alpha}, \quad (4.2.114)$$

$$D_{jk}^{(\omega^3)} = \frac{e^{6\pi i \alpha^2/M}}{\sqrt{M}} e^{-2\pi i \frac{(j+2\alpha) \cdot 2\alpha}{M}} e^{-2\pi i \frac{j \cdot k}{M}}. \quad (4.2.115)$$

• **The number of physical solutions on $\mathbb{T}^2/\mathbb{Z}_4$**

By investigating the rank of the matrix

$$M_{jk}^{(\mathbb{Z}_4; \eta)} = \frac{1}{4} \sum_{x=0}^3 \bar{\eta}^x D_{jk}^{(\omega^x)}, \quad \eta \in \{1, i, -1, -i\}, \quad (4.2.116)$$

we obtain Tables 4.5 and 4.6 of the number of physical solutions.

$ M $		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
η	+1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
	+i	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4
	-1	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4
	-i	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3
$ M $		16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
η	+1	5	5	5	5	6	6	6	6	7	7	7	7	8	8	8
	+i	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7
	-1	4	4	5	5	5	5	6	6	6	6	7	7	7	7	8
	-i	3	4	4	4	4	5	5	5	5	6	6	6	6	7	7

Table 4.5: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_4$ eigenvalue η for $(\alpha_1, \alpha_2) = (0, 0)$ on $\mathbb{T}^2/\mathbb{Z}_4$.

$ M $		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
η	+1	0	1	1	1	1	2	2	2	2	3	3	3	3	4	4
	+i	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4
	-1	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
	-i	0	0	1	1	1	1	2	2	2	2	3	3	3	3	4
$ M $		16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
η	+1	4	4	5	5	5	5	6	6	6	6	7	7	7	7	8
	+i	4	5	5	5	5	6	6	6	6	7	7	7	7	8	8
	-1	4	4	4	4	5	5	5	5	6	6	6	6	7	7	7
	-i	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7

Table 4.6: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_4$ eigenvalue η for $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ on $\mathbb{T}^2/\mathbb{Z}_4$.

4.2.5 $\mathbb{T}^2/\mathbb{Z}_6$

In this subsection, we discuss the $\mathbb{T}^2/\mathbb{Z}_6$ orbifold.

• **Allowed values of the Scherk–Schwarz phases on $\mathbb{T}^2/\mathbb{Z}_6$**

First, we investigate the allowed values of the Scherk–Schwarz phases on the $\mathbb{T}^2/\mathbb{Z}_6$. By acting the $\mathbb{Z}_6$ -twist operator $\hat{U}$ on the both-sides of the constraint conditions (4.1.24) and using formulae (A.1), (A.4), we find

$$\begin{aligned} e^{-2\pi i(\alpha_2 + \frac{M}{2})} e^{i\hat{\mathcal{P}}} e^{2\pi i M \hat{\mathcal{Y}}} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle, \\ e^{2\pi i(\alpha_1 - \alpha_2)} e^{2\pi i M \hat{\mathcal{Y}}} (\hat{U} |\psi\rangle) &= \hat{U} |\psi\rangle. \end{aligned} \quad (4.2.117)$$

These lead the conditions

$$\alpha_1 - 2\alpha_2 - \frac{M}{2} = n, \quad \alpha_1 - \alpha_2 = n', \quad (n, n' \in \mathbb{Z}), \quad (4.2.118)$$

Therefore, the allowed values of Scherk–Schwarz phases are follows:

$$\alpha := \alpha_1 = \alpha_2 = \begin{cases} 0 & (\text{for } M = \text{even}), \\ \frac{1}{2} & (\text{for } M = \text{odd}). \end{cases} \quad (4.2.119)$$

• **$\mathbb{Z}_6$ -rotated states**

Next, we find the matrix elements $D_{jk}^{(\omega^x)}$ defined in Eq. (4.2.16). For later convenience, we note the $\mathbb{Z}_6$ transformation properties of the operators $\hat{\mathcal{Y}}, \hat{\mathcal{P}}$ in Eq. (4.2.12) for the case $\alpha = \alpha_1 = \alpha_2$,

$$\hat{U} \hat{\mathcal{Y}} \hat{U} = \hat{\mathcal{Y}} + \frac{\hat{\mathcal{P}}}{2\pi M} - \frac{\alpha}{M}, \quad (4.2.120a)$$

$$\hat{U} \hat{\mathcal{P}} \hat{U} = -2\pi M \hat{\mathcal{Y}}, \quad (4.2.120b)$$

$$\hat{U}^{-1} \hat{\mathcal{Y}} \hat{U} = -\frac{\hat{\mathcal{P}}}{2\pi M}, \quad (4.2.120c)$$

$$\hat{U}^{-1} \hat{\mathcal{P}} \hat{U} = 2\pi M \hat{\mathcal{Y}} + \hat{\mathcal{P}} + 2\pi\alpha. \quad (4.2.120d)$$

By the calculation same as the $\mathbb{Z}_3$ case, we find that

$$\sum_{j=0}^{M-1} \hat{U} |j/M\rangle = \mathcal{A} e^{\pi i \eta} |0/M\rangle. \quad (4.2.121)$$

By operating $e^{-i\hat{\mathcal{P}} \frac{k}{M}}$ on the both sides of the above equation, the left-hand side becomes

$$\begin{aligned} e^{-i\hat{\mathcal{P}} \frac{k}{M}} \sum_{j=0}^{M-1} \hat{U} |j/M\rangle &= \hat{U} \sum_{j=0}^{M-1} e^{-i(\hat{U}^{-1} \hat{\mathcal{P}} \hat{U}) \frac{k}{M}} |j/M\rangle \\ &= \hat{U} \sum_{j=0}^{M-1} e^{-2\pi i \hat{\mathcal{Y}} k - i\hat{\mathcal{P}} \frac{k}{M} - 2\pi i \alpha \frac{k}{M}} |j/M\rangle \end{aligned}$$

$$\begin{aligned}
&= \hat{U} \sum_{j=0}^{M-1} e^{-\pi i \frac{k \cdot 2\alpha}{M}} \cdot e^{-i \hat{P} \frac{k}{M}} e^{-2\pi i \hat{Y} k} e^{-\frac{1}{2}[-i \hat{P} \frac{k}{M}, -2\pi i \hat{Y} k]} |j/M\rangle \\
&= e^{-\pi i \frac{k \cdot 2\alpha}{M} - \pi i \frac{k^2}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |(j+k)/M\rangle \\
&= e^{-\pi i \frac{k \cdot 2\alpha}{M} - \pi i \frac{k^2}{M}} \sum_{j=k}^{M-1+k} e^{-2\pi i \frac{(j-k) \cdot k}{M}} \hat{U} |j/M\rangle \\
&= e^{\pi i \frac{k(k-2\alpha)}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |j/M\rangle, \tag{4.2.122}
\end{aligned}$$

with help of the formulae (A.1), (A.4), and the transformation properties (4.2.120). On the other hand, the right-hand side becomes $\mathcal{A} e^{\pi i \eta} |k/M\rangle$. Therefore, we find that

$$e^{\pi i \frac{k(k-2\alpha)}{M}} \sum_{j=0}^{M-1} e^{-2\pi i \frac{j \cdot k}{M}} \hat{U} |j/M\rangle = \mathcal{A} e^{\pi i \eta} |k/M\rangle. \tag{4.2.123}$$

By multiplying $e^{-\pi i \frac{k(k-2\alpha)}{M}}$ on the both sides, summing over $k = 0, 1, \dots, |M| - 1$, and using the formula (A.6), we find that the twisted state of $|0/M\rangle$ is given by the following linear combination,

$$\hat{U} |0/M\rangle = \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{-\pi i \frac{k(k-2\alpha)}{M}} |k/M\rangle. \tag{4.2.124}$$

We can obtain the expression for the state $\hat{U} |j/M\rangle$ by using the translation property, $|j/M\rangle = e^{-i \hat{P} \frac{j}{M}} |0/M\rangle$, and the $\mathbb{Z}_6$ -twist transformation property, Eq. (4.2.120), as follows:

$$\begin{aligned}
\hat{U} |j/M\rangle &= \hat{U} (e^{-i \hat{P} \frac{j}{M}} |0/M\rangle) \\
&= e^{-i(\hat{U} \hat{P} \hat{U}^{-1}) \frac{j}{M}} \hat{U} |0/M\rangle \\
&= e^{2\pi i j \hat{Y}} \cdot \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{-\pi i \frac{k(k-2\alpha)}{M}} |k/M\rangle \\
&= \frac{\mathcal{A}}{M} e^{\pi i \eta} \sum_{k=0}^{M-1} e^{-\pi i \frac{k(k-2j-2\alpha)}{M}} |k/M\rangle. \tag{4.2.125}
\end{aligned}$$

Finally, we determine the magnitude factor $\mathcal{A}$ and the phase factor $e^{\pi i \eta}$ so that $\hat{U}^6$ is the identity operator, $\hat{U}^6 = \mathbf{1}$. This is done as follows: First we operate $\hat{U}^6$ on the state $|j/M\rangle$. By using the expression in Eq. (4.2.125) and the formulae (A.8), (A.6) in Appendix A, we find that

$$\begin{aligned}
\hat{U}^6 |j/M\rangle &= \sum_{k=0}^{M-1} \frac{\mathcal{A}^6}{M^6} e^{6\pi i \eta} \sum_{l_1=0}^{M-1} \sum_{l_2=0}^{M-1} \sum_{l_3=0}^{M-1} \sum_{l_4=0}^{M-1} \sum_{l_5=0}^{M-1} e^{-\pi i \frac{l_1(l_1-2j-2\alpha)+l_2(l_2-2l_1-2\alpha)}{M}} \\
&\quad \times e^{-\pi i \frac{l_3(l_3-2l_2-2\alpha)+l_4(l_4-2l_3-2\alpha)+l_5(l_5-2l_4-2\alpha)}{M}} |k/M\rangle \\
&= \frac{\mathcal{A}^6}{M^6} e^{6\pi i \eta} \cdot M^3 e^{-\frac{2}{4}\pi i} e^{\pi i \frac{2\alpha^2}{M}} |j/M\rangle, \tag{4.2.126}
\end{aligned}$$

where we used the formulae (A.8) and (A.6). Therefore, the factors $\mathcal{A}$ and η are determined by the $\mathbb{Z}_6$ consistency, $\hat{U}^6 = \mathbf{1}$, as follows:

$$\mathcal{A} = \sqrt{M}, \quad \eta = \frac{\alpha^2}{3M} - \frac{1}{12}. \quad (4.2.127)$$

As the result, we find the following expressions of the matrix elements:

$$D_{jk}^{(\omega)} = \frac{e^{\pi i (\frac{1}{12} + \frac{\alpha^2}{M})}}{\sqrt{M}} e^{-\pi i \frac{k(k-2j-2\alpha)}{M}}, \quad (4.2.128)$$

$$D_{jk}^{(\omega^2)} = \frac{e^{\pi i (\frac{2\alpha^2}{M} - \frac{1}{12})}}{\sqrt{M}} e^{\pi i \frac{(j+\alpha)^2}{M}} e^{2\pi i \frac{k(j+2\alpha)}{M}}, \quad (4.2.129)$$

$$D_{jk}^{(\omega^3)} = e^{-2\pi i \frac{(j+\alpha) \cdot 2\alpha}{M}} \delta_{k, -j-2\alpha}, \quad (4.2.130)$$

$$D_{jk}^{(\omega^4)} = \frac{e^{\pi i (\frac{1}{12} - \frac{4\alpha j + 3\alpha^2}{M})}}{\sqrt{M}} e^{-\pi i \frac{k(k+2j+2\alpha)}{M}}, \quad (4.2.131)$$

$$D_{jk}^{(\omega^5)} = \frac{e^{-\pi i (\frac{1}{12} + \frac{4\alpha j + 2\alpha^2}{M})}}{\sqrt{M}} e^{\pi i \frac{(j+\alpha)^2}{M}} e^{-2\pi i \frac{j \cdot k}{M}}. \quad (4.2.132)$$

• **The number of physical solutions on $\mathbb{T}^2/\mathbb{Z}_6$**

By investigating the rank of the matrix

$$M_{jk}^{(\mathbb{Z}_6; \eta)} = \frac{1}{6} \sum_{x=0}^5 \bar{\eta}^x D_{jk}^{(\omega^x)}, \quad \eta \in \{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5\}, \quad (4.2.133)$$

we obtain Tables 4.7 and 4.8 of the number of physical solutions.

$ M $		2	4	6	8	10	12	14	16	18	20	22
η	1	1	1	2	2	2	3	3	3	4	4	4
	ω	0	1	1	1	2	2	2	3	3	3	4
	ω^2	1	1	1	2	2	2	3	3	3	4	4
	ω^3	0	0	1	1	1	2	2	2	3	3	3
	ω^4	0	1	1	1	2	2	2	3	3	3	4
	ω^5	0	0	0	1	1	1	2	2	2	3	3
$ M $		24	26	28	30	32	34	36	38	40	42	44
η	1	5	5	5	6	6	6	7	7	7	8	8
	ω	4	4	5	5	5	6	6	6	7	7	7
	ω^2	4	5	5	5	6	6	6	7	7	7	8
	ω^3	4	4	4	5	5	5	6	6	6	7	7
	ω^4	4	4	5	5	5	6	6	6	7	7	7
	ω^5	3	4	4	4	5	5	5	6	6	6	7

Table 4.7: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_6$ eigenvalue η for $M = \text{even}$ and $(\alpha_1, \alpha_2) = (0, 0)$ on $\mathbb{T}^2/\mathbb{Z}_6$.

$ M $		1	3	5	7	9	11	13	15	17	19	21
η	1	0	1	1	1	2	2	2	3	3	3	4
	ω	1	1	1	2	2	2	3	3	3	4	4
	ω^2	0	0	1	1	1	2	2	2	3	3	3
	ω^3	0	1	1	1	2	2	2	3	3	3	4
	ω^4	0	0	0	1	1	1	2	2	2	3	3
	ω^5	0	0	1	1	1	2	2	2	3	3	3
$ M $		23	25	27	29	31	33	35	37	39	41	43
η	1	4	4	5	5	5	6	6	6	7	7	7
	ω	4	5	5	5	6	6	6	7	7	7	8
	ω^2	4	4	4	5	5	5	6	6	6	7	7
	ω^3	4	4	5	5	5	6	6	6	7	7	7
	ω^4	3	4	4	4	5	5	5	6	6	6	7
	ω^5	4	4	4	5	5	5	6	6	6	7	7

Table 4.8: The numbers of linearly independent zero-mode eigenstates with $\mathbb{Z}_6$ eigenvalue η for $M = \text{odd}$ and $(\alpha_1, \alpha_2) = (\frac{1}{2}, \frac{1}{2})$ on $\mathbb{T}^2/\mathbb{Z}_6$.

Chapter 5

Classification of Three Generation Models

In this chapter, based on the aspects which are obtained in the previous three chapters, we propose 10D SYM on magnetized orbifolds for solving the puzzles in the SM, the origin of the generation, the chirality, the mass hierarchy of the matters, and the gauge structure.

10D SYM is a well-motivated higher-dimensional field theory since this appears at low-energy limit of Type-I, IIB, and heterotic superstring theories. From phenomenological point of view, 10D SYM on magnetized orbifolds has the following advantages: First, we can use the advantages of higher-dimensional field theory and magnetized orbifold which have been mentioned in Chapter 3 and 4. That is, three generations of the matters and hierarchical Yukawa couplings can be realized. Second, various particles in the SM arise from only two fields, the gauge field and the gaugino field. Third, the gauge structure in the SM is obtained from a larger gauge group through explicit symmetry breaking caused by the magnetic flux.

In the following, we start with 10D U(8) SYM on magnetized orbifolds and classify all the possible configurations of the parameters which provide models with three generations of the chiral matter fermions in 4D effective field theory [49].

5.1 Setups: 10D SYM on magnetized orbifolds

In this section, we review 10D SYM on magnetized orbifold.

We consider 10 SYM on the background space-time $\mathbb{M}^4 \times (\mathbb{T}^2)^3$, where $\mathbb{M}^4$ is 4D Minkowski space-time described by the coordinates x^μ ($\mu = 0, 1, 2, 3$), and $(\mathbb{T}^2)^3$ is a factorizable 6-torus which consists in the product of three 2-tori. For three 2-tori, we describe the i -th torus $\mathbb{T}^{2(i)}$ ($i = 1, 2, 3$) by the complex coordinates z_i and $\bar{z}_i = (z_i)^*$ which are defined by $z_i = y^{2i+3} + i y^{2i+4}$ from the 2D orthogonal coordinates y^{2i+3}, y^{2i+4} . In addition, the coordinate is identified as $z_i \sim z_i + 1 \sim z_i + \tau_i$ where τ_i is the moduli parameter of $\mathbb{T}^{2(i)}$.

10D SYM consists of the 10D gauge field A_M ($M = 0, 1, 2, 3, z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3$) and the 10D Majorana–Weyl spinor field λ . λ is called the gaugino because this is the supersymmetric partner of A_M . The action of 10D SYM is given by

$$S = \int_{\mathbb{M}^4} d^4x \int_{(\mathbb{T}^2)^3} d^6z \left[-\frac{1}{4} \text{tr}(F^{MN} F_{MN}) + \frac{1}{2} \text{tr}(\bar{\lambda} i \Gamma^M D_M \lambda) \right], \quad (5.1.1)$$

where d^6z is a shorthand notation of $\prod_{i=1}^3 dz_i d\bar{z}_i$, and

$$F_{MN} = \partial_M A_N - \partial_N A_M - i g [A_M, A_N], \quad (5.1.2)$$

$$D_M \lambda = \partial_M \lambda - i g [A_M, \lambda], \quad (5.1.3)$$

are the gauge field strength and the covariant derivative of the gaugino field, respectively. g is the gauge coupling constant. Γ^M is the gamma matrices on 10D which are summarized in Appendix B.

The gauge field and the gaugino field are decomposed in Kaluza–Klein (KK) modes as

$$A_M(x, \{z_i\}) = \sum_{n_1, n_2, n_3} \varphi_{M; n_1, n_2, n_3}(x) \otimes \phi_{M; n_1}^{(1)}(z_1) \otimes \phi_{M; n_2}^{(2)}(z_2) \otimes \phi_{M; n_3}^{(3)}(z_3), \quad (5.1.4)$$

$$\lambda(x, \{z_i\}) = \sum_{n_1, n_2, n_3} \chi_{n_1, n_2, n_3}(x) \otimes \psi_{n_1}^{(1)}(z_1) \otimes \psi_{n_2}^{(2)}(z_2) \otimes \psi_{n_3}^{(3)}(z_3), \quad (5.1.5)$$

where n_i are KK mode indices on $\mathbb{T}^{2(i)}$. The fermion field $\psi_{n_i}^{(i)}(z_i)$ is a 2D spinor describing the n_i -th KK mode on $\mathbb{T}^{2(i)}$,

$$\psi_{n_i}^{(i)}(z_i) = \begin{pmatrix} \psi_{n_i, +}^{(i)}(z_i) \\ \psi_{n_i, -}^{(i)}(z_i) \end{pmatrix}, \quad (5.1.6)$$

and corresponding gamma matrices are given by the Pauli matrices σ_1, σ_2 (again, see Appendix B) in orthogonal coordinates,

$$\tilde{\gamma}^{2i+3} = i \sigma_1, \quad \tilde{\gamma}^{2i+4} = i \sigma_2. \quad (5.1.7)$$

The extra-dimensional components of the gauge field possess the following background configurations which are given by straightforward expansion of (3.1.9),

$$\begin{aligned} \langle A(\{z_i\}) \rangle &= \sum_{i=1}^3 \left(\langle A_{z_i}(z_i) \rangle dz_i + \langle A_{\bar{z}_i}(z_i) \rangle d\bar{z}_i \right) \\ &= \sum_{i=1}^3 \frac{\pi}{g \operatorname{Im} \tau_i} \mathbf{M}^{(i)} \operatorname{Im}(\bar{z}_i dz_i), \end{aligned} \quad (5.1.8)$$

where $\mathbf{M}^{(i)}$ is the $N \times N$ matrix given by

$$\mathbf{M}^{(i)} = \begin{pmatrix} M_1^{(i)} \mathbb{I}_{N_1} & & 0 \\ & \ddots & \\ 0 & & M_n^{(i)} \mathbb{I}_{N_n} \end{pmatrix}. \quad (5.1.9)$$

Here, $M_a^{(i)}$ ($a = 1, \dots, n$) are the magnetic fluxes on the i -th $\mathbb{T}^2/\mathbb{Z}_N$ and should be integers because of the quantization condition [45]. Moreover, we assume that $M_a^{(i)} \neq M_b^{(i)}$ (for $a \neq b$), and N_a s should satisfy $\sum_{a=1}^n N_a = N$.¹ This background configuration explicitly breaks the original gauge group down as $U(N) \rightarrow \prod_{a=1}^n U(N_a)$, and splits the gaugino into n^2 sectors $\lambda_{ab}(a, b = 1, \dots, n)$ as

$$\lambda(x, \{z_i\}) = \begin{pmatrix} \lambda_{11}(x, \{z_i\}) & \cdots & \lambda_{1n}(x, \{z_i\}) \\ \vdots & \ddots & \vdots \\ \lambda_{n1}(x, \{z_i\}) & \cdots & \lambda_{nn}(x, \{z_i\}) \end{pmatrix}. \quad (5.1.10)$$

¹ Here we put the Wilson line phases zero. If one wants to account them, replace $M_a^{(i)} z_i$ by $M_a^{(i)} (z_i + a_a^{(i)})$, where $a_a^{(i)}$ is the Wilson line phase on the i -th orbifold.

Correspondingly, KK decompositions of each sectors are also split as follows:

$$\lambda_{ab}(x, \{z_i\}) = \sum_{n_1, n_2, n_3} \chi_{n_1, n_2, n_3; ab}(x) \otimes \psi_{n_1; ab}^{(1)}(z_1) \otimes \psi_{n_2; ab}^{(2)}(z_2) \otimes \psi_{n_3; ab}^{(3)}(z_3), \quad (a, b = 1, \dots, n). \quad (5.1.11)$$

The diagonal components λ_{aa} correspond to the gaugino fields of the unbroken gauge group $\prod_{a=1}^n U(N_a)$ while the off-diagonal components $\lambda_{ab} (a \neq b)$ are the matter fields which belong the representation $(N_a, \bar{N}_b)$ of the group $U(N_a) \times U(N_b)$.

The background configuration (5.1.8) satisfies the relation

$$\begin{cases} \langle A(\{z_i + \delta_{ij} p_j\}) \rangle = \langle A(\{z_i\}) \rangle + d_j \chi_p(\{z_i\}), \\ \chi_p(\{z_i\}) = \sum_{i=1}^3 \chi_{p_i}(z_i) = \sum_{i=1}^3 \frac{\pi}{g \operatorname{Im} \tau_i} \operatorname{Im}(\bar{p}_i \mathbf{M}^{(i)}), \end{cases} \quad (5.1.12)$$

where d_j is the exterior derivative on the j -th $\mathbb{T}^2$, and we used shorthand notation $p_i = 1, \tau_i$. Correspondingly, the gaugino field should satisfy the following boundary conditions

$$\begin{cases} \lambda(x, \{z_i + \delta_{ij} p_j\}) = U_{p_j}(z_j) \lambda(x, \{z_i\}) U_{p_j}^{-1}(z_j), \\ U_{p_i}(z_i) = \exp[i g \chi_{p_i}(z_i) + 2\pi i \alpha_p^{(i)}], \end{cases} \quad (5.1.13)$$

where the matrices $\alpha_p^{(i)}$ are of the form

$$\alpha_p^{(i)} = \begin{pmatrix} \alpha_{p,1}^{(i)} \mathbb{I}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \alpha_{p,n}^{(i)} \mathbb{I}_{N_n} \end{pmatrix} \quad (5.1.14)$$

and the Scherk–Schwarz phases $\alpha_{p,a}^{(i)}$ are restricted to the allowed values on the i -th $\mathbb{T}^2/\mathbb{Z}_N$ as we have seen in Chapter 3.

For the $\mathbb{Z}_N$ -twist $z_i \rightarrow \omega z_i$, we specify the following boundary conditions for the fields

$$A_\mu(x, \{\omega z_i\}) = P A_\mu(x, \{z_i\}) P^{-1}, \quad (5.1.15)$$

$$A_{z_i}(x, \{\omega z_i\}) = \bar{\omega} P A_{z_i}(x, \{z_i\}) P^{-1}, \quad (5.1.16)$$

$$A_{\bar{z}_i}(x, \{\omega z_i\}) = \omega P A_{\bar{z}_i}(x, \{z_i\}) P^{-1}, \quad (5.1.17)$$

$$\lambda(x, \{\omega z_i\}) = \mathcal{S} P \lambda(x, \{z_i\}) P^{-1}, \quad (5.1.18)$$

where P describes a non-trivial gauge structure part and should satisfy $P^N = \mathbb{I}$. The spinorial matrix $\mathcal{S}$ is defined by Eq. (B.41) in Appendix B. For the KK mode of the gaugino, $\psi_{n_i}^{(i)}(z_i)$, the boundary conditions (5.1.18) is reduced to

$$\psi_{n_i}^{(i)}(\omega z_i) = \tilde{s} P^{(i)} \psi_{n_i}^{(i)}(z_i) P^{(i)-1}, \quad (5.1.19)$$

where $\tilde{s}$ is defined in Eq. (B.35), and $P^{(i)}$ is a matrix describing a non-trivial gauge structure on i -th orbifold. Now, we take the matrix $P^{(i)}$ in the form

$$P^{(i)} = \begin{pmatrix} \eta_1^{(i)} \mathbb{I}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \eta_n^{(i)} \mathbb{I}_{N_n} \end{pmatrix}, \quad (5.1.20)$$

so that the matrix $P^{(i)}$ causes no additional explicit symmetry breaking. Therefore, the total gauge structure matrix becomes

$$P = P^{(1)}P^{(2)}P^{(3)} = \begin{pmatrix} \eta_1 \mathbb{I}_{N_1} & & 0 \\ & \ddots & \\ 0 & & \eta_n \mathbb{I}_{N_n} \end{pmatrix}, \quad \eta_a = \prod_{i=1}^3 \eta_a^{(i)}. \quad (5.1.21)$$

On the gauge background, the zero-mode equations for the gaugino

$$\psi_{ab,\eta}^{(i)}(z_i) = \begin{pmatrix} \psi_{+,ab,\eta}^{(i)}(z_i) \\ \psi_{-,ab,\eta}^{(i)}(z_i) \end{pmatrix}$$

are obtained as follows:²

$$\begin{aligned} \left[\frac{\partial}{\partial \bar{z}_i} + \frac{\pi M_{ab}^{(i)}}{2 \operatorname{Im} \tau_i} z_i \right] \psi_{+,ab,\eta}^{(i)}(z_i) &= 0, \\ \left[\frac{\partial}{\partial z_i} - \frac{\pi M_{ab}^{(i)}}{2 \operatorname{Im} \tau_i} \bar{z}_i \right] \psi_{-,ab,\eta}^{(i)}(z_i) &= 0, \end{aligned} \quad (5.1.22)$$

with $M_{ab}^{(i)} := M_a^{(i)} - M_b^{(i)}$. That is, no magnetic flux exists on the diagonal parts $\psi_{aa}^{(i)}$, the gauginos of the unbroken gauge group. On the other hand, the off-diagonal parts $\psi_{ab}^{(i)} (a \neq b)$ are affected by the effective magnetic flux $M_{ab}^{(i)}$. These equations are equal to the U(1) case discussed in Chapter 3, except that the magnetic flux M is replaced by $M_{ab}^{(i)}$. In addition, the boundary conditions (5.1.13) and (5.1.19) are reduced to the effective ones on $\psi_{\pm,ab,\eta}^{(i)}(z_i)$ as

$$\psi_{\pm,ab,\eta}^{(i)}(z_i + p_i) = e^{i g \chi_{p_i}(z_i) + 2 \pi i \alpha_{p,ab}^{(i)}} \psi_{\pm,ab,\eta}^{(i)}(z_i), \quad (5.1.23)$$

$$\psi_{+,ab,\eta}^{(i)}(\omega z_i) = \eta_{ab}^{(i)} \psi_{+,ab,\eta}^{(i)}(z_i), \quad \psi_{-,ab,\eta}^{(i)}(\omega z_i) = (\omega \eta_{ab}^{(i)}) \psi_{-,ab,\eta}^{(i)}(z_i), \quad (5.1.24)$$

with

$$\alpha_{p,ab}^{(i)} := \alpha_{p,a}^{(i)} - \alpha_{p,b}^{(i)}, \quad (5.1.25)$$

$$\eta_{ab}^{(i)} := \eta_a^{(i)} \bar{\eta}_b^{(i)}. \quad (5.1.26)$$

These are also equal to U(1) case with the replacements

$$\alpha_p \rightarrow \alpha_{p,ab}^{(i)}, \quad \eta \rightarrow \eta_{ab}^{(i)}. \quad (5.1.27)$$

Therefore, we find immediately that the solutions are $|M_{ab}^{(i)}|$ -fold degenerated and are given by the function,

$$\Theta_{M,\eta}^{(j+\alpha_1,\alpha_\tau)}(z;\tau) := \frac{1}{N} \sum_{x=0}^{N-1} \bar{\eta}^x \Theta_M^{(j+\alpha_1,\alpha_\tau)}(\omega^x z;\tau) \quad (j = 0, 1, \dots, |M| - 1), \quad (5.1.28)$$

where $\Theta_M^{(j+\alpha_1,\alpha_\tau)}(z;\tau)$ is given in Eq. (3.1.48).

²Here, we again omit the subscript “₀” describing the zero-modes.

It is also equal to the $U(1)$ case that either $\psi_{+,ab}^{(i)}$ or $\psi_{-,ab}^{(i)}$ have physical solution according to the sign of the effective magnetic flux $M_{ab}^{(i)}$. In this case, the zero-mode solutions are the follows:

$$\begin{cases} \psi_{ab,\eta}^{(i)}(z_i) = \begin{pmatrix} \psi_{+,ab,\eta}^{(i)}(z_i) \\ \psi_{-,ab,\eta}^{(i)}(z_i) \end{pmatrix} = \begin{pmatrix} \Theta_{M_{ab}^{(i)}, \eta_{ab}^{(i)}}^{(j_i + \alpha_{1,ab}, \alpha_{\tau,ab}^{(i)})}(z_i; \tau_i) \\ 0 \end{pmatrix}, \\ \psi_{ba,\eta}^{(i)}(z_i) = \begin{pmatrix} \psi_{+,ba,\eta}^{(i)}(z_i) \\ \psi_{-,ba,\eta}^{(i)}(z_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \Theta_{M_{ba}^{(i)}, \eta_{ba}^{(i)}}^{(j_i + \alpha_{1,ba}, \alpha_{\tau,ba}^{(i)})}(\bar{z}_i; \bar{\tau}_i) \end{pmatrix}, \end{cases} \quad (\text{for } M_{ab}^{(i)} > 0), \quad (5.1.29)$$

$$\begin{cases} \psi_{ab,\eta}^{(i)}(z_i) = \begin{pmatrix} \psi_{+,ab,\eta}^{(i)}(z_i) \\ \psi_{-,ab,\eta}^{(i)}(z_i) \end{pmatrix} = \begin{pmatrix} 0 \\ \Theta_{M_{ab}^{(i)}, \eta_{ab}^{(i)}}^{(j_i + \alpha_{1,ab}, \alpha_{\tau,ab}^{(i)})}(\bar{z}_i; \bar{\tau}_i) \end{pmatrix}, \\ \psi_{ba,\eta}^{(i)}(z_i) = \begin{pmatrix} \psi_{+,ba,\eta}^{(i)}(z_i) \\ \psi_{-,ba,\eta}^{(i)}(z_i) \end{pmatrix} = \begin{pmatrix} \Theta_{M_{ba}^{(i)}, \eta_{ba}^{(i)}}^{(j_i + \alpha_{1,ba}, \alpha_{\tau,ba}^{(i)})}(z_i; \tau_i) \\ 0 \end{pmatrix}, \end{cases} \quad (\text{for } M_{ab}^{(i)} < 0). \quad (5.1.30)$$

The number of independent physical solutions on the i -th $\mathbb{T}^2/\mathbb{Z}_N$ is given by the rank of the matrix defined in Eq. (3.2.43),

$$\mathcal{K}_{jk}^{(\mathbb{Z}_N; \eta)} = \int_{\mathbb{T}^2(i)} d^2 z_i \left(\Theta_{M, \eta}^{(k + \alpha_1, \alpha_\tau)}(z_i; \tau_i) \right)^* \Theta_{M, \eta}^{(j + \alpha_1, \alpha_\tau)}(z_i; \tau_i) = M_{jk}^{(\mathbb{Z}_N; \eta)}. \quad (5.1.31)$$

The results are in the tables in Chapter 4. From KK decomposition (5.1.11), we find that the generation number of (a, b) -sector $\chi_{ab}(x)$, d_{ab} , is given by the product of the numbers of physical solutions of each zero-modes $\psi_{ab,\eta}^{(i)}(z_i)$ ($i = 1, 2, 3$), $d_{ab}^{(i)}$. That is,

$$d_{ab} = \prod_{i=1}^3 d_{ab}^{(i)}. \quad (5.1.32)$$

Therefore, it is necessary that the numbers of independent zero-modes $d_{ab}^{(i)}$ s are in combinations

$$(d_{ab}^{(1)}, d_{ab}^{(2)}, d_{ab}^{(3)}) = (3, 1, 1), (1, 3, 1), (1, 1, 3), \quad (5.1.33)$$

in order to realize three generations of 4D chiral fermion χ_{ab} . On the other hand, at least one of $d_{ab}^{(i)}$ ($i = 1, 2, 3$) must be zero in order to remove χ_{ab} , while all of $d_{ab}^{(i)}$ s must be not zero for leading to non-zero generation of χ_{ab} in effective theory.

5.1.1 Integral formula of three-point function

In this subsection, we present the integral formula of three-point function of mode functions on magnetized $\mathbb{T}^2$, which is developed in Ref. [45]. This formula is useful for computing the Yukawa couplings in 4D effective field theory.

In the magnetized torus model without orbifolding, 4D effective Yukawa couplings are obtained

by the following overlap integral,

$$\begin{aligned}
Y_{jkl} &= \int_{\mathbb{T}^2} d^2 z \, \Theta_{M_{ab}}^{(j+\alpha_{1,ab}, \alpha_{\tau,ab})}(z; \tau) \left(\Theta_{M_{bc}}^{(k+\alpha_{1,bc}, \alpha_{\tau,bc})}(z; \tau) \right)^* \Theta_{M_{ca}}^{(\ell+\alpha_{1,ca}, \alpha_{\tau,ca})}(z; \tau) \\
&= \frac{\mathcal{N}_{M_{ab}} \mathcal{N}_{M_{ca}}}{\mathcal{N}_{M_{bc}}} \sum_{m \in \mathbb{Z}_{M_{bc}}} \vartheta \left[\frac{M_{ca}(j + \alpha_{1,ab}) - M_{ab}(\ell + \alpha_{1,ca}) + m M_{ab} M_{ca}}{M_{ab} M_{bc} M_{ca}} \right] (X, Y) \\
&\quad \times \delta_{(j+\alpha_{1,ab})+(\ell+\alpha_{1,ca})+m M_{ab}, (k+\alpha_{1,bc})+n M_{bc}}.
\end{aligned} \tag{5.1.34}$$

Here, n is a integer (that is, the Kronecker's delta in the above equation is defined in mod M_{bc}), $\Theta_M^{(j+\alpha_1, \alpha_\tau)}(z; \tau)$ is the zero-mode function on the torus defined in Eq. (3.1.48), X and Y are defined by

$$X := M_{ab} \alpha_{\tau,ca} - M_{ca} \alpha_{\tau,ab} + M_{ab} M_{ca} (\alpha_{1,ab} - \alpha_{1,ca}), \quad Y := M_{ab} M_{bc} M_{ca} \tau. \tag{5.1.35}$$

In addition, the parameters must satisfy the following constraint conditions:

$$\begin{aligned}
M_{ab} + M_{ca} &= M_{bc}, \\
\alpha_{1,ab} + \alpha_{1,ca} &= \alpha_{1,bc}, \\
\alpha_{\tau,ab} + \alpha_{\tau,ca} &= \alpha_{\tau,bc}.
\end{aligned} \tag{5.1.36}$$

Although the integral formula, Eq. (5.1.34), is for 6D field theory, 4D effective Yukawa couplings in 10D field theory are given by the product of this formula as follows:

$$\mathbf{Y}_{jkl} = c Y_{j_1 k_1 \ell_1} Y_{j_2 k_2 \ell_2} Y_{j_3 k_3 \ell_3}. \tag{5.1.37}$$

Here, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{\ell}$ represent the combinations of the indices, for example, $\mathbf{j} = (j_1, j_2, j_3)$, $Y_{j_i k_i \ell_i}$ ($i = 1, 2, 3$) is the overlap integral on i -th torus, and the constant c is a factor arising from higher-dimensional coupling constant.

In the case with orbifolding, the Yukawa couplings $\mathcal{Y}_{jkl}$ are obtained by the linear combinations of the above overlap integral as

$$\mathcal{Y}_{jkl} = \sum_{j'=0}^{|M_{ab}|-1} \sum_{k'=0}^{|M_{bc}|-1} \sum_{\ell'=0}^{|M_{ca}|-1} M_{jj'}^{(\mathbb{Z}_N; \eta_{ab})} \left(M_{kk'}^{(\mathbb{Z}_N; \eta_{bc})} \right)^* M_{\ell\ell'}^{(\mathbb{Z}_N; \eta_{ca})} Y_{j'k'\ell'}. \tag{5.1.38}$$

Here, the matrix element $M_{jk}^{(\mathbb{Z}_N; \eta)}$ is defined by Eq. (4.2.20) in Chapter 4, and the parameters must satisfy the condition

$$\eta_{ab} \bar{\eta}_{bc} \eta_{ca} = 1, \tag{5.1.39}$$

in addition to the conditions in Eq. (5.1.36). For 10D field theory, of course, Yukawa couplings are in the same form of Eq. (5.1.37) as follows:

$$\mathcal{Y}_{jkl} = c \mathcal{Y}_{j_1 k_1 \ell_1} \mathcal{Y}_{j_2 k_2 \ell_2} \mathcal{Y}_{j_3 k_3 \ell_3}. \tag{5.1.40}$$

5.2 Search conditions

In our model, the background configuration is taken to be the form

$$\langle A(\{z_i\}) \rangle = \sum_{i=1}^3 \frac{\pi}{g \operatorname{Im} \tau_i} \begin{pmatrix} M_1^{(i)} \mathbb{I}_4 & 0 & 0 \\ 0 & M_2^{(i)} \mathbb{I}_2 & 0 \\ 0 & 0 & M_3^{(i)} \mathbb{I}_2 \end{pmatrix} \operatorname{Im}(\bar{z}_i dz_i), \quad (5.2.1)$$

which breaks the original gauge group $U(8)$ down to the Pati–Salam group $U(4) \times U(2)_L \times U(2)_R$. The gaugino is split into 9 sectors due to this background configuration. The $(1, 2)$ -sector, the $(3, 1)$ -sector, and the $(2, 3)$ -sector correspond to the left-handed matter, charge-conjugation of the right-handed matter, and MSSM-like (up-type and down-type) Higgsinos, respectively.

$$\begin{aligned} \lambda_{12} &\leftrightarrow \text{left-handed quarks and leptons}, \\ \lambda_{31} &\leftrightarrow (\text{charge-conjugation of}) \text{ right-handed quarks and leptons}, \\ \lambda_{23} &\leftrightarrow \text{up-type and down-type (MSSM-like) Higgsinos}. \end{aligned} \quad (5.2.2)$$

In what follows, we denote the parameters of these sectors by the sector symbol, L for the left-handed matter sector, R for the right-handed matter sector, and H for the Higgsino sector, instead of the sector number, such as $M_{12}^{(i)} \rightarrow M_L^{(i)}$.

What we perform in this section is to search for all the phenomenologically possible combinations of the parameters

$$(M_L^{(i)}, \alpha_{1,L}^{(i)}, \alpha_{\tau,L}^{(i)}, \eta_L^{(i)}), \quad (M_R^{(i)}, \alpha_{1,R}^{(i)}, \alpha_{\tau,R}^{(i)}, \eta_R^{(i)}), \quad (M_H^{(i)}, \alpha_{1,H}^{(i)}, \alpha_{\tau,H}^{(i)}, \eta_H^{(i)}), \quad (5.2.3)$$

using the results obtained in the chapter 3 and 4. In the searches, the parameters should satisfy the following conditions:

- **Providing three generations of the quarks and the leptons**

Three generations of the left-handed and the right-handed matters must be provided. That is, the parameters of the left-handed matter and the right-handed matter sector must be chosen as

$$(d_L^{(1)}, d_L^{(2)}, d_L^{(3)}), (d_R^{(1)}, d_R^{(2)}, d_R^{(3)}) = (3, 1, 1), (1, 3, 1), (3, 1, 1). \quad (5.2.4)$$

- **Satisfying invariant conditions**

From the definitions of the “double-indexed” parameters such as $M_{ab}^{(i)}$ in the last section, the parameters of the left-handed matter, the right-handed matter, and the Higgsino sector should satisfy the following constraint conditions:

$$\begin{aligned} M_L^{(i)} + M_R^{(i)} + M_H^{(i)} &= 0, \\ \alpha_{p,L}^{(i)} + \alpha_{p,R}^{(i)} + \alpha_{p,H}^{(i)} &= 0, \\ \eta_L^{(i)} \eta_R^{(i)} \eta_H^{(i)} &= 1. \end{aligned} \quad (5.2.5)$$

These conditions imply that the parameters of the Higgsino sector are determined automatically if the parameters of the other sectors are given. In addition, the combinations of the left-handed matter, right-handed matter and Higgsino sector must be invariant under the unbroken gauge group $U(4) \times U(2)_L \times U(2)_R$ in order to .

- **Avoiding rank-one Yukawa matrices**

The Yukawa couplings in 4D effective theory is given by the following overlap integrals:

$$Y_{jk\ell} = \int_{(\mathbb{T}^2)^3} d^6 z \, \psi_{Lj}(z) \psi_{Rk}(z) \phi_{H\ell}(z). \quad (5.2.6)$$

If the three generations of the left-handed and the right-handed matters originate from different tori, the above integral gives the Yukawa couplings of the factorized form

$$Y_{jk\ell} = a_{j\ell} b_{k\ell}. \quad (5.2.7)$$

The rank of the matrices of this form (assuming that j and k are column and row indices, respectively) is one. Thus, such a type of the Yukawa couplings cannot provide realistic flavor structure. This fact indicates that we focus only on single torus providing three generations originate in order to find realistic possibilities of the parameters providing rank-three Yukawa matrices. Thus, hereafter we omit the index i distinguishing three tori.

- **Additional condition**

The above conditions, Eq. (5.2.5), are invariant under the exchange of the parameters of the left-handed matter sector for the ones of the right-handed matter sector. This implies that we can technically impose the additional condition,

$$|M_L| \leq |M_R|. \quad (5.2.8)$$

In addition, from the constraint condition on the fluxes, the first line of (5.2.5), there are two possibilities on the signs of the three fluxes,

$$M_L < 0, \quad M_R < 0, \quad (\because M_H > 0), \quad (5.2.9a)$$

or

$$M_L < 0, \quad M_R > 0, \quad (\because M_H \leq 0). \quad (5.2.9b)$$

5.3 Results

Let us search possible configurations of the parameters on each orbifolds with the conditions discussed in the last section. We show only the number of possible configurations in $M_L < 0, M_R < 0$ case and $M_L < 0, M_R > 0$ case separately in this section, while the full results of the search are in Appendix D.

The conventions in the following tables are,

- “General cases” and “Trivial BC’s only” means the cases with and without nontrivial Scherk–Schwarz phases respectively,
- d_H show the numbers of the Higgs pairs for the parameter configurations,
- “1_{trivial}” for d_H show the configurations without magnetic flux, i.e., $M_H = 0$ and therefore the Higgsino wavefunction has non-localized flat profile. Since they are not in phenomenological concern, we distinguish them from the one Higgs cases with magnetic flux which cases are denoted by “1”.

General cases		Trivial BC's only	
$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$	$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$
41 ($d_H = 5$)	16 ($d_H = 1_{\text{trivial}}$)	5 ($d_H = 5$)	4 ($d_H = 1_{\text{trivial}}$)
56 ($d_H = 6$)	65 ($d_H = 1$)	2 ($d_H = 6$)	5 ($d_H = 1$)
30 ($d_H = 7$)			
8 ($d_H = 8$)		2 ($d_H = 8$)	
1 ($d_H = 9$)		1 ($d_H = 9$)	
136 + 81 = 217 in total		10 + 9 = 19 in total	

Table 5.1: Result in $\mathbb{Z}_2$ case. (Quoted from [49].)

General cases		Trivial BC's only	
$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$	$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$
11 ($d_H = 4$)	17 ($d_H = 1_{\text{trivial}}$)	1 ($d_H = 4$)	9 ($d_H = 1_{\text{trivial}}$)
83 ($d_H = 5$)	142 ($d_H = 1$)	6 ($d_H = 5$)	27 ($d_H = 1$)
190 ($d_H = 6$)	21 ($d_H = 2$)	7 ($d_H = 6$)	
83 ($d_H = 7$)		6 ($d_H = 7$)	
11 ($d_H = 8$)		1 ($d_H = 8$)	
378 + 180 = 558 in total		21 + 36 = 57 in total	

Table 5.2: Result in $\mathbb{Z}_3$ case. (Quoted from [49].)

General cases		Trivial BC's only	
$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$	$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$
9 ($d_H = 4$)	24 ($d_H = 1_{\text{trivial}}$)	3 ($d_H = 4$)	12 ($d_H = 1_{\text{trivial}}$)
128 ($d_H = 5$)	228 ($d_H = 1$)	37 ($d_H = 5$)	60 ($d_H = 1$)
254 ($d_H = 6$)	18 ($d_H = 2$)	59 ($d_H = 6$)	6 ($d_H = 2$)
120 ($d_H = 7$)		27 ($d_H = 7$)	
17 ($d_H = 8$)		10 ($d_H = 8$)	
528 + 270 = 798 in total		136 + 78 = 214 in total	

Table 5.3: Result in $\mathbb{Z}_4$ case. (Quoted from [49].)

General cases		Trivial BC's only	
$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$	$M_L < 0, M_R < 0$	$M_L < 0, M_R > 0$
14 ($d_H = 4$)	24 ($d_H = 1_{\text{trivial}}$)	4 ($d_H = 4$)	12 ($d_H = 1_{\text{trivial}}$)
156 ($d_H = 5$)	282 ($d_H = 1$)	45 ($d_H = 5$)	73 ($d_H = 1$)
326 ($d_H = 6$)	27 ($d_H = 2$)	76 ($d_H = 6$)	8 ($d_H = 2$)
150 ($d_H = 7$)		36 ($d_H = 7$)	
20 ($d_H = 8$)		10 ($d_H = 8$)	
666 + 333 = 999 in total		171 + 93 = 264 in total	

Table 5.4: Result in $\mathbb{Z}_6$ case. (Quoted from [49].)

We show the results in the form of histogram, Figure 5.1, and make comments on the results.

1. In the “Trivial BC’s only” of $\mathbb{Z}_2$, our analyze reconstructs perfectly the result which was obtained in Ref. [41].
2. In “General cases”, the histograms 5.1 manifestly show that the number of possible configurations centers in the configurations with $d_H = 5, 6, 7$ when M_L and M_R have same sign, while $d_H = 1$ when M_L and M_R have different sign. On the other hand, there are no possible configurations with $d_H = 3$ or $d_H \geq 9$, except for one configuration with $d_H = 9$ on $\mathbb{T}^2/\mathbb{Z}_2$. This trend is common for all of the orbifolds. Therefore, we can claim that the three generations of the matters favors the certain number of Higgs generation.

In this Chapter, we found that 10D SYM theory on magnetized orbifolds is very desirable framework since various puzzles in the SM, the generation, the chirality, the mass hierarchy of the matters, and the gauge structure, originate from relatively simple geometrical background. In addition, we obtained total 2572 possible configurations of the parameters which lead to three generations of the matters by starting with $U(8)$ gauge group and considering the breaking pattern $U(8) \rightarrow U(4) \times U(2)_L \times U(2)_R$.

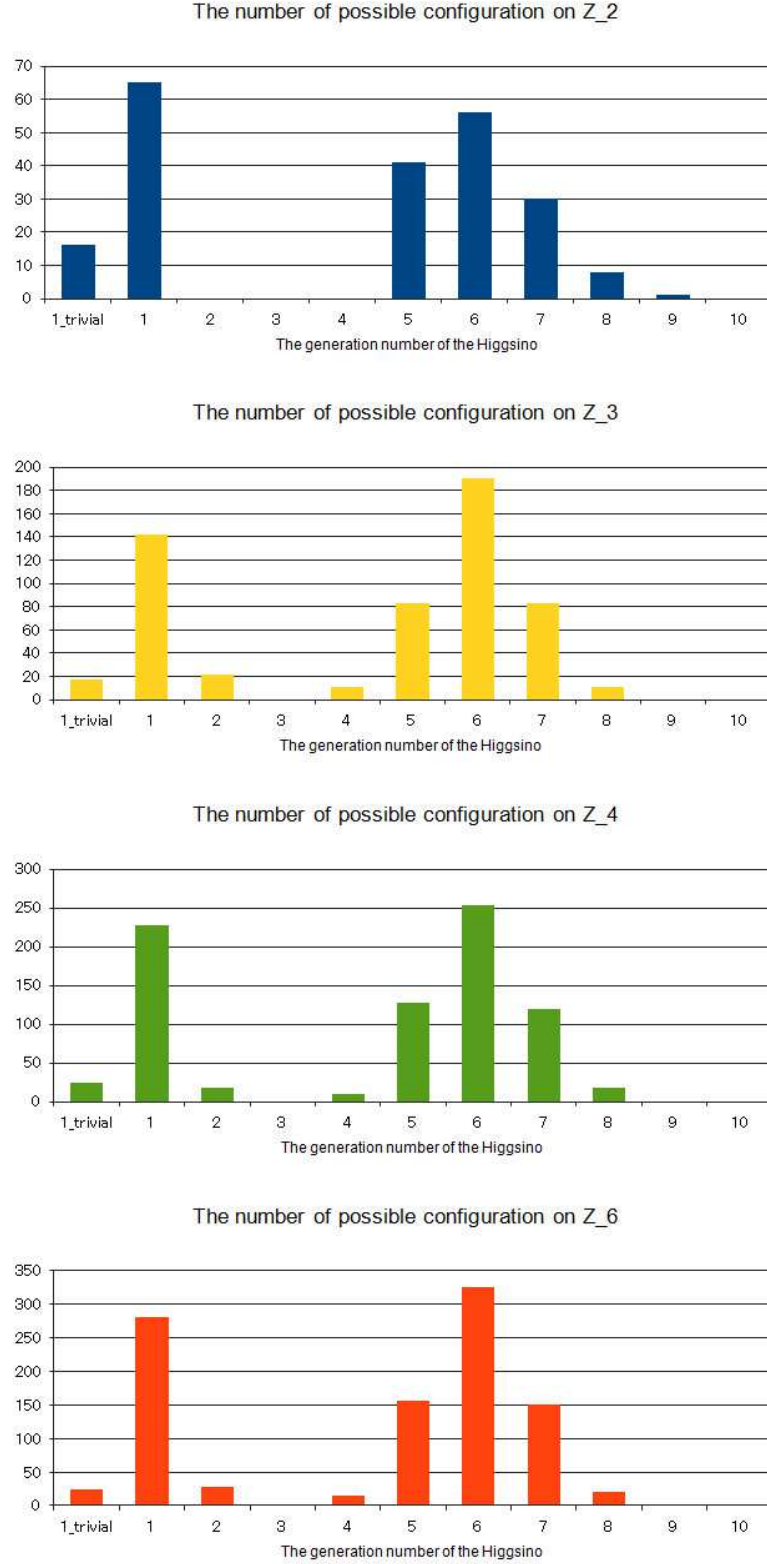


Figure 5.1: These show the results of parameter search in the form of histogram. Horizontal-axes represent the number of Higgsino generation, and vertical-axes describe the number of the possible configurations in “General cases”. The top histogram is for $\mathbb{Z}_2$, the second is for $\mathbb{Z}_3$, the third is for $\mathbb{Z}_4$, and the last is for $\mathbb{Z}_6$. On any orbifolds, the histogram shapes “double-mountain” whose peaks at $d_H = 1$ and $d_H = 6$.

Chapter 6

Summary and future works

In this thesis, we have focused on the 10D SYM theory on the magnetized twisted orbifolds and proposed a desirable framework solving the puzzles in the SM.

We have reviewed briefly the SM in Chapter 2. We have mentioned the puzzles remaining in the SM, the generation, the chirality, the hierarchy of the fermion masses, and the gauge structure.

In Chapter 3, we have investigated the phenomenological aspects of magnetized twisted orbifold by studying the $U(1)$ gauge theory on two-dimensional twisted orbifold $\mathbb{T}^2/\mathbb{Z}_N$ ($N = 2, 3, 4, 6$) with homogeneous magnetic field. By solving the zero-mode equation with non-trivial Wilson-line phase and the Scherk-Schwarz phases on magnetized torus, we have found that the zero-mode solution is degenerate, chiral, and quasi-localized. We have found also that the number of independent zero-mode solution is determined by the magnetic flux. These properties of the zero-mode functions lead to the generation, the chirality, and the hierarchical Yukawa couplings in 4D effective field theory. Furthermore, we have constructed $\mathbb{Z}_N$ eigenfunctions by the linear combinations of the zero-modes on $\mathbb{T}^2$. Since the zero-modes on $\mathbb{T}^2$ are given by Jacobi's theta function, as is seen in Eq. (3.1.33), it is difficult to analyze the $\mathbb{Z}_N$ twisted orbifolds with magnetic flux. In addition, we have found the properties of the Wilson-line phase and the Scherk-Schwarz phases as follows: First, the Wilson-line phase can be absorbed into the Scherk-Schwarz phases by the redefinition of the fields and vice versa. Second, these phases are restricted to the particular values on twisted orbifolds. In particular, we have found that non-zero values of these phases are needed for making the theory consistent on certain orbifold. Third, the number of independent zero-mode on magnetized twisted orbifold is determined by these phases in addition to the magnetic flux and the $\mathbb{Z}_N$ eigenvalue.

In order to resolve the difficulty of magnetized twisted orbifold, we have studied the operator formalism analysis in Chapter 4. This is the dual description to the wavefunction formalism which is used in Chapter 3. We have confirmed the correspondence between the operator formalism and the wavefunction formalism. By using this method of analysis, we have reproduced the limitation on the Scherk-Schwarz phases which is obtained in Chapter 3, and found exact and analytic expression of $\mathbb{Z}_N$ -twisted eigenstates. We have investigated the number of independent zero-mode on magnetized twisted orbifold, thanks to the analytic expression of $\mathbb{Z}_N$ eigenstates.

In Chapter 5, we have proposed 10D SYM theory on magnetized twisted orbifolds, based on the aspects which we have seen in Chapter 3 and 4. By starting with $U(8)$ gauge group and introducing Abelian magnetic flux as in Eq. (5.2.1), we have classified all the possible parameter configurations which lead to three generations of chiral matters in 4D effective theory. We have

obtained total 2572 possible configurations of the parameters by this classification. In addition, we have obtained Pati–Salam gauge group $U(4) \times U(2)_L \times U(2)_R$ by explicit symmetry breakdown due to the magnetic flux. This gauge group is one of well-motivated extension of the SM gauge group.

As we have seen in this thesis, this framework has the following advantages:

First, the biggest advantage is that we can obtain three generation chiral fermions in 4D effective theory because of magnetized torus with/without twisted orbifolding. In other words, the origin of the generation and the chirality, which are the puzzles in the SM as are mentioned in Chapter 2, can be solved by the language of the gauge field configuration and the extra-dimensional geometry.

The second advantage is that exotic particles, which often appear in model-building of beyond the SM, can be projected out by taking appropriate configuration of the parameters. Such configuration can be realized at the same time as a configuration providing three generation of SM fermions.

The third advantage is that we can obtain analytic zero-mode function by solving equation of motion. This is favorable phenomenologically because physical quantities (in particular, Yukawa couplings) in the 4D effective theory can be calculated analytically through overlap integrals. Moreover, such overlap integrals can provide hierarchical Yukawa couplings in the 4D effective theory as are in the SM. In addition, the result of this calculation helps us checking the validity of the model by comparing the obtained quantities with experimental results.

The fourth advantage is that the start point of this framework is 10D SYM theory. It is favorable from the theoretical point of view because 10D SYM theory is a low-energy effective theory of the superstring theories, which are the strongest candidates of the theory describing all the particles and the interactions consistently.

These advantages motivate us to research the framework proposed in this thesis, which solves the puzzles in the SM and gives us a route from the SM to the superstring theories.

Some future works which is necessary for actual model-building are the followings.

1. Investigating the flavor structure:

Now we have all the possible configurations of the parameters, therefore we can compute the Yukawa matrices. These are given by overlap integrals as in Eqs. (5.1.34), (5.1.37), (5.1.38), and (5.1.40). This analysis is important because we can check the validity of models by comparing Yukawa matrices in Eq. (5.1.38) with ones observed in experiments.

2. Searching the parameter configurations on the other two tori:

We have concentrated on the parameters only on a single torus because the rank-three Yukawa matrices are obtained only when three generations of both the left-handed and the right-handed matters originate from the same torus. On the other hand, we should investigate the parameter configurations on the other two tori in order to remove exotic particles. In other words, for constructing realistic models, we need the parameter configurations of exotic sectors which lead to no independent zero-mode on at least one orbifold. Furthermore, such configurations should be consistent with (i) the configurations of the matter sectors

providing three generations, and (ii) the configurations of the higgsino sector providing non-zero generation. The condition (i) is in Eq. (5.1.33), and the condition (ii) requires that all $d^{(i)}$ ($i = 1, 2, 3$) of the higgsino sector are not zero.

3. Investigating tachyonic modes:

In model building, we are interested in tachyonic modes arising from extra-dimensional components of the gauge field. As is mentioned in Appendix C, tachyonic modes may appear in the field $A_{2i+3} + i A_{2i+4}$ depending on the configuration of $M^{(i)}$ and $\mathcal{A}^{(i)}$, where $M^{(i)}$ and $\mathcal{A}^{(i)}$ are the magnetic flux on i -th torus and the area of i -th torus, respectively. Since the Higgs fields arise from extra-dimensional components of the gauge field in the framework presented in this thesis, the negative mass-squared of the Higgs potential (which is the essential for Higgs mechanism) can originate from this tachyonic mode. However, this tachyonic mass is of order of the compactification scale, therefore the EWSB scenario like this becomes unrealistic depending on the compactification scale. We have to investigate this problem in detail.

4. Investigating supersymmetry in effective theory:

It is also phenomenologically interesting whether the supersymmetry remains in 4D effective theory. The method for seeking 4D $\mathcal{N} = 1$ supersymmetry is given in Refs. [46, 56]. If the supersymmetry is preserved, 4D effective theory becomes MSSM with (possibly multiple pairs of) up-type and down-type higgsinos, and there are no tachyonic modes. In such a situation, EWSB occurs if the Higgs scalar potential satisfies certain conditions (see, for example, Ref. [57]).

The framework which we proposed in this thesis is desirable since this solves the puzzles in the SM and is obtained from the superstring theories with simple background configurations. Therefore, this framework is valuable for researching.

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Appendix A

Some useful mathematical formulas

In this Appendix, we list some of the formula useful for calculations of non-commutative quantities and summations in Chapter 3 and 4.

For any invertible matrices A and B , the formula

$$A e^B A^{-1} = e^{ABA^{-1}} \quad (\text{A.1})$$

holds. Baker–Campbell–Hausdorff’s formula is

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B] + \cdots\right). \quad (\text{A.2})$$

If $[A, B]$ commutes with both A and B , i.e., $[A, [A, B]] = [B, [A, B]] = 0$, the following formulas hold

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A, B]}, \quad (\text{A.3})$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]}, \quad (\text{A.4})$$

$$e^A e^B = e^B e^A e^{[A, B]}, \quad (\text{A.5})$$

The summation formula

$$\sum_{j=0}^{N-1} e^{2\pi i \frac{j \cdot k}{N}} = N \delta_{k,0} \quad (\text{for } N \in \mathbb{Z}; k = 0, 1, \dots, N-1), \quad (\text{A.6})$$

is proven by using the property of N -th root of unity.

The formula

$$\sum_{k=0}^{N-1} e^{2\pi i \frac{k^2}{N}} = \sqrt{\frac{N}{2}} e^{\frac{1}{4}\pi i} (1 + e^{-\frac{N}{2}\pi i}), \quad (\text{for } N \in \mathbb{N}) \quad (\text{A.7})$$

is derived by using residue theorem. By using the above, the formula

$$\sum_{j=0}^{N-1} e^{\pi i \frac{(j+s+\zeta_N)^2}{N}} = \sqrt{N} e^{\frac{1}{4}\pi i}, \quad \text{for } N \in \mathbb{N}; s \in \mathbb{Z}; \zeta_N = \begin{cases} 0 & \text{for } N = \text{even}, \\ \frac{1}{2} & \text{for } N = \text{odd}, \end{cases} \quad (\text{A.8})$$

is proven.¹ This formula is rewritten in the following form which is useful for calculations in Chapter 4:

$$\sum_{j=0}^{N-1} e^{\pi i \frac{j(j+2s+2\zeta_N)}{N}} = \sqrt{N} e^{\frac{1}{4}\pi i} e^{-\pi i \frac{(s+\zeta_N)^2}{N}} . \quad (\text{A.9})$$

¹Proof of these formulas is in Appendix B.2 for (A.7) and Appendix B.1 for (A.8), of [43].

Appendix B

Clifford algebra

In this Appendix, we summarize the gamma matrices of 6D and 10D which are used in Chapter 3 and Chapter 5.

B.1 6D Clifford algebra

The 6D gamma matrices Γ^M ($M = 0, 1, 2, 3, 5, 6$) is defined by the Clifford algebra,

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \quad \Gamma_7 := -\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 \Gamma^6, \quad (\text{B.1})$$

where η^{MN} is the higher-dimensional Minkowski metric

$$\eta^{MN} = \text{diag}(+1, -1, \dots, -1). \quad (\text{B.2})$$

We choose the representation of the gamma matrices satisfying (B.1) as

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathbb{I}_2 = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \quad (\mu = 0, 1, 2, 3), \\ \Gamma^5 &= \gamma^5 \otimes i\sigma^1 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix}, \quad \Gamma^6 = \gamma^5 \otimes i\sigma^2 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix}, \end{aligned} \quad (\text{B.3})$$

and define the matrix determining 6D chirality

$$\Gamma_7 = \gamma^5 \otimes \sigma^3 = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}, \quad (\text{B.4})$$

where γ^μ ($\mu = 0, 1, 2, 3$) are the 4D gamma matrices, and $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. The gamma matrices in the complex coordinate are defined by

$$\begin{aligned} \Gamma^z &:= \Gamma^5 + i\Gamma^6 = \gamma^5 \otimes i\sigma^+ = \begin{pmatrix} 0 & 2i\gamma^5 \\ 0 & 0 \end{pmatrix}, \\ \Gamma^{\bar{z}} &:= \Gamma^5 - i\Gamma^6 = \gamma^5 \otimes i\sigma^- = \begin{pmatrix} 0 & 0 \\ 2i\gamma^5 & 0 \end{pmatrix}, \end{aligned} \quad (\text{B.5})$$

with

$$\sigma^+ := \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- := \sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (\text{B.6})$$

We find the $\mathbb{Z}_N$ -twist boundary of a 6D spinor field $\Psi(z)$ interacting with gauge field. Here and hereafter, we use the complex coordinate $z = y^5 + i y^6$ where y^5, y^6 are 2D orthogonal coordinate. We denote the $\mathbb{Z}_N$ -twisted boundary condition as

$$\Psi(\omega z) = \eta \mathcal{S} \Psi(z), \quad (\text{B.7})$$

where $\omega = e^{2\pi i/N}$, η is a numerical factor, and $\mathcal{S}$ is a spinorial matrix. We determine η and $\mathcal{S}$ as the condition (B.7) makes the term $\Gamma^M D_M \Psi$ covariant.

For later convenience, we prepare a little. From the representation (B.3),

$$\Gamma^z = \gamma^5 \otimes i \sigma^+, \quad \Gamma^{\bar{z}} = \gamma^5 \otimes i \sigma^-, \quad (\text{B.8})$$

we immediately find that

$$\Gamma^z \Gamma^{\bar{z}} = -4 \mathbb{I}_4 \otimes \delta_+, \quad \Gamma^{\bar{z}} \Gamma^z = -4 \mathbb{I}_4 \otimes \delta_-, \quad (\text{B.9})$$

$$(\Gamma^z)^2 = (\Gamma^{\bar{z}})^2 = 0, \quad (\text{B.10})$$

$$\{\Gamma^z, \Gamma^{\bar{z}}\} = -4 \mathbb{I}_8, \quad (\text{B.11})$$

where we used $(\gamma^5)^2 = \mathbb{I}_4$ and

$$\begin{aligned} \sigma^+ \sigma^- &= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} =: 4 \delta_+, \quad \sigma^- \sigma^+ = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} =: 4 \delta_-, \\ (\sigma^+)^2 &= (\sigma^-)^2 = 0, \\ \delta_{\pm} \sigma^{\pm} &= \sigma^{\pm} \delta_{\mp} = \sigma^{\pm}, \quad \delta_{\pm} \sigma^{\mp} = \sigma^{\pm} \delta_{\pm} = 0. \end{aligned} \quad (\text{B.12})$$

It is convenient to define the matrices

$$S_+ := -\frac{1}{4} \Gamma^z \Gamma^{\bar{z}} = \mathbb{I}_4 \otimes \delta_+, \quad S_- := -\frac{1}{4} \Gamma^{\bar{z}} \Gamma^z = \mathbb{I}_4 \otimes \delta_-, \quad (\text{B.13})$$

which satisfy the following relations:

$$\begin{aligned} (S_+)^2 &= S_+, \quad (S_-)^2 = S_-, \quad S_{\pm} S_{\mp} = 0, \\ S_- \Gamma^z &= \Gamma^z S_+ = 0, \quad S_+ \Gamma^{\bar{z}} = \Gamma^{\bar{z}} S_- = 0, \\ S_+ \Gamma^z &= \Gamma^z S_- = \Gamma^z, \quad S_- \Gamma^{\bar{z}} = \Gamma^{\bar{z}} S_+ = \Gamma^{\bar{z}}. \end{aligned} \quad (\text{B.14})$$

Now, let us discuss the $\mathbb{Z}_N$ -twist transformation, $z \rightarrow \omega z$. Under this transformation, the covariant derivative term transforms as

$$(\Gamma^\mu D_\mu + \Gamma^z D_z + \Gamma^{\bar{z}} D_{\bar{z}}) \Psi(z) \rightarrow \eta (\Gamma^\mu \mathcal{S} D_\mu + \bar{\omega} \Gamma^z \mathcal{S} D_z + \omega \Gamma^{\bar{z}} \mathcal{S} D_{\bar{z}}) \Psi(z), \quad (\text{B.15})$$

where we used the $\mathbb{Z}_N$ transformation properties of the covariant derivative operators

$$D_\mu \rightarrow D_\mu, \quad D_z \rightarrow \bar{\omega} D_z, \quad D_{\bar{z}} \rightarrow \omega D_{\bar{z}}. \quad (\text{B.16})$$

To make this term covariant, the matrix $\mathcal{S}$ should satisfy the conditions

$$\Gamma^\mu \mathcal{S} = \mathcal{S} \Gamma^\mu, \quad (\text{B.17a})$$

$$\bar{\omega} \Gamma^z \mathcal{S} = \mathcal{S} \Gamma^z, \quad (\text{B.17b})$$

$$\omega \Gamma^{\bar{z}} \mathcal{S} = \mathcal{S} \Gamma^{\bar{z}}. \quad (\text{B.17c})$$

Here, the third condition is the Hermitian-conjugate of the second condition. The Lagrangian transforms under the $\mathbb{Z}_N$ -twist as

$$\begin{aligned} \mathcal{L}(\omega z) &= (\eta \mathcal{S} \Psi)^\dagger \Gamma^0 \cdot i (\Gamma^\mu D_\mu + \Gamma^z \cdot \bar{\omega} D_z + \Gamma^{\bar{z}} \cdot \omega D_{\bar{z}}) \cdot \eta \mathcal{S} \Psi \\ &= \eta^* \Psi^\dagger \mathcal{S}^\dagger \Gamma^0 \cdot i \eta \mathcal{S} \Gamma^M D_M \Psi \\ &= \eta^* \eta (\Psi^\dagger \Gamma^0) \mathcal{S}^\dagger \mathcal{S} \cdot i \Gamma^M D_M \Psi, \end{aligned} \quad (\text{B.18})$$

where we used (B.17). The invariance $\mathcal{L}(\omega z) = \mathcal{L}(z)$ requires the condition

$$\eta^* \eta = 1, \quad \mathcal{S}^\dagger \mathcal{S} = \mathbb{I}_8. \quad (\text{B.19})$$

In addition, since $\omega^N = 1$,

$$\Psi(z) = \Psi(\omega^N z) = \eta^N \mathcal{S}^N \Psi(z). \quad (\text{B.20})$$

Therefore, we find the conditions on η and $\mathcal{S}$,

$$\eta^N = 1, \quad \mathcal{S}^N = \mathbb{I}_8. \quad (\text{B.21})$$

Firstly, we find that the matrix $\mathcal{S}$ consists of even-power terms of $\Gamma^z, \Gamma^{\bar{z}}$, and Γ_7 , because of the commutativity with Γ^μ , (B.17a). So the possible forms of $\mathcal{S}$ are

$$S_+, \quad S_-, \quad \Gamma^z \Gamma_7, \quad \Gamma^{\bar{z}} \Gamma_7, \quad (\text{B.22})$$

and their products. However, the products are reduced in the above four matrices because of the relations (B.14) and

$$(\Gamma^z \Gamma_7)^2 = -\Gamma_7 (\Gamma^z)^2 \Gamma_7 = 0, \quad (\text{B.23})$$

$$(\Gamma^{\bar{z}} \Gamma_7)^2 = -\Gamma_7 (\Gamma^{\bar{z}})^2 \Gamma_7 = 0, \quad (\text{B.24})$$

$$\Gamma^z \Gamma_7 \cdot \Gamma^{\bar{z}} \Gamma_7 = -\Gamma^z \Gamma^{\bar{z}} (\Gamma_7)^2 = 4 S_+, \quad (\text{B.25})$$

$$\Gamma^{\bar{z}} \Gamma_7 \cdot \Gamma^z \Gamma_7 = -\Gamma^{\bar{z}} \Gamma^z (\Gamma_7)^2 = 4 S_-. \quad (\text{B.26})$$

Therefore the possible form of $\mathcal{S}$ is

$$\mathcal{S} = \zeta (S_+ + \xi S_-) + \beta_+ \Gamma^z \Gamma_7 + \beta_- \Gamma^{\bar{z}} \Gamma_7, \quad (\text{B.27})$$

where $\zeta, \xi, \beta_\pm$ are numerical factors. For this form, the both sides of the condition (B.17b) become

$$\begin{aligned} \text{L.H.S.} &= \bar{\omega} \zeta \xi \Gamma^z + \bar{\omega} \beta_- \Gamma^z \Gamma^{\bar{z}} \Gamma_7 \\ &= \bar{\omega} \zeta \xi \Gamma^z + \bar{\omega} \beta_- \left(\Gamma^{\bar{z}} \Gamma_7 \Gamma^z + [\Gamma^z, \Gamma^{\bar{z}} \Gamma_7] \right) \\ &= \bar{\omega} \zeta \xi \Gamma^z + \bar{\omega} \beta_- \Gamma^{\bar{z}} \Gamma_7 \Gamma^z - 4 \bar{\omega} \beta_- \Gamma_7, \end{aligned} \quad (\text{B.28})$$

$$\text{R.H.S.} = \zeta \Gamma^z + \beta_- \Gamma^{\bar{z}} \Gamma_7 \Gamma^z, \quad (\text{B.29})$$

where we used the relation (B.11) to calculate the commutator $[\Gamma^z, \Gamma^{\bar{z}}\Gamma_7]$. Therefore the following two properties are necessary for the condition (B.17b) hold,

$$\bar{\omega}\xi = 1, \quad \beta_- = 0. \quad (\text{B.30})$$

Similarly, the other condition (B.17c) requires the following two properties,

$$\omega\xi^{-1} = 1, \quad \beta_+ = 0. \quad (\text{B.31})$$

Therefore we find that the only possible form of $\mathcal{S}$ is

$$\mathcal{S} = \zeta(S_+ + \omega S_-) = \zeta \mathbb{I}_4 \otimes \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}. \quad (\text{B.32})$$

Furthermore, we find immediately that this form of $\mathcal{S}$ satisfies the unitary condition (B.19) only when ζ is a phase factor satisfying $\zeta^N = 1$,

$$\begin{cases} \mathcal{S}^\dagger \mathcal{S} = \mathbb{I}_8 \iff \zeta^* \zeta = 1, \\ \mathcal{S}^N = \mathbb{I}_8 \iff \zeta^N = 1. \end{cases} \quad (\text{B.33})$$

Therefore ζ can be absorbed into η in (B.7).

Finally, we obtain the result

$$\mathcal{S} = -\frac{1}{4}(\Gamma^z \Gamma^{\bar{z}} + \omega \Gamma^{\bar{z}} \Gamma^z) = \mathbb{I}_4 \otimes \tilde{s}, \quad (\text{B.34})$$

where

$$\tilde{s} := \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}. \quad (\text{B.35})$$

B.2 10D Clifford algebra

On 10D, the gamma matrices are similar to 6D ones. They satisfy the Clifford algebra

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \quad (M, N = 0, 1, 2, 3, 5, \dots, 10), \quad (\text{B.36})$$

and their explicit forms are

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, \quad (\mu = 0, 1, 2, 3), \\ \Gamma^5 &= \gamma^5 \otimes i\sigma^1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, \quad \Gamma^6 = \gamma^5 \otimes i\sigma^2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, \\ \Gamma^7 &= \gamma^5 \otimes \mathbb{I}_2 \otimes i\sigma^1 \otimes \mathbb{I}_2, \quad \Gamma^8 = \gamma^5 \otimes \mathbb{I}_2 \otimes i\sigma^2 \otimes \mathbb{I}_2, \\ \Gamma^9 &= \gamma^5 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes i\sigma^1, \quad \Gamma^{10} = \gamma^5 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes i\sigma^2. \end{aligned} \quad (\text{B.37})$$

We also define

$$\Gamma_{11} = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^5 \Gamma^6 \Gamma^7 \Gamma^8 \Gamma^9 \Gamma^{10} = \gamma^5 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \quad (\text{B.38})$$

which corresponds to Γ_7 on 6D. The gamma matrices in the complex coordinates are defined by

$$\Gamma^{z_i} = \Gamma^{2i+3} + i\Gamma^{2i+4}, \quad \Gamma^{\bar{z}_i} = \Gamma^{2i+3} - i\Gamma^{2i+4}, \quad (i = 1, 2, 3). \quad (\text{B.39})$$

Here, we find the $\mathbb{Z}_N$ -twist boundary of a 10D spinor field $\Psi(\{z_i\}) = \Psi(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)$ interacting with gauge field. Since the $\mathbb{Z}_N$ -twist transformation on $\mathbb{M}^4 \times \mathbb{T}^2$ is described by the direct product of the matrices, $\mathbb{I}_4$ for $\mathbb{M}^4$ and $\tilde{s}$ for $\mathbb{T}^2$, we immediately find that the $\mathbb{Z}_N$ twist matrix $\mathcal{S}$ on $\mathbb{M}^4 \times (\mathbb{T}^2)^3$, which is defined by

$$\Psi(\{\omega z_i\}) = \eta \mathcal{S} \Psi(\{z_i\}), \quad (\text{B.40})$$

is given by

$$\mathcal{S} = \mathbb{I}_4 \otimes \tilde{s} \otimes \tilde{s} \otimes \tilde{s}. \quad (\text{B.41})$$

This matrix indeed satisfies the followings:

- The relations

$$\begin{cases} \Gamma^\mu \mathcal{S} = \mathcal{S} \Gamma^\mu, \\ \bar{\omega} \Gamma^{z_i} \mathcal{S} = \mathcal{S} \Gamma^{z_i}, \\ \omega \Gamma^{\bar{z}_i} \mathcal{S} = \mathcal{S} \Gamma^{\bar{z}_i}, \end{cases} \quad (i = 1, 2, 3), \quad (\text{B.42})$$

which are necessary for the term $\Gamma^M D_M \Psi(\{z_i\}) = [\Gamma^\mu D_\mu + \sum_{i=1}^3 (\Gamma^{z_i} D_{z_i} + \Gamma^{\bar{z}_i} D_{\bar{z}_i})] \Psi(\{z_i\})$ to be covariant under the $\mathbb{Z}_N$ transformation,

- The unitarity condition $\mathcal{S}^\dagger \mathcal{S} = \mathbb{I}$ which is required by the invariance of the Lagrangian,
- The $\mathbb{Z}_N$ consistency condition $\mathcal{S}^N = \mathbb{I}$.

Using the gamma matrices, this matrix can be written as

$$\mathcal{S} = \prod_{i=1}^3 \mathcal{S}^{(i)}, \quad \mathcal{S}^{(i)} := -\frac{1}{4} (\Gamma^{z_i} \Gamma^{\bar{z}_i} + \omega \Gamma^{\bar{z}_i} \Gamma^{z_i}). \quad (\text{B.43})$$

This result can be extended immediately to the higher dimensions, $\mathbb{M}^4 \times (\mathbb{T}^2)^n$.

Appendix C

KK higher modes

In this Appendix, we discuss KK higher modes of spinor, scalar, gauge fields. We first discuss higher modes on 6D magnetized torus/orbifold. As we will see in the following, higher modes can be analyzed by using same method as a harmonic oscillator. In addition, we show that tachyonic mode appears in the zero-mode of gauge field. In the later of this Appendix, we discuss higher modes on 10D magnetized tori/orbifolds. The mass spectra of higher modes on 10D are important in actual model-building, as is mentioned in Chapter 6. The discussions in this Appendix are based on Refs. [51, 58].

C.1 Higher modes on 6D

C.1.1 Higher modes of fermion field

On 6D magnetized torus/orbifold, fermion KK modes $\psi_{n,\pm}(z)$ ($n \geq 0$) satisfy the equation of motion, Eq. (3.1.6),

$$2\langle D_{\bar{z}} \rangle \psi_{n,+}(z) = m_n \psi_{n,-}(z), \quad 2\langle D_z \rangle \psi_{n,-}(z) = -m_n \psi_{n,+}(z), \quad (\text{C.1})$$

where are the covariant derivative in the background gauge configurations $\langle A_z(z) \rangle, \langle A_{\bar{z}}(z) \rangle$, and defined by

$$\langle D_z \rangle = \partial - ie \langle A_z(z) \rangle, \quad \langle D_{\bar{z}} \rangle = \bar{\partial} - ie \langle A_{\bar{z}}(z) \rangle. \quad (\text{C.2})$$

The background gauge configurations $\langle A_z(z) \rangle, \langle A_{\bar{z}}(z) \rangle$ are given by

$$\langle A_z(z) \rangle = -\frac{\pi i M}{2e \operatorname{Im} \tau}(\bar{z} + \bar{a}), \quad \langle A_{\bar{z}}(z) \rangle = \frac{\pi i M}{2e \operatorname{Im} \tau}(z + a). \quad (\text{C.3})$$

The equation of motion, Eq. (C.1), leads to the following equations

$$\begin{aligned} -4\langle D_z \rangle \langle D_{\bar{z}} \rangle \psi_{n,+}(z) &= m_n^2 \psi_{n,+}(z), \\ -4\langle D_{\bar{z}} \rangle \langle D_z \rangle \psi_{n,-}(z) &= m_n^2 \psi_{n,-}(z). \end{aligned} \quad (\text{C.4})$$

In order to solve these equations, we define the operator Δ by

$$\Delta := -2(\langle D_z \rangle \langle D_{\bar{z}} \rangle + \langle D_{\bar{z}} \rangle \langle D_z \rangle). \quad (\text{C.5})$$

By using the relation

$$[\langle D_z \rangle, \langle D_{\bar{z}} \rangle] = \frac{\pi M}{\mathcal{A}}, \quad (\text{C.6})$$

where $\mathcal{A}$ is the torus area, we find that the operator Δ satisfy the following commutation relations

$$[\Delta, \langle D_z \rangle] = \frac{4\pi M}{\mathcal{A}} \langle D_z \rangle, \quad [\Delta, \langle D_{\bar{z}} \rangle] = -\frac{4\pi M}{\mathcal{A}} \langle D_{\bar{z}} \rangle. \quad (\text{C.7})$$

These relations are equivalent to the algebra of single harmonic oscillator with the angular frequency $\frac{4\pi|M|}{\mathcal{A}}$. Therefore, we define the raising and lowering operators by

$$a_+ := i \sqrt{\frac{\mathcal{A}}{\pi|M|}} \langle D_{\bar{z}} \rangle, \quad a_+^\dagger := i \sqrt{\frac{\mathcal{A}}{\pi|M|}} \langle D_z \rangle, \quad (\text{for } M > 0), \quad (\text{C.8})$$

$$a_- := i \sqrt{\frac{\mathcal{A}}{\pi|M|}} \langle D_z \rangle, \quad a_-^\dagger := i \sqrt{\frac{\mathcal{A}}{\pi|M|}} \langle D_{\bar{z}} \rangle, \quad (\text{for } M < 0), \quad (\text{C.9})$$

and the operator Δ is rewritten as follows:

$$\Delta = \begin{cases} \frac{4\pi|M|}{\mathcal{A}} \left(a_+^\dagger a_+ + \frac{1}{2} \right) & (\text{for } M > 0), \\ \frac{4\pi|M|}{\mathcal{A}} \left(a_-^\dagger a_- + \frac{1}{2} \right) & (\text{for } M < 0). \end{cases} \quad (\text{C.10})$$

As is well known, the eigenvalue of this operator is given by

$$E_n = \frac{4\pi|M|}{\mathcal{A}} \left(n + \frac{1}{2} \right), \quad (n = 0, 1, 2, \dots). \quad (\text{C.11})$$

Therefore, we find that the fermion mass-squared is given by

$$m_n^2 = \frac{4\pi|M|}{\mathcal{A}} n, \quad (n = 0, 1, 2, \dots), \quad (\text{C.12})$$

since Eq.(C.4) is rewritten as

$$\left(\Delta - \frac{2\pi|M|}{\mathcal{A}} \right) \psi_{n,\pm}(z) = m_n^2 \psi_{n,\pm}(z). \quad (\text{C.13})$$

Higher modes are constructed by operating the raising operators on the zero-mode states,

$$\begin{cases} \psi_{n,+,j}(z) = \frac{1}{\sqrt{n!}} (a_+^\dagger)^n \psi_{0,+,j}(z), \\ \psi_{n,-,j}(z) = \frac{2}{m_n} \langle D_{\bar{z}} \rangle \psi_{n,+,j}(z), \end{cases} \quad (\text{for } M > 0), \quad (\text{C.14})$$

$$\begin{cases} \psi_{n,-,j}(z) = \frac{1}{\sqrt{n!}} (a_-^\dagger)^n \psi_{0,-,j}(z), \\ \psi_{n,+,j}(z) = -\frac{2}{m_n} \langle D_z \rangle \psi_{n,-,j}(z), \end{cases} \quad (\text{for } M < 0). \quad (\text{C.15})$$

where zero-modes $\psi_{0,\pm,j}(z)$ are given by Eqs. (3.1.33) and (3.1.34). The explicit form of these higher modes is given by the Hermite polynomial. Here, it should be noted that the higher modes of negative-sign (positive-sign) chirality $\psi_{n,-,j}(z)$ ($\psi_{n,+,j}(z)$) exist also for $M > 0$ ($M < 0$), that is, higher modes are not chiral. This is in contrast to the zero-modes, where only $\psi_{0,+,j}(z)$ ($\psi_{0,-,j}(z)$) are well-defined for $M > 0$ ($M < 0$).

On the orbifold $\mathbb{T}^2/\mathbb{Z}_N$, we can construct higher modes similarly to the case of $\mathbb{T}^2$ as

$$\tilde{\psi}_{n,+,j;\mathbb{Z}_N,\eta}(z) = \frac{1}{\sqrt{n!}} (a_+^\dagger)^n \psi_{0,+,j;\mathbb{Z}_N,\eta}(z) = \frac{1}{\sqrt{n!}} \sum_{x=0}^{N-1} \bar{\eta}^x (a_+^\dagger)^n \psi_{0,+,j}(\omega^x z), \quad (\text{C.16a})$$

$$\tilde{\psi}_{n,-,j;\mathbb{Z}_N,\eta}(z) = \frac{1}{\sqrt{n!}} (a_-^\dagger)^n \psi_{0,-,j;\mathbb{Z}_N,\eta}(z) = \frac{1}{\sqrt{n!}} \sum_{x=0}^{N-1} \bar{\omega} \bar{\eta}^x (a_-^\dagger)^n \psi_{0,-,j}(\omega^x z). \quad (\text{C.16b})$$

Here, the higher modes of twisted zero-mode functions $(a_\pm^\dagger)^n \psi_{0,\pm,j}(\omega^x z)$ are given by, e.g.,

$$a_+^\dagger \psi_{0,+,j}(\omega^x z) = i \sqrt{\frac{\mathcal{A}}{\pi|M|}} \left(\frac{\partial}{\partial(\omega^x z)} - \frac{\pi M}{2 \text{Im } \tau} \bar{\omega}^x z \right) \psi_{0,+,j}(\omega^x z). \quad (\text{C.17})$$

Note that, neither η nor $\omega \eta$ are the eigenvalues of $\tilde{\psi}_{n,\pm,j;\mathbb{Z}_N,\eta}(z)$. This is because of the following reason: From the $\mathbb{Z}_N$ -twist transformation properties of $\langle D_z \rangle$ and $\langle D_{\bar{z}} \rangle$, the raising operators $a_\pm^\dagger$ transform as

$$\mathbb{Z}_N : \quad a_+^\dagger \rightarrow \bar{\omega} a_+^\dagger, \quad a_-^\dagger \rightarrow \omega a_-^\dagger. \quad (\text{C.18})$$

Therefore, the higher modes have the $\mathbb{Z}_N$ eigenvalue as follows:

$$\tilde{\psi}_{n,+,j;\mathbb{Z}_N,\eta}(\omega z) = (\bar{\omega}^n \cdot \eta) \tilde{\psi}_{n,+,j;\mathbb{Z}_N,\eta}(z), \quad (\text{C.19})$$

$$\tilde{\psi}_{n,-,j;\mathbb{Z}_N,\eta}(\omega z) = (\omega^n \cdot \omega \eta) \tilde{\psi}_{n,-,j;\mathbb{Z}_N,\eta}(z). \quad (\text{C.20})$$

As an example, the fermion mass spectra for $M = 2$ on $\mathbb{T}^2$ and $\mathbb{T}^2/\mathbb{Z}_3$ are illustrated in the Figures C.1 and C.2. In both figures,

- The red crosses mean the absence of zero-mode solutions,
- The blue (green) filled circles correspond to a zero-mode and its Kaluza–Klein modes of $\psi_{n,\pm,0}(z)$ ($\psi_{n,\pm,1}(z)$),
- The blue (green) arrows mean that $a^\dagger$ operates on the n -th modes $\psi_{n,+,j}(z)$ and the next modes $\psi_{n+1,+,j}(z)$ are made by it,
- Two modes in each black oval, $\psi_{n,+,j}$ and $\psi_{n,-,j}$, make a pair to have a mass term.

C.1.2 Higher modes of scalar and gauge fields

Now, let us consider a 6D complex scalar field $\Phi(x, y)$, which is expanded in KK modes as follows:

$$\Phi(x, y) = \sum_n \varphi_n(x) \times \phi_n(y). \quad (\text{C.21})$$

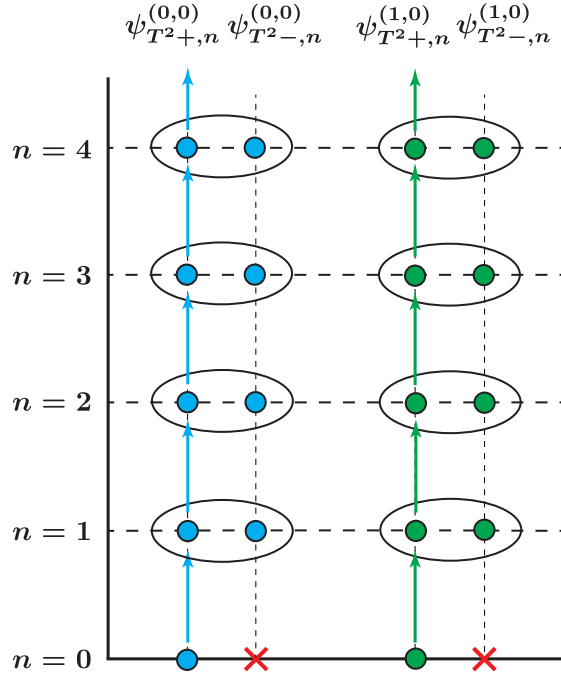


Figure C.1: The mass spectrum of $\psi_{n,j}(z)$ for $M = 2$. (Quoted from [42].)

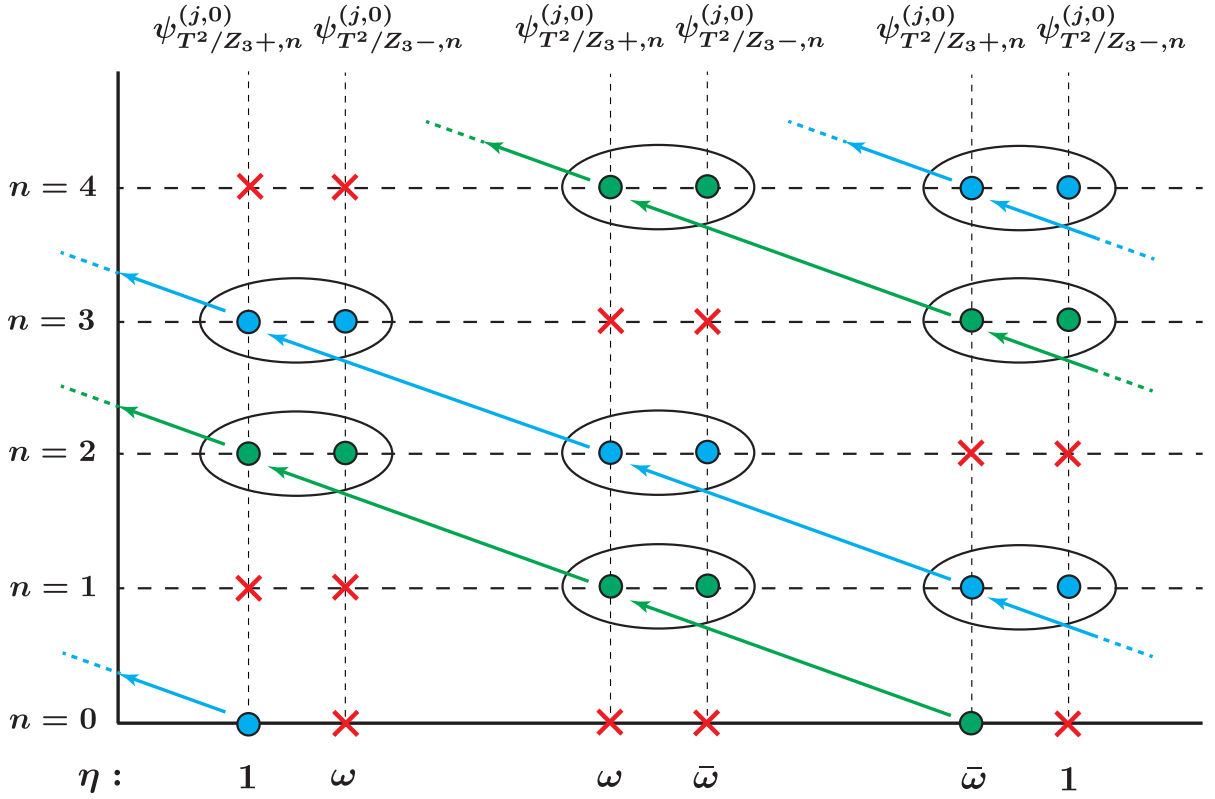


Figure C.2: The mass spectrum of $\psi_{\pm,n,j;\mathbb{Z}_N,\eta}(z)$ for $M = 2$. (Quoted from [42].)

The equation of motion of mode function $\phi_n(y)$ is

$$\Delta \phi_n(y) = m_n^2 \phi_n(y), \quad (\text{C.22})$$

where Δ is defined by Eq. (C.5) and equal to

$$\Delta = -\langle D^m \rangle \langle D_m \rangle, \quad (m = 5, 6). \quad (\text{C.23})$$

Therefore, the mass-squared is immediately found as follows:

$$m_n^2 = \frac{4\pi|M|}{\mathcal{A}} \left(n + \frac{1}{2} \right), \quad (n = 0, 1, 2, \dots). \quad (\text{C.24})$$

For the gauge field, the quadratic term of the KK modes are

$$(\phi_{5;n}, \phi_{6;n}) \begin{pmatrix} \Delta & -i \frac{4\pi M}{\mathcal{A}} \\ i \frac{4\pi M}{\mathcal{A}} & \Delta \end{pmatrix} \begin{pmatrix} \phi_{5;n} \\ \phi_{6;n} \end{pmatrix}, \quad (\text{C.25})$$

which arises from the Lagrangian

$$\mathcal{L}_{6\text{D gauge}} = -\frac{1}{4} F^{MN} F_{MN}. \quad (\text{C.26})$$

The mass matrix in Eq. (C.25) can be diagonalized in the complex basis,

$$\phi_{z;n} := \frac{1}{\sqrt{2}}(\phi_{5;n} + i \phi_{6;n}), \quad \phi_{\bar{z};n} := \frac{1}{\sqrt{2}}(\phi_{5;n} - i \phi_{6;n}), \quad (\text{C.27})$$

and the equation of motion become

$$\left(\Delta - \frac{4\pi M}{\mathcal{A}} \right) \phi_{z;n}(y) = m_{z;n}^2 \phi_{z;n}(y), \quad (\text{C.28a})$$

$$\left(\Delta + \frac{4\pi M}{\mathcal{A}} \right) \phi_{\bar{z};n}(y) = m_{\bar{z};n}^2 \phi_{\bar{z};n}(y). \quad (\text{C.28b})$$

Therefore, the mass spectra of these fields are given by

$$m_{z;n}^2 = \frac{4\pi|M|}{\mathcal{A}} \left(n + \frac{1}{2} \right) - \frac{4\pi M}{\mathcal{A}}, \quad (\text{C.29a})$$

$$m_{\bar{z};n}^2 = \frac{4\pi|M|}{\mathcal{A}} \left(n + \frac{1}{2} \right) + \frac{4\pi M}{\mathcal{A}}. \quad (\text{C.29b})$$

These spectra imply the followings:

1. If $M > 0$, the zero-mode $\phi_{z;0}$ becomes tachyonic, while all the mode $\phi_{z;n}$ ($n = 0, 1, 2, \dots$) are massive.
2. If $M < 0$, the zero-mode $\phi_{\bar{z};0}$ becomes tachyonic, while all the mode $\phi_{\bar{z};n}$ ($n = 0, 1, 2, \dots$) are massive.

C.2 Higher modes on 10D

On 10D magnetized tori/orbifolds, the equations of motion for KK modes are obtained by replacing the Laplace operator with

$$\Delta = \sum_{i=1}^3 \Delta^{(i)}, \quad (\text{C.1})$$

where $\Delta^{(i)}$ is the Laplace operator on i -th torus/orbifold. Therefore, the mass spectra of fermion and scalar fields are given by the summation of the mass eigenvalues on single torus. That is, the fermion mass-squared is given by

$$m_{n_1, n_2, n_3}^2 = \sum_{i=1}^3 \frac{4\pi |M^{(i)}|}{\mathcal{A}^{(i)}} n_i, \quad (n_i = 1, 2, 3), \quad (\text{C.2})$$

and the scalar mass-squared is given by

$$m_{n_1, n_2, n_3}^2 = \sum_{i=1}^3 \frac{4\pi |M^{(i)}|}{\mathcal{A}^{(i)}} \left(n_i + \frac{1}{2} \right), \quad (i = 1, 2, 3). \quad (\text{C.3})$$

Here, $M^{(i)}$ and $\mathcal{A}^{(i)}$ are the magnetic flux on i -th torus and the area of i -th torus, respectively.

For the vector KK modes, the equations of motion are

$$\left(\Delta - \frac{4\pi M^{(i)}}{\mathcal{A}^{(i)}} \right) \left[\prod_{j=1}^3 \phi_{z_i; n_j}^{(j)}(y^{2j+3}, y^{2j+4}) \right] = m_{z_i; n_1, n_2, n_3}^2 \left[\prod_{j=1}^3 \phi_{z_i; n_j}^{(j)}(y^{2j+3}, y^{2j+4}) \right], \quad (\text{C.4a})$$

$$\left(\Delta + \frac{4\pi M^{(i)}}{\mathcal{A}^{(i)}} \right) \left[\prod_{j=1}^3 \phi_{\bar{z}_i; n_j}^{(j)}(y^{2j+3}, y^{2j+4}) \right] = m_{\bar{z}_i; n_1, n_2, n_3}^2 \left[\prod_{j=1}^3 \phi_{\bar{z}_i; n_j}^{(j)}(y^{2j+3}, y^{2j+4}) \right]. \quad (\text{C.4b})$$

Here, $\phi_{z_i; n_j}^{(j)}$ and $\phi_{\bar{z}_i; n_j}^{(j)}$ are defined in the same manner as Eq. (C.27) by

$$\begin{aligned} \phi_{z_i; n_j}^{(j)} &= \frac{1}{\sqrt{2}} (\phi_{2i+3; n_j}^{(j)} + i \phi_{2i+4; n_j}^{(j)}), \\ \phi_{\bar{z}_i; n_j}^{(j)} &= \frac{1}{\sqrt{2}} (\phi_{2i+3; n_j}^{(j)} - i \phi_{2i+4; n_j}^{(j)}), \end{aligned} \quad (\text{C.5})$$

where $\phi_{2i+3; n_j}^{(j)}$ and $\phi_{2i+4; n_j}^{(j)}$ are n_j -th KK modes on j -th torus associated with the extra-dimensional components of gauge field A_{2i+3} and A_{2i+4} , respectively. The mass spectra are given by

$$m_{z_i; n_1, n_2, n_3}^2 = \sum_{j=1}^3 \frac{4\pi |M^{(j)}|}{\mathcal{A}^{(j)}} \left(n_j + \frac{1}{2} \right) - \frac{4\pi M^{(i)}}{\mathcal{A}^{(i)}}, \quad (\text{C.6a})$$

$$m_{\bar{z}_i; n_1, n_2, n_3}^2 = \sum_{j=1}^3 \frac{4\pi |M^{(j)}|}{\mathcal{A}^{(j)}} \left(n_j + \frac{1}{2} \right) + \frac{4\pi M^{(i)}}{\mathcal{A}^{(i)}}. \quad (\text{C.6b})$$

In particular, the mass-squared of zero-modes ($n_1 = n_2 = n_3 = 0$) are

$$m_{z_i; 0}^2 = 2\pi \left[\sum_{j \neq i} \frac{|M^{(j)}|}{\mathcal{A}^{(j)}} - \frac{M^{(i)}}{\mathcal{A}^{(i)}} \right], \quad (\text{C.7a})$$

$$m_{\bar{z}_i;0}^2 = 2\pi \left[\sum_{j \neq i} \frac{|M^{(j)}|}{\mathcal{A}^{(j)}} + \frac{M^{(i)}}{\mathcal{A}^{(i)}} \right]. \quad (\text{C.7b})$$

Therefore, tachyonic mode can be removed depending on the parameters $M^{(i)}$ and $\mathcal{A}^{(i)}$; There is a tachyonic mode in 4D scalars, $\varphi_{z_i;n_1,n_2,n_2}$ or $\varphi_{\bar{z}_i;n_1,n_2,n_3}$, when

$$\delta_i := \sum_{j \neq i} \frac{|M^{(j)}|}{\mathcal{A}^{(j)}} - \frac{|M^{(i)}|}{\mathcal{A}^{(i)}} < 0. \quad (\text{C.8})$$

All the scalars $\varphi_{z_i;n_1,n_2,n_3}$ and $\varphi_{\bar{z}_i;n_1,n_2,n_3}$ are massive when $\delta_i > 0$, while one massless mode exists in the scalars $\varphi_{z_i;n_1,n_2,n_3}$ or $\varphi_{\bar{z}_i;n_1,n_2,n_3}$ when $\delta_i = 0$.

Appendix D

Possible parameter configurations

In this appendix, we list all the possible configurations of the parameters that satisfy the search conditions given in Section 5.2.

The conventions used in the following tables are as follows:

1. The results of the two cases $M_L < 0, M_R < 0$ and $M_L < 0, M_R > 0$ are shown separately.
2. The parameters of the Higgs sector are not shown except for the number of generation, because these are automatically determined by the parameters of the matter sectors through the constraint conditions (5.2.5).
3. Since the Scherk–Schwarz phases α_1 and α_τ related to the two basis of the torus are the same on the orbifolds $\mathbb{T}^2/\mathbb{Z}_{3,4,6}$, we denote the Scherk–Schwarz phases by α .
4. The $\mathbb{Z}_N$ eigenvalue is of the form of $\eta = \omega^x$ ($x = 0, 1, \dots, N-1$) where $\omega = e^{2\pi i/N}$. We represent the $\mathbb{Z}_N$ eigenvalues as a function of x .
5. We denote one generation of the Higgs with non-localized wavefunction by “ $\boxed{1}$ ”, instead of “ 1_{trivial} ” in Chapter 5.

D.1 $\mathbb{T}^2/\mathbb{Z}_2$

Table D.1: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_2$ with $M_L < 0, M_R < 0$.

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-4	0	0	0	-4	0	0	0	5
-4	0	0	0	-5	0	0	0	5
-4	0	0	0	-5	0	0	1/2	5
-4	0	0	0	-5	0	1/2	0	5
-4	0	0	0	-5	1	1/2	1/2	5
-5	0	0	0	-5	0	0	0	6
-5	0	0	0	-5	0	0	1/2	5
-5	0	0	0	-5	0	1/2	0	5
-5	0	0	1/2	-5	0	0	1/2	6
-5	0	0	1/2	-5	0	1/2	0	5
-5	0	1/2	0	-5	0	1/2	0	6
-5	0	0	0	-5	1	1/2	1/2	5
-5	0	0	1/2	-5	1	1/2	1/2	5
-5	0	1/2	0	-5	1	1/2	1/2	5
-5	1	1/2	1/2	-5	1	1/2	1/2	6

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-4	0	0	0	-6	0	0	1/2	5
-4	0	0	0	-6	0	1/2	0	5
-4	0	0	0	-6	0	1/2	1/2	5
-4	0	0	0	-6	1	0	1/2	5
-4	0	0	0	-6	1	1/2	0	5
-4	0	0	0	-6	1	1/2	1/2	5
-5	0	0	0	-6	0	0	1/2	6
-5	0	0	0	-6	0	1/2	0	6
-5	0	0	0	-6	0	1/2	1/2	5
-5	0	0	1/2	-6	0	0	1/2	6
-5	0	0	1/2	-6	0	1/2	0	5
-5	0	0	1/2	-6	0	1/2	1/2	6
-5	0	1/2	0	-6	0	0	1/2	5
-5	0	1/2	0	-6	0	1/2	0	6
-5	0	1/2	0	-6	0	1/2	1/2	6
-5	0	0	0	-6	1	0	1/2	5
-5	0	0	0	-6	1	1/2	0	5
-5	0	0	0	-6	1	1/2	1/2	6
-5	0	0	1/2	-6	1	0	1/2	5
-5	0	0	1/2	-6	1	1/2	0	6
-5	0	0	1/2	-6	1	1/2	1/2	5
-5	0	1/2	0	-6	1	0	1/2	6
-5	0	1/2	0	-6	1	1/2	0	5
-5	0	1/2	0	-6	1	1/2	1/2	5
-5	1	1/2	1/2	-6	0	0	1/2	5
-5	1	1/2	1/2	-6	0	1/2	0	5
-5	1	1/2	1/2	-6	0	1/2	1/2	5
-5	1	1/2	1/2	-6	1	0	1/2	6
-5	1	1/2	1/2	-6	1	1/2	0	6
-5	1	1/2	1/2	-6	1	1/2	1/2	6
-6	0	0	1/2	-6	0	0	1/2	7
-6	0	0	1/2	-6	0	1/2	0	6
-6	0	0	1/2	-6	0	1/2	1/2	6
-6	0	1/2	0	-6	0	1/2	0	7
-6	0	1/2	0	-6	0	1/2	1/2	6
-6	0	1/2	1/2	-6	0	1/2	1/2	7
-6	0	0	1/2	-6	1	0	1/2	5
-6	0	0	1/2	-6	1	1/2	0	6
-6	0	0	1/2	-6	1	1/2	1/2	6
-6	0	1/2	0	-6	1	0	1/2	6
-6	0	1/2	0	-6	1	1/2	0	5
-6	0	1/2	0	-6	1	1/2	1/2	6
-6	0	1/2	1/2	-6	1	0	1/2	6
-6	0	1/2	1/2	-6	1	1/2	0	6
-6	0	1/2	1/2	-6	1	1/2	1/2	5
-6	1	0	1/2	-6	1	0	1/2	7
-6	1	0	1/2	-6	1	1/2	0	6
-6	1	0	1/2	-6	1	1/2	1/2	6
-6	1	1/2	0	-6	1	1/2	0	7
-6	1	1/2	0	-6	1	1/2	1/2	6
-6	1	1/2	1/2	-6	1	1/2	1/2	7
-4	0	0	0	-7	0	1/2	1/2	5
-4	0	0	0	-7	1	0	0	5
-4	0	0	0	-7	1	0	1/2	5
-4	0	0	0	-7	1	1/2	0	5
-5	0	0	0	-7	0	1/2	1/2	6
-5	0	0	1/2	-7	0	1/2	1/2	6
-5	0	1/2	0	-7	0	1/2	1/2	6
-5	0	0	0	-7	1	0	0	5
-5	0	0	0	-7	1	0	1/2	6
-5	0	0	0	-7	1	1/2	0	6
-5	0	0	1/2	-7	1	0	0	6
-5	0	0	1/2	-7	1	0	1/2	5

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-5	0	0	1/2	-7	1	1/2	0	6
-5	0	1/2	0	-7	1	0	0	6
-5	0	1/2	0	-7	1	0	1/2	6
-5	0	1/2	0	-7	1	1/2	0	5
-5	1	1/2	1/2	-7	0	1/2	1/2	5
-5	1	1/2	1/2	-7	1	0	0	6
-5	1	1/2	1/2	-7	1	0	1/2	6
-5	1	1/2	1/2	-7	1	1/2	0	6
-6	0	0	1/2	-7	0	1/2	1/2	7
-6	0	1/2	0	-7	0	1/2	1/2	7
-6	0	1/2	1/2	-7	0	1/2	1/2	7
-6	0	0	1/2	-7	1	0	0	6
-6	0	0	1/2	-7	1	0	1/2	6
-6	0	0	1/2	-7	1	1/2	0	7
-6	0	1/2	0	-7	1	0	0	6
-6	0	1/2	0	-7	1	0	1/2	7
-6	0	1/2	0	-7	1	1/2	0	6
-6	0	1/2	1/2	-7	1	0	0	7
-6	0	1/2	1/2	-7	1	0	1/2	6
-6	0	1/2	1/2	-7	1	1/2	0	6
-6	1	0	1/2	-7	0	1/2	1/2	6
-6	1	1/2	0	-7	0	1/2	1/2	6
-6	1	1/2	1/2	-7	0	1/2	1/2	6
-6	1	0	1/2	-7	1	0	0	7
-6	1	0	1/2	-7	1	0	1/2	7
-6	1	0	1/2	-7	1	1/2	0	6
-6	1	1/2	0	-7	1	0	0	7
-6	1	1/2	0	-7	1	0	1/2	6
-6	1	1/2	0	-7	1	1/2	0	7
-6	1	1/2	1/2	-7	1	0	0	6
-6	1	1/2	1/2	-7	1	0	1/2	7
-6	1	1/2	1/2	-7	1	1/2	0	7
-7	0	1/2	1/2	-7	0	1/2	1/2	8
-7	0	1/2	1/2	-7	1	0	0	7
-7	0	1/2	1/2	-7	1	0	1/2	7
-7	0	1/2	1/2	-7	1	1/2	0	7
-7	1	0	0	-7	1	0	0	8
-7	1	0	0	-7	1	0	1/2	7
-7	1	0	0	-7	1	1/2	0	7
-7	1	0	1/2	-7	1	0	1/2	8
-7	1	0	1/2	-7	1	1/2	0	7
-7	1	1/2	0	-7	1	1/2	0	8
-4	0	0	0	-8	1	0	0	5
-5	0	0	0	-8	1	0	0	6
-5	0	0	1/2	-8	1	0	0	6
-5	0	1/2	0	-8	1	0	0	6
-5	1	1/2	1/2	-8	1	0	0	6
-6	0	0	1/2	-8	1	0	0	7
-6	0	1/2	0	-8	1	0	0	7
-6	0	1/2	1/2	-8	1	0	0	7
-6	1	0	1/2	-8	1	0	0	7
-6	1	1/2	0	-8	1	0	0	7
-6	1	1/2	1/2	-8	1	0	0	7
-7	0	1/2	1/2	-8	1	0	0	8
-7	1	0	0	-8	1	0	0	8
-7	1	0	1/2	-8	1	0	0	8
-7	1	1/2	0	-8	1	0	0	8
-8	1	0	0	-8	1	0	0	9
-4	0	0	0	-4	0	0	0	5
-4	0	0	0	-5	0	0	0	5
-4	0	0	0	-5	0	0	1/2	5
-4	0	0	0	-5	0	1/2	0	5
-4	0	0	0	-5	1	1/2	1/2	5

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-5	0	0	0	-5	0	0	0	6
-5	0	0	0	-5	0	0	1/2	5
-5	0	0	0	-5	0	1/2	0	5
-5	0	0	1/2	-5	0	0	1/2	6
-5	0	0	1/2	-5	0	1/2	0	5
-5	0	1/2	0	-5	0	1/2	0	6
-5	0	0	0	-5	1	1/2	1/2	5
-5	0	0	1/2	-5	1	1/2	1/2	5
-5	0	1/2	0	-5	1	1/2	1/2	5
-5	1	1/2	1/2	-5	1	1/2	1/2	6
-4	0	0	0	-6	0	0	1/2	5
-4	0	0	0	-6	0	1/2	0	5
-4	0	0	0	-6	0	1/2	1/2	5
-4	0	0	0	-6	1	0	1/2	5
-4	0	0	0	-6	1	1/2	0	5
-4	0	0	0	-6	1	1/2	1/2	5
-5	0	0	0	-6	0	0	1/2	6
-5	0	0	0	-6	0	1/2	0	6
-5	0	0	0	-6	0	1/2	1/2	5
-5	0	0	1/2	-6	0	0	1/2	6
-5	0	0	1/2	-6	0	1/2	0	5
-5	0	0	1/2	-6	0	1/2	1/2	6
-5	0	1/2	0	-6	0	0	1/2	5
-5	0	1/2	0	-6	0	1/2	0	6
-5	0	1/2	0	-6	0	1/2	1/2	6
-5	0	0	0	-6	1	0	1/2	5
-5	0	0	0	-6	1	1/2	0	5
-5	0	0	0	-6	1	1/2	1/2	6
-5	0	0	1/2	-6	1	0	1/2	5
-5	0	0	1/2	-6	1	1/2	0	6
-5	0	0	1/2	-6	1	1/2	1/2	5
-5	0	1/2	0	-6	1	0	1/2	6
-5	0	1/2	0	-6	1	1/2	0	5
-5	0	1/2	0	-6	1	1/2	1/2	5
-5	1	1/2	1/2	-6	0	0	1/2	5
-5	1	1/2	1/2	-6	0	1/2	0	5
-5	1	1/2	1/2	-6	0	1/2	1/2	5
-5	1	1/2	1/2	-6	1	0	1/2	6
-5	1	1/2	1/2	-6	1	1/2	0	6
-5	1	1/2	1/2	-6	1	1/2	1/2	6
-6	0	0	1/2	-6	0	0	1/2	7
-6	0	0	1/2	-6	0	1/2	0	6
-6	0	0	1/2	-6	0	1/2	1/2	6
-6	0	1/2	0	-6	0	1/2	0	7
-6	0	1/2	0	-6	0	1/2	1/2	6
-6	0	1/2	1/2	-6	0	1/2	1/2	7
-6	0	0	1/2	-6	1	0	1/2	5
-6	0	0	1/2	-6	1	1/2	0	6
-6	0	0	1/2	-6	1	1/2	1/2	6
-6	0	1/2	0	-6	1	0	1/2	6
-6	0	1/2	0	-6	1	1/2	0	5
-6	0	1/2	0	-6	1	1/2	1/2	6
-6	0	1/2	1/2	-6	1	0	1/2	6
-6	0	1/2	1/2	-6	1	1/2	0	6
-6	0	1/2	1/2	-6	1	1/2	1/2	5
-6	1	0	1/2	-6	1	0	1/2	7
-6	1	0	1/2	-6	1	1/2	0	6
-6	1	0	1/2	-6	1	1/2	1/2	6
-6	1	1/2	0	-6	1	1/2	0	7
-6	1	1/2	0	-6	1	1/2	1/2	6
-6	1	1/2	1/2	-6	1	1/2	1/2	7
-4	0	0	0	-7	0	1/2	1/2	5
-4	0	0	0	-7	1	0	0	5

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-4	0	0	0	-7	1	0	1/2	5
-4	0	0	0	-7	1	1/2	0	5
-5	0	0	0	-7	0	1/2	1/2	6
-5	0	0	1/2	-7	0	1/2	1/2	6
-5	0	1/2	0	-7	0	1/2	1/2	6
-5	0	0	0	-7	1	0	0	5
-5	0	0	0	-7	1	0	1/2	6
-5	0	0	0	-7	1	1/2	0	6
-5	0	0	1/2	-7	1	0	0	6
-5	0	0	1/2	-7	1	0	1/2	5
-5	0	0	1/2	-7	1	1/2	0	6
-5	0	1/2	0	-7	1	0	0	6
-5	0	1/2	0	-7	1	0	1/2	6
-5	0	1/2	0	-7	1	1/2	0	5
-5	1	1/2	1/2	-7	0	1/2	1/2	5
-5	1	1/2	1/2	-7	1	0	0	6
-5	1	1/2	1/2	-7	1	0	1/2	6
-5	1	1/2	1/2	-7	1	1/2	0	6
-6	0	0	1/2	-7	0	1/2	1/2	7
-6	0	1/2	0	-7	0	1/2	1/2	7
-6	0	1/2	1/2	-7	0	1/2	1/2	7
-6	0	0	1/2	-7	1	0	0	6
-6	0	0	1/2	-7	1	0	1/2	6
-6	0	0	1/2	-7	1	1/2	0	7
-6	0	1/2	0	-7	1	0	0	6
-6	0	1/2	0	-7	1	0	1/2	7
-6	0	1/2	0	-7	1	1/2	0	6
-6	0	1/2	1/2	-7	1	0	0	7
-6	0	1/2	1/2	-7	1	0	1/2	6
-6	0	1/2	1/2	-7	1	1/2	0	6
-6	1	0	1/2	-7	0	1/2	1/2	6
-6	1	1/2	0	-7	0	1/2	1/2	6
-6	1	1/2	1/2	-7	0	1/2	1/2	6
-6	1	0	1/2	-7	1	0	0	7
-6	1	0	1/2	-7	1	0	1/2	7
-6	1	0	1/2	-7	1	1/2	0	6
-6	1	1/2	0	-7	1	0	0	7
-6	1	1/2	0	-7	1	0	1/2	6
-6	1	1/2	0	-7	1	1/2	0	7
-6	1	1/2	1/2	-7	1	0	0	6
-6	1	1/2	1/2	-7	1	0	1/2	7
-6	1	1/2	1/2	-7	1	1/2	0	7
-7	0	1/2	1/2	-7	0	1/2	1/2	8
-7	0	1/2	1/2	-7	1	0	0	7
-7	0	1/2	1/2	-7	1	0	1/2	7
-7	0	1/2	1/2	-7	1	1/2	0	7
-7	1	0	0	-7	1	0	0	8
-7	1	0	0	-7	1	0	1/2	7
-7	1	0	0	-7	1	1/2	0	7
-7	1	0	1/2	-7	1	0	1/2	8
-7	1	0	1/2	-7	1	1/2	0	7
-7	1	1/2	0	-7	1	1/2	0	8
-4	0	0	0	-8	1	0	0	5
-5	0	0	0	-8	1	0	0	6
-5	0	0	1/2	-8	1	0	0	6
-5	0	1/2	0	-8	1	0	0	6
-5	1	1/2	1/2	-8	1	0	0	6
-6	0	0	1/2	-8	1	0	0	7
-6	0	1/2	0	-8	1	0	0	7
-6	0	1/2	1/2	-8	1	0	0	7
-6	1	0	1/2	-8	1	0	0	7
-6	1	1/2	0	-8	1	0	0	7
-6	1	1/2	1/2	-8	1	0	0	7

L-sector				R-sector				H-sector
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs
-7	0	1/2	1/2	-8	1	0	0	8
-7	1	0	0	-8	1	0	0	8
-7	1	0	1/2	-8	1	0	0	8
-7	1	1/2	0	-8	1	0	0	8
-8	1	0	0	-8	1	0	0	9

Table D.2: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_2$ with $M_L < 0$, $M_R > 0$.

L-sector				R-sector				H-sector	
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs	
-4	0	0	0	4	0	0	0	1	
-4	0	0	0	5	0	0	0	1	
-4	0	0	0	5	0	0	1/2	1	
-4	0	0	0	5	0	1/2	0	1	
-4	0	0	0	5	1	1/2	1/2	1	
-5	0	0	0	5	0	0	0	1	
-5	0	0	1/2	5	0	0	1/2	1	
-5	0	1/2	0	5	0	1/2	0	1	
-5	1	1/2	1/2	5	1	1/2	1/2	1	
-4	0	0	0	6	0	0	1/2	1	
-4	0	0	0	6	0	1/2	0	1	
-4	0	0	0	6	0	1/2	1/2	1	
-4	0	0	0	6	1	0	1/2	1	
-4	0	0	0	6	1	1/2	0	1	
-4	0	0	0	6	1	1/2	1/2	1	
-5	0	0	0	6	0	0	1/2	1	
-5	0	0	0	6	0	1/2	0	1	
-5	0	0	1/2	6	0	0	1/2	1	
-5	0	0	1/2	6	0	1/2	1/2	1	
-5	0	1/2	0	6	0	1/2	0	1	
-5	0	1/2	0	6	0	1/2	1/2	1	
-5	0	0	0	6	1	1/2	1/2	1	
-5	0	0	1/2	6	1	1/2	0	1	
-5	0	1/2	0	6	1	0	1/2	1	
-5	1	1/2	1/2	6	1	0	1/2	1	
-5	1	1/2	1/2	6	1	1/2	0	1	
-5	1	1/2	1/2	6	1	1/2	1/2	1	
-6	0	0	1/2	6	0	0	1/2	1	
-6	0	1/2	0	6	0	1/2	0	1	
-6	0	1/2	1/2	6	0	1/2	1/2	1	
-6	1	0	1/2	6	1	0	1/2	1	
-6	1	1/2	0	6	1	1/2	0	1	
-6	1	1/2	1/2	6	1	1/2	1/2	1	
-4	0	0	0	7	0	1/2	1/2	1	
-4	0	0	0	7	1	0	0	1	
-4	0	0	0	7	1	0	1/2	1	
-4	0	0	0	7	1	1/2	0	1	
-5	0	0	0	7	0	1/2	1/2	1	
-5	0	0	1/2	7	0	1/2	1/2	1	
-5	0	1/2	0	7	0	1/2	1/2	1	
-5	0	0	0	7	1	0	1/2	1	
-5	0	0	0	7	1	1/2	0	1	
-5	0	0	1/2	7	1	0	0	1	
-5	0	0	1/2	7	1	1/2	0	1	
-5	0	1/2	0	7	1	0	0	1	
-5	0	1/2	0	7	1	0	1/2	1	
-5	1	1/2	1/2	7	1	0	0	1	
-5	1	1/2	1/2	7	1	0	1/2	1	
-5	1	1/2	1/2	7	1	1/2	0	1	
-6	0	0	1/2	7	0	1/2	1/2	1	
-6	0	1/2	0	7	0	1/2	1/2	1	
-6	0	1/2	1/2	7	0	1/2	1/2	1	
-6	0	0	1/2	7	1	1/2	0	1	
-6	0	1/2	0	7	1	0	1/2	1	
-6	0	1/2	1/2	7	1	0	0	1	
-6	1	0	1/2	7	1	0	0	1	
-6	1	0	1/2	7	1	0	1/2	1	
-6	1	1/2	0	7	1	0	0	1	
-6	1	1/2	0	7	1	1/2	0	1	
-6	1	1/2	1/2	7	1	0	1/2	1	
-6	1	1/2	1/2	7	1	1/2	0	1	
-7	0	1/2	1/2	7	0	1/2	1/2	1	

L-sector				R-sector				H-sector	
M_L	x_L	$\alpha_{1,L}$	$\alpha_{\tau,L}$	M_R	x_R	$\alpha_{1,R}$	$\alpha_{\tau,R}$	# of Higgs	
-7	1	0	0	7	1	0	0		1
-7	1	0	1/2	7	1	0	1/2		1
-7	1	1/2	0	7	1	1/2	0		1
-4	0	0	0	8	1	0	0		1
-5	0	0	0	8	1	0	0		1
-5	0	0	1/2	8	1	0	0		1
-5	0	1/2	0	8	1	0	0		1
-5	1	1/2	1/2	8	1	0	0		1
-6	0	0	1/2	8	1	0	0		1
-6	0	1/2	0	8	1	0	0		1
-6	0	1/2	1/2	8	1	0	0		1
-6	1	0	1/2	8	1	0	0		1
-6	1	1/2	0	8	1	0	0		1
-6	1	1/2	1/2	8	1	0	0		1
-7	0	1/2	1/2	8	1	0	0		1
-7	1	0	0	8	1	0	0		1
-7	1	0	1/2	8	1	0	0		1
-7	1	1/2	0	8	1	0	0		1
-8	1	0	0	8	1	0	0		1

D.2 $\mathbb{T}^2/\mathbb{Z}_3$ Table D.3: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_3$ with $M_L < 0$, $M_R < 0$.

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-6	0	0	-6	0	0	5
-6	0	0	-7	0	1/6	5
-6	0	0	-7	0	5/6	5
-6	0	0	-7	1	1/2	4
-7	0	1/6	-7	0	1/6	5
-7	0	1/6	-7	0	5/6	5
-7	0	5/6	-7	0	5/6	5
-7	0	1/6	-7	1	1/2	4
-7	0	5/6	-7	1	1/2	4
-7	1	1/2	-7	1	1/2	4
-6	0	0	-8	0	0	5
-6	0	0	-8	0	1/3	5
-6	0	0	-8	0	2/3	5
-6	0	0	-8	1	1/3	4
-6	0	0	-8	1	2/3	4
-6	0	0	-8	2	0	4
-7	0	1/6	-8	0	0	5
-7	0	1/6	-8	0	1/3	6
-7	0	1/6	-8	0	2/3	5
-7	0	5/6	-8	0	0	5
-7	0	5/6	-8	0	1/3	5
-7	0	5/6	-8	0	2/3	6
-7	0	1/6	-8	1	1/3	4
-7	0	1/6	-8	1	2/3	5
-7	0	5/6	-8	1	1/3	5
-7	0	5/6	-8	1	2/3	4
-7	0	1/6	-8	2	0	5
-7	0	5/6	-8	2	0	5
-7	1	1/2	-8	0	0	4
-7	1	1/2	-8	0	1/3	5
-7	1	1/2	-8	0	2/3	5
-7	1	1/2	-8	1	1/3	5
-7	1	1/2	-8	1	2/3	5
-7	1	1/2	-8	2	0	6
-8	0	0	-8	0	0	5
-8	0	0	-8	0	1/3	6
-8	0	0	-8	0	2/3	6
-8	0	1/3	-8	0	1/3	6
-8	0	1/3	-8	0	2/3	5
-8	0	2/3	-8	0	2/3	6
-8	0	0	-8	1	1/3	5
-8	0	0	-8	1	2/3	5
-8	0	1/3	-8	1	1/3	5
-8	0	1/3	-8	1	2/3	5
-8	0	2/3	-8	1	1/3	5
-8	0	2/3	-8	1	2/3	5
-8	0	0	-8	2	0	6
-8	0	1/3	-8	2	0	5
-8	0	2/3	-8	2	0	5
-8	1	1/3	-8	1	1/3	5
-8	1	1/3	-8	1	2/3	6
-8	1	2/3	-8	1	2/3	5
-8	1	1/3	-8	2	0	6
-8	1	2/3	-8	2	0	6
-8	2	0	-8	2	0	5
-6	0	0	-9	0	1/6	5
-6	0	0	-9	0	5/6	5
-6	0	0	-9	1	1/6	5
-6	0	0	-9	1	1/2	4

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-6	0	0	-9	1	5/6	5
-6	0	0	-9	2	1/6	5
-6	0	0	-9	2	5/6	5
-7	0	1/6	-9	0	1/6	6
-7	0	1/6	-9	0	5/6	5
-7	0	5/6	-9	0	1/6	5
-7	0	5/6	-9	0	5/6	6
-7	0	1/6	-9	1	1/6	5
-7	0	1/6	-9	1	1/2	5
-7	0	1/6	-9	1	5/6	5
-7	0	5/6	-9	1	1/6	5
-7	0	5/6	-9	1	1/2	5
-7	0	5/6	-9	1	5/6	5
-7	0	1/6	-9	2	1/6	5
-7	0	1/6	-9	2	5/6	6
-7	0	5/6	-9	2	1/6	6
-7	0	5/6	-9	2	5/6	5
-7	1	1/2	-9	0	1/6	5
-7	1	1/2	-9	0	5/6	5
-7	1	1/2	-9	1	1/6	5
-7	1	1/2	-9	1	1/2	6
-7	1	1/2	-9	1	5/6	5
-7	1	1/2	-9	2	1/6	6
-7	1	1/2	-9	2	5/6	6
-8	0	0	-9	0	1/6	6
-8	0	0	-9	0	5/6	6
-8	0	1/3	-9	0	1/6	6
-8	0	1/3	-9	0	5/6	6
-8	0	2/3	-9	0	1/6	6
-8	0	2/3	-9	0	5/6	6
-8	0	0	-9	1	1/6	5
-8	0	0	-9	1	1/2	6
-8	0	0	-9	1	5/6	5
-8	0	1/3	-9	1	1/6	6
-8	0	1/3	-9	1	1/2	5
-8	0	1/3	-9	1	5/6	5
-8	0	2/3	-9	1	1/6	5
-8	0	2/3	-9	1	1/2	5
-8	0	2/3	-9	1	5/6	6
-8	0	0	-9	2	1/6	6
-8	0	0	-9	2	5/6	6
-8	0	1/3	-9	2	1/6	5
-8	0	1/3	-9	2	5/6	6
-8	0	2/3	-9	2	1/6	6
-8	0	2/3	-9	2	5/6	5
-8	1	1/3	-9	0	1/6	6
-8	1	1/3	-9	0	5/6	5
-8	1	2/3	-9	0	1/6	5
-8	1	2/3	-9	0	5/6	6
-8	1	1/3	-9	1	1/6	5
-8	1	1/3	-9	1	1/2	6
-8	1	1/3	-9	1	5/6	6
-8	1	2/3	-9	1	1/6	6
-8	1	2/3	-9	1	1/2	6
-8	1	2/3	-9	1	5/6	5
-8	1	1/3	-9	2	1/6	6
-8	1	1/3	-9	2	5/6	6
-8	1	2/3	-9	2	1/6	6
-8	1	2/3	-9	2	5/6	6
-8	2	0	-9	0	1/6	6
-8	2	0	-9	0	5/6	6
-8	2	0	-9	1	1/6	6
-8	2	0	-9	1	1/2	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-8	2	0	-9	1	5/6	6
-8	2	0	-9	2	1/6	5
-8	2	0	-9	2	5/6	5
-9	0	1/6	-9	0	1/6	6
-9	0	1/6	-9	0	5/6	7
-9	0	5/6	-9	0	5/6	6
-9	0	1/6	-9	1	1/6	6
-9	0	1/6	-9	1	1/2	6
-9	0	1/6	-9	1	5/6	5
-9	0	5/6	-9	1	1/6	5
-9	0	5/6	-9	1	1/2	6
-9	0	5/6	-9	1	5/6	6
-9	0	1/6	-9	2	1/6	6
-9	0	1/6	-9	2	5/6	6
-9	0	5/6	-9	2	1/6	6
-9	0	5/6	-9	2	5/6	6
-9	1	1/6	-9	1	1/6	6
-9	1	1/6	-9	1	1/2	6
-9	1	1/6	-9	1	5/6	6
-9	1	1/2	-9	1	1/2	6
-9	1	1/2	-9	1	5/6	6
-9	1	5/6	-9	1	5/6	6
-9	1	1/6	-9	2	1/6	6
-9	1	1/6	-9	2	5/6	7
-9	1	1/2	-9	2	1/6	6
-9	1	1/2	-9	2	5/6	6
-9	1	5/6	-9	2	1/6	7
-9	1	5/6	-9	2	5/6	6
-9	2	1/6	-9	2	1/6	6
-9	2	1/6	-9	2	5/6	5
-9	2	5/6	-9	2	5/6	6
-6	0	0	-10	0	0	5
-6	0	0	-10	1	1/3	5
-6	0	0	-10	1	2/3	5
-6	0	0	-10	2	0	6
-6	0	0	-10	2	1/3	5
-6	0	0	-10	2	2/3	5
-7	0	1/6	-10	0	0	6
-7	0	5/6	-10	0	0	6
-7	0	1/6	-10	1	1/3	6
-7	0	1/6	-10	1	2/3	5
-7	0	5/6	-10	1	1/3	5
-7	0	5/6	-10	1	2/3	6
-7	0	1/6	-10	2	0	6
-7	0	1/6	-10	2	1/3	5
-7	0	1/6	-10	2	2/3	6
-7	0	5/6	-10	2	0	6
-7	0	5/6	-10	2	1/3	6
-7	0	5/6	-10	2	2/3	5
-7	1	1/2	-10	0	0	6
-7	1	1/2	-10	1	1/3	6
-7	1	1/2	-10	1	2/3	6
-7	1	1/2	-10	2	0	6
-7	1	1/2	-10	2	1/3	6
-7	1	1/2	-10	2	2/3	6
-8	0	0	-10	0	0	7
-8	0	1/3	-10	0	0	6
-8	0	2/3	-10	0	0	6
-8	0	0	-10	1	1/3	6
-8	0	0	-10	1	2/3	6
-8	0	1/3	-10	1	1/3	6
-8	0	1/3	-10	1	2/3	5
-8	0	2/3	-10	1	1/3	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-8	0	2/3	-10	1	2/3	6
-8	0	0	-10	2	0	6
-8	0	0	-10	2	1/3	6
-8	0	0	-10	2	2/3	6
-8	0	1/3	-10	2	0	6
-8	0	1/3	-10	2	1/3	6
-8	0	1/3	-10	2	2/3	6
-8	0	2/3	-10	2	0	6
-8	0	2/3	-10	2	1/3	6
-8	0	2/3	-10	2	2/3	6
-8	1	1/3	-10	0	0	6
-8	1	2/3	-10	0	0	6
-8	1	1/3	-10	1	1/3	6
-8	1	1/3	-10	1	2/3	6
-8	1	2/3	-10	1	1/3	6
-8	1	2/3	-10	1	2/3	6
-8	1	1/3	-10	2	0	6
-8	1	1/3	-10	2	1/3	6
-8	1	1/3	-10	2	2/3	7
-8	1	2/3	-10	2	0	6
-8	1	2/3	-10	2	1/3	7
-8	1	2/3	-10	2	2/3	6
-8	2	0	-10	0	0	6
-8	2	0	-10	1	1/3	6
-8	2	0	-10	1	2/3	6
-8	2	0	-10	2	0	5
-8	2	0	-10	2	1/3	6
-8	2	0	-10	2	2/3	6
-9	0	1/6	-10	0	0	7
-9	0	5/6	-10	0	0	7
-9	0	1/6	-10	1	1/3	6
-9	0	1/6	-10	1	2/3	6
-9	0	5/6	-10	1	1/3	6
-9	0	5/6	-10	1	2/3	6
-9	0	1/6	-10	2	0	6
-9	0	1/6	-10	2	1/3	7
-9	0	1/6	-10	2	2/3	6
-9	0	5/6	-10	2	0	6
-9	0	5/6	-10	2	1/3	6
-9	0	5/6	-10	2	2/3	7
-9	1	1/6	-10	0	0	6
-9	1	1/2	-10	0	0	6
-9	1	5/6	-10	0	0	6
-9	1	1/6	-10	1	1/3	7
-9	1	1/6	-10	1	2/3	6
-9	1	1/2	-10	1	1/3	6
-9	1	1/2	-10	1	2/3	6
-9	1	5/6	-10	1	1/3	6
-9	1	5/6	-10	1	2/3	7
-9	1	1/6	-10	2	0	7
-9	1	1/6	-10	2	1/3	6
-9	1	1/6	-10	2	2/3	7
-9	1	1/2	-10	2	0	6
-9	1	1/2	-10	2	1/3	7
-9	1	1/2	-10	2	2/3	7
-9	1	5/6	-10	2	0	7
-9	1	5/6	-10	2	1/3	7
-9	1	5/6	-10	2	2/3	6
-9	2	1/6	-10	0	0	6
-9	2	5/6	-10	0	0	6
-9	2	1/6	-10	1	1/3	6
-9	2	1/6	-10	1	2/3	7
-9	2	5/6	-10	1	1/3	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-9	2	5/6	-10	1	2/3	6
-9	2	1/6	-10	2	0	6
-9	2	1/6	-10	2	1/3	6
-9	2	1/6	-10	2	2/3	6
-9	2	5/6	-10	2	0	6
-9	2	5/6	-10	2	1/3	6
-9	2	5/6	-10	2	2/3	6
-10	0	0	-10	0	0	7
-10	0	0	-10	1	1/3	6
-10	0	0	-10	1	2/3	6
-10	0	0	-10	2	0	6
-10	0	0	-10	2	1/3	7
-10	0	0	-10	2	2/3	7
-10	1	1/3	-10	1	1/3	7
-10	1	1/3	-10	1	2/3	6
-10	1	2/3	-10	1	2/3	7
-10	1	1/3	-10	2	0	7
-10	1	1/3	-10	2	1/3	7
-10	1	1/3	-10	2	2/3	7
-10	1	2/3	-10	2	0	7
-10	1	2/3	-10	2	1/3	7
-10	1	2/3	-10	2	2/3	7
-10	2	0	-10	2	0	7
-10	2	0	-10	2	1/3	6
-10	2	0	-10	2	2/3	6
-10	2	1/3	-10	2	1/3	6
-10	2	1/3	-10	2	2/3	7
-10	2	2/3	-10	2	2/3	6
-6	0	0	-11	1	1/2	6
-6	0	0	-11	2	1/6	6
-6	0	0	-11	2	5/6	6
-7	0	1/6	-11	1	1/2	6
-7	0	5/6	-11	1	1/2	6
-7	0	1/6	-11	2	1/6	6
-7	0	1/6	-11	2	5/6	6
-7	0	5/6	-11	2	1/6	6
-7	0	5/6	-11	2	5/6	6
-7	1	1/2	-11	1	1/2	6
-7	1	1/2	-11	2	1/6	6
-7	1	1/2	-11	2	5/6	6
-8	0	0	-11	1	1/2	6
-8	0	1/3	-11	1	1/2	6
-8	0	2/3	-11	1	1/2	6
-8	0	0	-11	2	1/6	6
-8	0	0	-11	2	5/6	6
-8	0	1/3	-11	2	1/6	7
-8	0	1/3	-11	2	5/6	6
-8	0	2/3	-11	2	1/6	6
-8	0	2/3	-11	2	5/6	7
-8	1	1/3	-11	1	1/2	6
-8	1	2/3	-11	1	1/2	6
-8	1	1/3	-11	2	1/6	6
-8	1	1/3	-11	2	5/6	7
-8	1	2/3	-11	2	1/6	7
-8	1	2/3	-11	2	5/6	6
-8	2	0	-11	1	1/2	6
-8	2	0	-11	2	1/6	6
-8	2	0	-11	2	5/6	6
-9	0	1/6	-11	1	1/2	6
-9	0	5/6	-11	1	1/2	6
-9	0	1/6	-11	2	1/6	7
-9	0	1/6	-11	2	5/6	6
-9	0	5/6	-11	2	1/6	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-9	0	5/6	-11	2	5/6	7
-9	1	1/6	-11	1	1/2	7
-9	1	1/2	-11	1	1/2	6
-9	1	5/6	-11	1	1/2	7
-9	1	1/6	-11	2	1/6	7
-9	1	1/6	-11	2	5/6	7
-9	1	1/2	-11	2	1/6	7
-9	1	1/2	-11	2	5/6	7
-9	1	5/6	-11	2	1/6	7
-9	1	5/6	-11	2	5/6	7
-9	2	1/6	-11	1	1/2	7
-9	2	5/6	-11	1	1/2	7
-9	2	1/6	-11	2	1/6	6
-9	2	1/6	-11	2	5/6	7
-9	2	5/6	-11	2	1/6	7
-9	2	5/6	-11	2	5/6	6
-10	0	0	-11	1	1/2	6
-10	0	0	-11	2	1/6	7
-10	0	0	-11	2	5/6	7
-10	1	1/3	-11	1	1/2	7
-10	1	2/3	-11	1	1/2	7
-10	1	1/3	-11	2	1/6	8
-10	1	1/3	-11	2	5/6	7
-10	1	2/3	-11	2	1/6	7
-10	1	2/3	-11	2	5/6	8
-10	2	0	-11	1	1/2	8
-10	2	1/3	-11	1	1/2	7
-10	2	2/3	-11	1	1/2	7
-10	2	0	-11	2	1/6	7
-10	2	0	-11	2	5/6	7
-10	2	1/3	-11	2	1/6	6
-10	2	1/3	-11	2	5/6	7
-10	2	2/3	-11	2	1/6	7
-10	2	2/3	-11	2	5/6	6
-11	1	1/2	-11	1	1/2	8
-11	1	1/2	-11	2	1/6	8
-11	1	1/2	-11	2	5/6	8
-11	2	1/6	-11	2	1/6	7
-11	2	1/6	-11	2	5/6	7
-11	2	5/6	-11	2	5/6	7
-6	0	0	-12	2	0	6
-7	0	1/6	-12	2	0	6
-7	0	5/6	-12	2	0	6
-7	1	1/2	-12	2	0	6
-8	0	0	-12	2	0	6
-8	0	1/3	-12	2	0	7
-8	0	2/3	-12	2	0	7
-8	1	1/3	-12	2	0	7
-8	1	2/3	-12	2	0	7
-8	2	0	-12	2	0	7
-9	0	1/6	-12	2	0	7
-9	0	5/6	-12	2	0	7
-9	1	1/6	-12	2	0	7
-9	1	1/2	-12	2	0	8
-9	1	5/6	-12	2	0	7
-9	2	1/6	-12	2	0	7
-9	2	5/6	-12	2	0	7
-10	0	0	-12	2	0	8
-10	1	1/3	-12	2	0	8
-10	1	2/3	-12	2	0	8
-10	2	0	-12	2	0	7
-10	2	1/3	-12	2	0	7
-10	2	2/3	-12	2	0	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-11	1	1/2	-12	2	0	8
-11	2	1/6	-12	2	0	7
-11	2	5/6	-12	2	0	7
-12	2	0	-12	2	0	7

Table D.4: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_3$ with $M_L < 0$, $M_R > 0$.

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-6	0	0	6	0	0	1	
-6	0	0	7	1/6	0	1	
-6	0	0	7	5/6	0	1	
-7	1/6	0	7	5/6	0	1	
-7	5/6	0	7	1/6	0	1	
-6	0	0	8	0	0	1	
-6	0	0	8	1/3	0	1	
-6	0	0	8	2/3	0	1	
-7	1/6	0	8	0	0	1	
-7	1/6	0	8	2/3	0	1	
-7	5/6	0	8	0	0	1	
-7	5/6	0	8	1/3	0	1	
-8	0	0	8	0	0	1	
-8	1/3	0	8	2/3	0	1	
-8	2/3	0	8	1/3	0	1	
-6	0	0	9	1/6	0	1	
-6	0	0	9	5/6	0	1	
-6	0	0	9	1/6	1	1	
-6	0	0	9	5/6	1	1	
-6	0	0	9	1/6	2	1	
-6	0	0	9	5/6	2	1	
-7	1/6	0	9	1/6	0	1	
-7	1/6	0	9	5/6	0	1	
-7	5/6	0	9	1/6	0	1	
-7	5/6	0	9	5/6	0	1	
-7	1/6	0	9	5/6	1	1	
-7	5/6	0	9	1/6	1	1	
-7	1/6	0	9	1/6	2	1	
-7	5/6	0	9	5/6	2	1	
-7	1/2	1	9	1/6	1	1	
-7	1/2	1	9	5/6	1	1	
-7	1/2	1	9	1/6	2	1	
-7	1/2	1	9	5/6	2	1	
-8	0	0	9	1/6	0	1	
-8	0	0	9	5/6	0	1	
-8	1/3	0	9	5/6	0	1	
-8	2/3	0	9	1/6	0	1	
-8	1/3	0	9	1/6	2	1	
-8	2/3	0	9	5/6	2	1	
-8	1/3	1	9	1/6	1	1	
-8	2/3	1	9	5/6	1	1	
-8	1/3	1	9	5/6	2	1	
-8	2/3	1	9	1/6	2	1	
-8	0	2	9	1/6	1	1	
-8	0	2	9	5/6	1	1	
-9	1/6	0	9	5/6	0	1	
-9	5/6	0	9	1/6	0	1	
-9	1/6	1	9	5/6	2	1	
-9	5/6	1	9	1/6	2	1	
-9	1/6	2	9	5/6	1	1	
-9	5/6	2	9	1/6	1	1	
-6	0	0	10	0	0	1	
-6	0	0	10	1/3	1	1	
-6	0	0	10	2/3	1	1	
-6	0	0	10	0	2	2	
-6	0	0	10	1/3	2	1	
-6	0	0	10	2/3	2	1	
-7	1/6	0	10	0	0	1	
-7	5/6	0	10	0	0	1	
-7	1/6	0	10	2/3	1	1	
-7	5/6	0	10	1/3	1	1	
-7	1/6	0	10	0	2	1	

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-7	1/6	0	10	1/3	2	1	
-7	1/6	0	10	2/3	2	1	
-7	5/6	0	10	0	2	1	
-7	5/6	0	10	1/3	2	1	
-7	5/6	0	10	2/3	2	1	
-7	1/2	1	10	1/3	1	1	
-7	1/2	1	10	2/3	1	1	
-7	1/2	1	10	0	2	2	
-7	1/2	1	10	1/3	2	1	
-7	1/2	1	10	2/3	2	1	
-8	0	0	10	0	0	1	
-8	1/3	0	10	0	0	1	
-8	2/3	0	10	0	0	1	
-8	1/3	0	10	2/3	1	1	
-8	2/3	0	10	1/3	1	1	
-8	0	0	10	1/3	2	1	
-8	0	0	10	2/3	2	1	
-8	1/3	0	10	0	2	1	
-8	1/3	0	10	1/3	2	1	
-8	2/3	0	10	0	2	1	
-8	2/3	0	10	2/3	2	1	
-8	1/3	1	10	1/3	1	1	
-8	2/3	1	10	2/3	1	1	
-8	1/3	1	10	0	2	1	
-8	1/3	1	10	1/3	2	1	
-8	1/3	1	10	2/3	2	1	
-8	2/3	1	10	0	2	1	
-8	2/3	1	10	1/3	2	1	
-8	2/3	1	10	2/3	2	1	
-8	0	2	10	1/3	1	1	
-8	0	2	10	2/3	1	1	
-8	0	2	10	0	2	1	
-9	1/6	0	10	0	0	1	
-9	5/6	0	10	0	0	1	
-9	1/6	0	10	1/3	2	1	
-9	5/6	0	10	2/3	2	1	
-9	1/6	1	10	1/3	1	1	
-9	5/6	1	10	2/3	1	1	
-9	1/6	1	10	0	2	1	
-9	1/6	1	10	2/3	2	1	
-9	1/2	1	10	1/3	2	1	
-9	1/2	1	10	2/3	2	1	
-9	5/6	1	10	0	2	1	
-9	5/6	1	10	1/3	2	1	
-9	1/6	2	10	2/3	1	1	
-9	5/6	2	10	1/3	1	1	
-10	0	0	10	0	0	1	
-10	1/3	1	10	2/3	2	1	
-10	2/3	1	10	1/3	2	1	
-10	1/3	2	10	2/3	1	1	
-10	2/3	2	10	1/3	1	1	
-6	0	0	11	1/2	1	2	
-6	0	0	11	1/6	2	2	
-6	0	0	11	5/6	2	2	
-7	1/6	0	11	1/2	1	1	
-7	5/6	0	11	1/2	1	1	
-7	1/6	0	11	1/6	2	1	
-7	1/6	0	11	5/6	2	2	
-7	5/6	0	11	1/6	2	2	
-7	5/6	0	11	5/6	2	1	
-7	1/2	1	11	1/2	1	2	
-7	1/2	1	11	1/6	2	2	
-7	1/2	1	11	5/6	2	2	

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-8	1/3	0	11	1/2	1	1
-8	2/3	0	11	1/2	1	1
-8	0	0	11	1/6	2	1
-8	0	0	11	5/6	2	1
-8	1/3	0	11	1/6	2	1
-8	1/3	0	11	5/6	2	1
-8	2/3	0	11	1/6	2	1
-8	2/3	0	11	5/6	2	1
-8	1/3	1	11	1/2	1	1
-8	2/3	1	11	1/2	1	1
-8	1/3	1	11	1/6	2	2
-8	1/3	1	11	5/6	2	1
-8	2/3	1	11	1/6	2	1
-8	2/3	1	11	5/6	2	2
-8	0	2	11	1/2	1	2
-8	0	2	11	1/6	2	1
-8	0	2	11	5/6	2	1
-9	1/6	0	11	1/6	2	1
-9	5/6	0	11	5/6	2	1
-9	1/6	1	11	1/2	1	1
-9	5/6	1	11	1/2	1	1
-9	1/6	1	11	1/6	2	1
-9	1/6	1	11	5/6	2	1
-9	1/2	1	11	1/6	2	1
-9	1/2	1	11	5/6	2	1
-9	5/6	1	11	1/6	2	1
-9	5/6	1	11	5/6	2	1
-9	1/6	2	11	1/2	1	1
-9	5/6	2	11	1/2	1	1
-9	1/6	2	11	5/6	2	1
-9	5/6	2	11	1/6	2	1
-10	1/3	1	11	5/6	2	1
-10	2/3	1	11	1/6	2	1
-10	1/3	2	11	1/2	1	1
-10	2/3	2	11	1/2	1	1
-6	0	0	12	0	2	2
-7	1/6	0	12	0	2	2
-7	5/6	0	12	0	2	2
-7	1/2	1	12	0	2	2
-8	0	0	12	0	2	2
-8	1/3	0	12	0	2	1
-8	2/3	0	12	0	2	1
-8	1/3	1	12	0	2	2
-8	2/3	1	12	0	2	2
-8	0	2	12	0	2	1
-9	1/6	0	12	0	2	1
-9	5/6	0	12	0	2	1
-9	1/6	1	12	0	2	1
-9	1/2	1	12	0	2	2
-9	5/6	1	12	0	2	1
-9	1/6	2	12	0	2	1
-9	5/6	2	12	0	2	1
-10	1/3	1	12	0	2	1
-10	2/3	1	12	0	2	1
-10	0	2	12	0	2	1

D.3 $\mathbb{T}^2/\mathbb{Z}_4$ Table D.5: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_4$ with $M_L < 0$, $M_R < 0$.

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-8	0	0	-8	0	0	5
-8	0	0	-9	0	0	5
-8	0	0	-9	1	1/2	4
-9	0	0	-9	0	0	5
-9	0	0	-9	1	1/2	4
-9	1	1/2	-9	1	1/2	5
-8	0	0	-10	0	0	5
-8	0	0	-10	0	1/2	5
-8	0	0	-10	1	1/2	4
-8	0	0	-10	2	0	5
-9	0	0	-10	0	0	5
-9	0	0	-10	0	1/2	5
-9	0	0	-10	1	1/2	5
-9	0	0	-10	2	0	5
-9	1	1/2	-10	0	0	5
-9	1	1/2	-10	0	1/2	4
-9	1	1/2	-10	1	1/2	5
-9	1	1/2	-10	2	0	5
-10	0	0	-10	0	0	6
-10	0	0	-10	0	1/2	5
-10	0	1/2	-10	0	1/2	6
-10	0	0	-10	1	1/2	5
-10	0	1/2	-10	1	1/2	4
-10	0	0	-10	2	0	5
-10	0	1/2	-10	2	0	5
-10	1	1/2	-10	1	1/2	5
-10	1	1/2	-10	2	0	5
-10	2	0	-10	2	0	6
-8	0	0	-11	0	0	5
-8	0	0	-11	0	1/2	5
-8	0	0	-11	1	0	4
-8	0	0	-11	1	1/2	5
-8	0	0	-11	2	0	5
-8	0	0	-11	3	1/2	5
-9	0	0	-11	0	0	6
-9	0	0	-11	0	1/2	5
-9	0	0	-11	1	0	4
-9	0	0	-11	1	1/2	5
-9	0	0	-11	2	0	5
-9	0	0	-11	3	1/2	5
-9	1	1/2	-11	0	0	5
-9	1	1/2	-11	0	1/2	4
-9	1	1/2	-11	1	0	5
-9	1	1/2	-11	1	1/2	5
-9	1	1/2	-11	2	0	5
-9	1	1/2	-11	3	1/2	6
-10	0	0	-11	0	0	6
-10	0	0	-11	0	1/2	5
-10	0	1/2	-11	0	0	5
-10	0	1/2	-11	0	1/2	6
-10	0	0	-11	1	0	5
-10	0	0	-11	1	1/2	5
-10	0	1/2	-11	1	0	5
-10	0	1/2	-11	1	1/2	5
-10	0	0	-11	2	0	5
-10	0	1/2	-11	2	0	5
-10	0	0	-11	3	1/2	6
-10	0	1/2	-11	3	1/2	5
-10	1	1/2	-11	0	0	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-10	1	1/2	-11	0	1/2	5
-10	1	1/2	-11	1	0	5
-10	1	1/2	-11	1	1/2	5
-10	1	1/2	-11	2	0	6
-10	1	1/2	-11	3	1/2	6
-10	2	0	-11	0	0	5
-10	2	0	-11	0	1/2	5
-10	2	0	-11	1	0	5
-10	2	0	-11	1	1/2	6
-10	2	0	-11	2	0	6
-10	2	0	-11	3	1/2	5
-11	0	0	-11	0	0	6
-11	0	0	-11	0	1/2	6
-11	0	1/2	-11	0	1/2	6
-11	0	0	-11	1	0	5
-11	0	0	-11	1	1/2	5
-11	0	1/2	-11	1	0	5
-11	0	1/2	-11	1	1/2	5
-11	0	0	-11	2	0	6
-11	0	1/2	-11	2	0	5
-11	0	0	-11	3	1/2	6
-11	0	1/2	-11	3	1/2	5
-11	1	0	-11	1	0	6
-11	1	0	-11	1	1/2	5
-11	1	1/2	-11	1	1/2	6
-11	1	0	-11	2	0	5
-11	1	1/2	-11	2	0	6
-11	1	0	-11	3	1/2	6
-11	1	1/2	-11	3	1/2	6
-11	2	0	-11	2	0	6
-11	2	0	-11	3	1/2	5
-11	3	1/2	-11	3	1/2	6
-8	0	0	-12	0	1/2	5
-8	0	0	-12	1	0	4
-8	0	0	-12	1	1/2	5
-8	0	0	-12	2	0	5
-8	0	0	-12	2	1/2	5
-8	0	0	-12	3	1/2	5
-9	0	0	-12	0	1/2	5
-9	0	0	-12	1	0	5
-9	0	0	-12	1	1/2	5
-9	0	0	-12	2	0	5
-9	0	0	-12	2	1/2	5
-9	0	0	-12	3	1/2	6
-9	1	1/2	-12	0	1/2	5
-9	1	1/2	-12	1	0	5
-9	1	1/2	-12	1	1/2	5
-9	1	1/2	-12	2	0	6
-9	1	1/2	-12	2	1/2	5
-9	1	1/2	-12	3	1/2	6
-10	0	0	-12	0	1/2	6
-10	0	1/2	-12	0	1/2	6
-10	0	0	-12	1	0	5
-10	0	0	-12	1	1/2	5
-10	0	1/2	-12	1	0	5
-10	0	1/2	-12	1	1/2	5
-10	0	0	-12	2	0	6
-10	0	0	-12	2	1/2	5
-10	0	1/2	-12	2	0	5
-10	0	1/2	-12	2	1/2	6
-10	0	0	-12	3	1/2	6
-10	0	1/2	-12	3	1/2	5
-10	1	1/2	-12	0	1/2	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-10	1	1/2	-12	1	0	5
-10	1	1/2	-12	1	1/2	6
-10	1	1/2	-12	2	0	6
-10	1	1/2	-12	2	1/2	5
-10	1	1/2	-12	3	1/2	6
-10	2	0	-12	0	1/2	5
-10	2	0	-12	1	0	5
-10	2	0	-12	1	1/2	6
-10	2	0	-12	2	0	6
-10	2	0	-12	2	1/2	6
-10	2	0	-12	3	1/2	5
-11	0	0	-12	0	1/2	6
-11	0	1/2	-12	0	1/2	6
-11	0	0	-12	1	0	5
-11	0	0	-12	1	1/2	6
-11	0	1/2	-12	1	0	6
-11	0	1/2	-12	1	1/2	5
-11	0	0	-12	2	0	6
-11	0	0	-12	2	1/2	5
-11	0	1/2	-12	2	0	5
-11	0	1/2	-12	2	1/2	6
-11	0	0	-12	3	1/2	6
-11	0	1/2	-12	3	1/2	6
-11	1	0	-12	0	1/2	6
-11	1	1/2	-12	0	1/2	5
-11	1	0	-12	1	0	6
-11	1	0	-12	1	1/2	5
-11	1	1/2	-12	1	0	5
-11	1	1/2	-12	1	1/2	6
-11	1	0	-12	2	0	6
-11	1	0	-12	2	1/2	6
-11	1	1/2	-12	2	0	6
-11	1	1/2	-12	2	1/2	6
-11	1	0	-12	3	1/2	6
-11	1	1/2	-12	3	1/2	6
-11	2	0	-12	0	1/2	5
-11	2	0	-12	1	0	6
-11	2	0	-12	1	1/2	6
-11	2	0	-12	2	0	6
-11	2	0	-12	2	1/2	6
-11	2	0	-12	3	1/2	6
-11	3	1/2	-12	0	1/2	6
-11	3	1/2	-12	1	0	6
-11	3	1/2	-12	1	1/2	6
-11	3	1/2	-12	2	0	6
-11	3	1/2	-12	2	1/2	5
-11	3	1/2	-12	3	1/2	6
-12	0	1/2	-12	0	1/2	7
-12	0	1/2	-12	1	0	6
-12	0	1/2	-12	1	1/2	5
-12	0	1/2	-12	2	0	6
-12	0	1/2	-12	2	1/2	6
-12	0	1/2	-12	3	1/2	6
-12	1	0	-12	1	0	6
-12	1	0	-12	1	1/2	6
-12	1	1/2	-12	1	1/2	6
-12	1	0	-12	2	0	6
-12	1	0	-12	2	1/2	6
-12	1	1/2	-12	2	0	6
-12	1	1/2	-12	2	1/2	6
-12	1	0	-12	3	1/2	6
-12	1	1/2	-12	3	1/2	7
-12	2	0	-12	2	0	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-12	2	0	-12	2	1/2	6
-12	2	1/2	-12	2	1/2	7
-12	2	0	-12	3	1/2	6
-12	2	1/2	-12	3	1/2	5
-12	3	1/2	-12	3	1/2	6
-8	0	0	-13	0	1/2	5
-8	0	0	-13	1	0	5
-8	0	0	-13	2	0	5
-8	0	0	-13	2	1/2	5
-8	0	0	-13	3	0	5
-8	0	0	-13	3	1/2	6
-9	0	0	-13	0	1/2	6
-9	0	0	-13	1	0	5
-9	0	0	-13	2	0	6
-9	0	0	-13	2	1/2	5
-9	0	0	-13	3	0	5
-9	0	0	-13	3	1/2	6
-9	1	1/2	-13	0	1/2	5
-9	1	1/2	-13	1	0	5
-9	1	1/2	-13	2	0	6
-9	1	1/2	-13	2	1/2	5
-9	1	1/2	-13	3	0	6
-9	1	1/2	-13	3	1/2	6
-10	0	0	-13	0	1/2	6
-10	0	1/2	-13	0	1/2	6
-10	0	0	-13	1	0	5
-10	0	1/2	-13	1	0	6
-10	0	0	-13	2	0	6
-10	0	0	-13	2	1/2	5
-10	0	1/2	-13	2	0	5
-10	0	1/2	-13	2	1/2	6
-10	0	0	-13	3	0	6
-10	0	0	-13	3	1/2	6
-10	0	1/2	-13	3	0	6
-10	0	1/2	-13	3	1/2	6
-10	1	1/2	-13	0	1/2	5
-10	1	1/2	-13	1	0	5
-10	1	1/2	-13	2	0	6
-10	1	1/2	-13	2	1/2	6
-10	1	1/2	-13	3	0	6
-10	1	1/2	-13	3	1/2	6
-10	2	0	-13	0	1/2	5
-10	2	0	-13	1	0	6
-10	2	0	-13	2	0	6
-10	2	0	-13	2	1/2	6
-10	2	0	-13	3	0	5
-10	2	0	-13	3	1/2	6
-11	0	0	-13	0	1/2	6
-11	0	1/2	-13	0	1/2	7
-11	0	0	-13	1	0	5
-11	0	1/2	-13	1	0	6
-11	0	0	-13	2	0	6
-11	0	0	-13	2	1/2	6
-11	0	1/2	-13	2	0	6
-11	0	1/2	-13	2	1/2	6
-11	0	0	-13	3	0	6
-11	0	0	-13	3	1/2	6
-11	0	1/2	-13	3	0	6
-11	0	1/2	-13	3	1/2	6
-11	1	0	-13	0	1/2	6
-11	1	1/2	-13	0	1/2	5
-11	1	0	-13	1	0	6
-11	1	1/2	-13	1	0	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-11	1	0	-13	2	0	6
-11	1	0	-13	2	1/2	6
-11	1	1/2	-13	2	0	6
-11	1	1/2	-13	2	1/2	6
-11	1	0	-13	3	0	7
-11	1	0	-13	3	1/2	6
-11	1	1/2	-13	3	0	6
-11	1	1/2	-13	3	1/2	7
-11	2	0	-13	0	1/2	6
-11	2	0	-13	1	0	6
-11	2	0	-13	2	0	7
-11	2	0	-13	2	1/2	6
-11	2	0	-13	3	0	5
-11	2	0	-13	3	1/2	6
-11	3	1/2	-13	0	1/2	6
-11	3	1/2	-13	1	0	6
-11	3	1/2	-13	2	0	6
-11	3	1/2	-13	2	1/2	5
-11	3	1/2	-13	3	0	6
-11	3	1/2	-13	3	1/2	6
-12	0	1/2	-13	0	1/2	7
-12	0	1/2	-13	1	0	6
-12	0	1/2	-13	2	0	6
-12	0	1/2	-13	2	1/2	6
-12	0	1/2	-13	3	0	7
-12	0	1/2	-13	3	1/2	6
-12	1	0	-13	0	1/2	6
-12	1	1/2	-13	0	1/2	6
-12	1	0	-13	1	0	6
-12	1	1/2	-13	1	0	6
-12	1	0	-13	2	0	6
-12	1	0	-13	2	1/2	7
-12	1	1/2	-13	2	0	7
-12	1	1/2	-13	2	1/2	6
-12	1	0	-13	3	0	7
-12	1	0	-13	3	1/2	6
-12	1	1/2	-13	3	0	6
-12	1	1/2	-13	3	1/2	7
-12	2	0	-13	0	1/2	6
-12	2	1/2	-13	0	1/2	6
-12	2	0	-13	1	0	6
-12	2	1/2	-13	1	0	7
-12	2	0	-13	2	0	7
-12	2	0	-13	2	1/2	6
-12	2	1/2	-13	2	0	6
-12	2	1/2	-13	2	1/2	7
-12	2	0	-13	3	0	6
-12	2	0	-13	3	1/2	6
-12	2	1/2	-13	3	0	6
-12	2	1/2	-13	3	1/2	6
-12	3	1/2	-13	0	1/2	6
-12	3	1/2	-13	1	0	6
-12	3	1/2	-13	2	0	6
-12	3	1/2	-13	2	1/2	6
-12	3	1/2	-13	3	0	6
-12	3	1/2	-13	3	1/2	6
-13	0	1/2	-13	0	1/2	7
-13	0	1/2	-13	1	0	6
-13	0	1/2	-13	2	0	6
-13	0	1/2	-13	2	1/2	7
-13	0	1/2	-13	3	0	7
-13	0	1/2	-13	3	1/2	6
-13	1	0	-13	1	0	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-13	1	0	-13	2	0	6
-13	1	0	-13	2	1/2	7
-13	1	0	-13	3	0	7
-13	1	0	-13	3	1/2	7
-13	2	0	-13	2	0	7
-13	2	0	-13	2	1/2	7
-13	2	1/2	-13	2	1/2	7
-13	2	0	-13	3	0	6
-13	2	0	-13	3	1/2	6
-13	2	1/2	-13	3	0	6
-13	2	1/2	-13	3	1/2	6
-13	3	0	-13	3	0	7
-13	3	0	-13	3	1/2	6
-13	3	1/2	-13	3	1/2	7
-8	0	0	-14	1	0	5
-8	0	0	-14	2	1/2	5
-8	0	0	-14	3	0	5
-8	0	0	-14	3	1/2	6
-9	0	0	-14	1	0	5
-9	0	0	-14	2	1/2	5
-9	0	0	-14	3	0	6
-9	0	0	-14	3	1/2	6
-9	1	1/2	-14	1	0	5
-9	1	1/2	-14	2	1/2	6
-9	1	1/2	-14	3	0	6
-9	1	1/2	-14	3	1/2	6
-10	0	0	-14	1	0	5
-10	0	1/2	-14	1	0	6
-10	0	0	-14	2	1/2	6
-10	0	1/2	-14	2	1/2	6
-10	0	0	-14	3	0	6
-10	0	0	-14	3	1/2	6
-10	0	1/2	-14	3	0	6
-10	0	1/2	-14	3	1/2	6
-10	1	1/2	-14	1	0	6
-10	1	1/2	-14	2	1/2	6
-10	1	1/2	-14	3	0	6
-10	1	1/2	-14	3	1/2	7
-10	2	0	-14	1	0	6
-10	2	0	-14	2	1/2	6
-10	2	0	-14	3	0	5
-10	2	0	-14	3	1/2	6
-11	0	0	-14	1	0	6
-11	0	1/2	-14	1	0	6
-11	0	0	-14	2	1/2	6
-11	0	1/2	-14	2	1/2	6
-11	0	0	-14	3	0	6
-11	0	0	-14	3	1/2	7
-11	0	1/2	-14	3	0	7
-11	0	1/2	-14	3	1/2	6
-11	1	0	-14	1	0	6
-11	1	1/2	-14	1	0	6
-11	1	0	-14	2	1/2	7
-11	1	1/2	-14	2	1/2	6
-11	1	0	-14	3	0	7
-11	1	0	-14	3	1/2	6
-11	1	1/2	-14	3	0	6
-11	1	1/2	-14	3	1/2	7
-11	2	0	-14	1	0	6
-11	2	0	-14	2	1/2	6
-11	2	0	-14	3	0	6
-11	2	0	-14	3	1/2	6
-11	3	1/2	-14	1	0	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-11	3	1/2	-14	2	1/2	6
-11	3	1/2	-14	3	0	6
-11	3	1/2	-14	3	1/2	6
-12	0	1/2	-14	1	0	6
-12	0	1/2	-14	2	1/2	7
-12	0	1/2	-14	3	0	7
-12	0	1/2	-14	3	1/2	6
-12	1	0	-14	1	0	7
-12	1	1/2	-14	1	0	6
-12	1	0	-14	2	1/2	7
-12	1	1/2	-14	2	1/2	6
-12	1	0	-14	3	0	7
-12	1	0	-14	3	1/2	7
-12	1	1/2	-14	3	0	7
-12	1	1/2	-14	3	1/2	7
-12	2	0	-14	1	0	6
-12	2	1/2	-14	1	0	7
-12	2	0	-14	2	1/2	7
-12	2	1/2	-14	2	1/2	7
-12	2	0	-14	3	0	6
-12	2	0	-14	3	1/2	6
-12	2	1/2	-14	3	0	6
-12	2	1/2	-14	3	1/2	6
-12	3	1/2	-14	1	0	7
-12	3	1/2	-14	2	1/2	6
-12	3	1/2	-14	3	0	6
-12	3	1/2	-14	3	1/2	7
-13	0	1/2	-14	1	0	7
-13	0	1/2	-14	2	1/2	7
-13	0	1/2	-14	3	0	7
-13	0	1/2	-14	3	1/2	7
-13	1	0	-14	1	0	7
-13	1	0	-14	2	1/2	7
-13	1	0	-14	3	0	7
-13	1	0	-14	3	1/2	7
-13	2	0	-14	1	0	7
-13	2	1/2	-14	1	0	7
-13	2	0	-14	2	1/2	7
-13	2	1/2	-14	2	1/2	7
-13	2	0	-14	3	0	6
-13	2	0	-14	3	1/2	7
-13	2	1/2	-14	3	0	7
-13	2	1/2	-14	3	1/2	6
-13	3	0	-14	1	0	7
-13	3	1/2	-14	1	0	7
-13	3	0	-14	2	1/2	7
-13	3	1/2	-14	2	1/2	6
-13	3	0	-14	3	0	7
-13	3	0	-14	3	1/2	6
-13	3	1/2	-14	3	0	6
-13	3	1/2	-14	3	1/2	7
-14	1	0	-14	1	0	7
-14	1	0	-14	2	1/2	7
-14	1	0	-14	3	0	8
-14	1	0	-14	3	1/2	7
-14	2	1/2	-14	2	1/2	8
-14	2	1/2	-14	3	0	7
-14	2	1/2	-14	3	1/2	6
-14	3	0	-14	3	0	7
-14	3	0	-14	3	1/2	7
-14	3	1/2	-14	3	1/2	7
-8	0	0	-15	2	1/2	5
-8	0	0	-15	3	0	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-9	0	0	-15	2	1/2	6
-9	0	0	-15	3	0	6
-9	1	1/2	-15	2	1/2	6
-9	1	1/2	-15	3	0	6
-10	0	0	-15	2	1/2	6
-10	0	1/2	-15	2	1/2	6
-10	0	0	-15	3	0	6
-10	0	1/2	-15	3	0	7
-10	1	1/2	-15	2	1/2	6
-10	1	1/2	-15	3	0	6
-10	2	0	-15	2	1/2	6
-10	2	0	-15	3	0	6
-11	0	0	-15	2	1/2	6
-11	0	1/2	-15	2	1/2	7
-11	0	0	-15	3	0	6
-11	0	1/2	-15	3	0	7
-11	1	0	-15	2	1/2	7
-11	1	1/2	-15	2	1/2	6
-11	1	0	-15	3	0	7
-11	1	1/2	-15	3	0	7
-11	2	0	-15	2	1/2	7
-11	2	0	-15	3	0	6
-11	3	1/2	-15	2	1/2	6
-11	3	1/2	-15	3	0	6
-12	0	1/2	-15	2	1/2	7
-12	0	1/2	-15	3	0	7
-12	1	0	-15	2	1/2	7
-12	1	1/2	-15	2	1/2	7
-12	1	0	-15	3	0	7
-12	1	1/2	-15	3	0	7
-12	2	0	-15	2	1/2	7
-12	2	1/2	-15	2	1/2	7
-12	2	0	-15	3	0	6
-12	2	1/2	-15	3	0	7
-12	3	1/2	-15	2	1/2	6
-12	3	1/2	-15	3	0	6
-13	0	1/2	-15	2	1/2	7
-13	0	1/2	-15	3	0	7
-13	1	0	-15	2	1/2	7
-13	1	0	-15	3	0	8
-13	2	0	-15	2	1/2	7
-13	2	1/2	-15	2	1/2	8
-13	2	0	-15	3	0	6
-13	2	1/2	-15	3	0	7
-13	3	0	-15	2	1/2	7
-13	3	1/2	-15	2	1/2	6
-13	3	0	-15	3	0	7
-13	3	1/2	-15	3	0	7
-14	1	0	-15	2	1/2	8
-14	1	0	-15	3	0	8
-14	2	1/2	-15	2	1/2	8
-14	2	1/2	-15	3	0	7
-14	3	0	-15	2	1/2	7
-14	3	1/2	-15	2	1/2	7
-14	3	0	-15	3	0	7
-14	3	1/2	-15	3	0	7
-15	2	1/2	-15	2	1/2	8
-15	2	1/2	-15	3	0	7
-15	3	0	-15	3	0	8
-8	0	0	-16	3	0	6
-9	0	0	-16	3	0	6
-9	1	1/2	-16	3	0	6
-10	0	0	-16	3	0	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-10	0	1/2	-16	3	0	7
-10	1	1/2	-16	3	0	7
-10	2	0	-16	3	0	6
-11	0	0	-16	3	0	7
-11	0	1/2	-16	3	0	7
-11	1	0	-16	3	0	7
-11	1	1/2	-16	3	0	7
-11	2	0	-16	3	0	6
-11	3	1/2	-16	3	0	6
-12	0	1/2	-16	3	0	7
-12	1	0	-16	3	0	8
-12	1	1/2	-16	3	0	7
-12	2	0	-16	3	0	6
-12	2	1/2	-16	3	0	7
-12	3	1/2	-16	3	0	7
-13	0	1/2	-16	3	0	8
-13	1	0	-16	3	0	8
-13	2	0	-16	3	0	7
-13	2	1/2	-16	3	0	7
-13	3	0	-16	3	0	7
-13	3	1/2	-16	3	0	7
-14	1	0	-16	3	0	8
-14	2	1/2	-16	3	0	7
-14	3	0	-16	3	0	8
-14	3	1/2	-16	3	0	7
-15	2	1/2	-16	3	0	8
-15	3	0	-16	3	0	8
-16	3	0	-16	3	0	8

Table D.6: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_4$ with $M_L < 0$, $M_R > 0$.

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-8	0	0	8	0	0	1	
-8	0	0	9	0	0	1	
-9	0	0	9	0	0	1	
-8	0	0	10	0	0	1	
-8	0	0	10	0	1/2	1	
-8	0	0	10	2	0	1	
-9	0	0	10	0	0	1	
-9	1	1/2	10	2	0	1	
-10	0	0	10	0	0	1	
-10	0	1/2	10	0	1/2	1	
-10	2	0	10	2	0	1	
-8	0	0	11	0	0	1	
-8	0	0	11	0	1/2	1	
-8	0	0	11	1	1/2	1	
-8	0	0	11	2	0	1	
-8	0	0	11	3	1/2	1	
-9	0	0	11	0	0	1	
-9	0	0	11	0	1/2	1	
-9	0	0	11	2	0	1	
-9	0	0	11	3	1/2	1	
-9	1	1/2	11	1	1/2	1	
-9	1	1/2	11	2	0	1	
-9	1	1/2	11	3	1/2	1	
-10	0	0	11	0	0	1	
-10	0	1/2	11	0	1/2	1	
-10	0	0	11	3	1/2	1	
-10	1	1/2	11	2	0	1	
-10	1	1/2	11	3	1/2	1	
-10	2	0	11	1	1/2	1	
-10	2	0	11	2	0	1	
-11	0	0	11	0	0	1	
-11	0	1/2	11	0	1/2	1	
-11	1	1/2	11	3	1/2	1	
-11	2	0	11	2	0	1	
-11	3	1/2	11	1	1/2	1	
-8	0	0	12	0	1/2	1	
-8	0	0	12	1	1/2	1	
-8	0	0	12	2	0	1	
-8	0	0	12	2	1/2	1	
-8	0	0	12	3	1/2	1	
-9	0	0	12	0	1/2	1	
-9	0	0	12	1	1/2	1	
-9	0	0	12	2	0	1	
-9	0	0	12	3	1/2	1	
-9	1	1/2	12	1	1/2	1	
-9	1	1/2	12	2	0	1	
-9	1	1/2	12	2	1/2	1	
-9	1	1/2	12	3	1/2	1	
-10	0	0	12	0	1/2	1	
-10	0	1/2	12	0	1/2	1	
-10	0	0	12	2	0	1	
-10	0	1/2	12	2	1/2	1	
-10	0	0	12	3	1/2	1	
-10	1	1/2	12	1	1/2	1	
-10	1	1/2	12	2	0	1	
-10	1	1/2	12	3	1/2	1	
-10	2	0	12	1	1/2	1	
-10	2	0	12	2	0	1	
-10	2	0	12	2	1/2	1	
-11	0	1/2	12	0	1/2	1	
-11	0	0	12	3	1/2	1	
-11	1	0	12	2	1/2	1	

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-11	1	1/2	12	2	0	1
-11	1	1/2	12	3	1/2	1
-11	2	0	12	1	1/2	1
-11	2	0	12	2	0	1
-11	3	1/2	12	1	1/2	1
-12	0	1/2	12	0	1/2	1
-12	1	1/2	12	3	1/2	1
-12	2	0	12	2	0	1
-12	2	1/2	12	2	1/2	1
-12	3	1/2	12	1	1/2	1
-8	0	0	13	0	1/2	1
-8	0	0	13	1	0	1
-8	0	0	13	2	0	1
-8	0	0	13	2	1/2	1
-8	0	0	13	3	0	1
-8	0	0	13	3	1/2	2
-9	0	0	13	0	1/2	1
-9	0	0	13	2	0	1
-9	0	0	13	2	1/2	1
-9	0	0	13	3	0	1
-9	0	0	13	3	1/2	1
-9	1	1/2	13	1	0	1
-9	1	1/2	13	2	0	1
-9	1	1/2	13	2	1/2	1
-9	1	1/2	13	3	0	1
-9	1	1/2	13	3	1/2	2
-10	0	0	13	0	1/2	1
-10	0	1/2	13	0	1/2	1
-10	0	1/2	13	1	0	1
-10	0	0	13	2	0	1
-10	0	1/2	13	2	1/2	1
-10	0	0	13	3	0	1
-10	0	0	13	3	1/2	1
-10	0	1/2	13	3	0	1
-10	0	1/2	13	3	1/2	1
-10	1	1/2	13	2	0	1
-10	1	1/2	13	2	1/2	1
-10	1	1/2	13	3	0	1
-10	1	1/2	13	3	1/2	1
-10	2	0	13	1	0	1
-10	2	0	13	2	0	1
-10	2	0	13	2	1/2	1
-10	2	0	13	3	1/2	1
-11	0	0	13	0	1/2	1
-11	0	1/2	13	0	1/2	1
-11	0	0	13	2	0	1
-11	0	1/2	13	2	1/2	1
-11	0	0	13	3	1/2	1
-11	0	1/2	13	3	0	1
-11	1	0	13	1	0	1
-11	1	0	13	2	1/2	1
-11	1	1/2	13	2	0	1
-11	1	0	13	3	0	1
-11	1	0	13	3	1/2	1
-11	1	1/2	13	3	0	1
-11	1	1/2	13	3	1/2	1
-11	2	0	13	2	0	1
-11	2	0	13	2	1/2	1
-11	3	1/2	13	1	0	1
-11	3	1/2	13	3	1/2	1
-12	0	1/2	13	0	1/2	1
-12	0	1/2	13	3	0	1
-12	1	0	13	2	1/2	1

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-12	1	1/2	13	2	0	1
-12	1	0	13	3	0	1
-12	1	1/2	13	3	1/2	1
-12	2	1/2	13	1	0	1
-12	2	0	13	2	0	1
-12	2	1/2	13	2	1/2	1
-13	0	1/2	13	0	1/2	1
-13	1	0	13	3	0	1
-13	2	0	13	2	0	1
-13	2	1/2	13	2	1/2	1
-13	3	0	13	1	0	1
-8	0	0	14	1	0	1
-8	0	0	14	2	1/2	1
-8	0	0	14	3	0	1
-8	0	0	14	3	1/2	2
-9	0	0	14	1	0	1
-9	0	0	14	2	1/2	1
-9	0	0	14	3	0	1
-9	0	0	14	3	1/2	2
-9	1	1/2	14	1	0	1
-9	1	1/2	14	2	1/2	1
-9	1	1/2	14	3	0	1
-9	1	1/2	14	3	1/2	2
-10	0	1/2	14	1	0	1
-10	0	0	14	2	1/2	1
-10	0	1/2	14	2	1/2	1
-10	0	0	14	3	0	1
-10	0	0	14	3	1/2	1
-10	0	1/2	14	3	0	1
-10	0	1/2	14	3	1/2	1
-10	1	1/2	14	1	0	1
-10	1	1/2	14	2	1/2	1
-10	1	1/2	14	3	0	1
-10	1	1/2	14	3	1/2	2
-10	2	0	14	1	0	1
-10	2	0	14	2	1/2	1
-10	2	0	14	3	1/2	1
-11	0	1/2	14	1	0	1
-11	0	1/2	14	2	1/2	1
-11	0	0	14	3	0	1
-11	0	0	14	3	1/2	1
-11	0	1/2	14	3	0	1
-11	0	1/2	14	3	1/2	1
-11	1	0	14	1	0	1
-11	1	0	14	2	1/2	1
-11	1	1/2	14	2	1/2	1
-11	1	0	14	3	0	1
-11	1	0	14	3	1/2	1
-11	1	1/2	14	3	0	1
-11	1	1/2	14	3	1/2	1
-11	2	0	14	1	0	1
-11	2	0	14	2	1/2	1
-11	2	0	14	3	1/2	1
-11	3	1/2	14	1	0	1
-11	3	1/2	14	3	1/2	1
-12	0	1/2	14	2	1/2	1
-12	0	1/2	14	3	0	1
-12	1	0	14	1	0	1
-12	1	0	14	2	1/2	1
-12	1	0	14	3	0	1
-12	1	0	14	3	1/2	1
-12	1	1/2	14	3	0	1
-12	1	1/2	14	3	1/2	1

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-12	2	1/2	14	1	0	1
-12	2	0	14	2	1/2	1
-12	2	1/2	14	2	1/2	1
-12	3	1/2	14	1	0	1
-12	3	1/2	14	3	1/2	1
-13	0	1/2	14	3	0	1
-13	1	0	14	2	1/2	1
-13	1	0	14	3	0	1
-13	2	1/2	14	1	0	1
-13	2	1/2	14	2	1/2	1
-13	3	0	14	1	0	1
-14	1	0	14	3	0	1
-14	2	1/2	14	2	1/2	1
-14	3	0	14	1	0	1
-8	0	0	15	2	1/2	1
-8	0	0	15	3	0	2
-9	0	0	15	2	1/2	1
-9	0	0	15	3	0	1
-9	1	1/2	15	2	1/2	1
-9	1	1/2	15	3	0	2
-10	0	0	15	2	1/2	1
-10	0	1/2	15	2	1/2	1
-10	0	0	15	3	0	1
-10	0	1/2	15	3	0	2
-10	1	1/2	15	2	1/2	1
-10	1	1/2	15	3	0	1
-10	2	0	15	2	1/2	1
-10	2	0	15	3	0	1
-11	0	0	15	2	1/2	1
-11	0	1/2	15	2	1/2	1
-11	0	0	15	3	0	1
-11	0	1/2	15	3	0	1
-11	1	0	15	2	1/2	1
-11	1	1/2	15	2	1/2	1
-11	1	0	15	3	0	2
-11	1	1/2	15	3	0	1
-11	2	0	15	2	1/2	1
-11	3	1/2	15	3	0	1
-12	0	1/2	15	2	1/2	1
-12	0	1/2	15	3	0	1
-12	1	0	15	2	1/2	1
-12	1	1/2	15	2	1/2	1
-12	1	0	15	3	0	1
-12	1	1/2	15	3	0	1
-12	2	0	15	2	1/2	1
-12	2	1/2	15	2	1/2	1
-12	2	1/2	15	3	0	1
-13	0	1/2	15	2	1/2	1
-13	0	1/2	15	3	0	1
-13	1	0	15	2	1/2	1
-13	1	0	15	3	0	1
-13	2	0	15	2	1/2	1
-13	2	1/2	15	2	1/2	1
-13	3	0	15	3	0	1
-14	1	0	15	2	1/2	1
-14	1	0	15	3	0	1
-14	2	1/2	15	2	1/2	1
-15	2	1/2	15	2	1/2	1
-8	0	0	16	3	0	2
-9	0	0	16	3	0	2
-9	1	1/2	16	3	0	2
-10	0	0	16	3	0	1
-10	0	1/2	16	3	0	2

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-10	1	1/2	16	3	0	2
-10	2	0	16	3	0	1
-11	0	0	16	3	0	1
-11	0	1/2	16	3	0	2
-11	1	0	16	3	0	2
-11	1	1/2	16	3	0	1
-11	2	0	16	3	0	1
-11	3	1/2	16	3	0	1
-12	0	1/2	16	3	0	1
-12	1	0	16	3	0	2
-12	1	1/2	16	3	0	1
-12	2	1/2	16	3	0	1
-12	3	1/2	16	3	0	1
-13	0	1/2	16	3	0	1
-13	1	0	16	3	0	1
-13	2	1/2	16	3	0	1
-13	3	0	16	3	0	1
-14	1	0	16	3	0	1
-14	3	0	16	3	0	1

D.4 $\mathbb{T}^2/\mathbb{Z}_6$ Table D.7: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_6$ with $M_L < 0$, $M_R < 0$.

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-12	0	0	-12	0	0	5
-12	0	0	-13	1	1/2	4
-13	1	1/2	-13	1	1/2	4
-12	0	0	-14	0	0	5
-12	0	0	-14	2	0	4
-13	1	1/2	-14	0	0	4
-13	1	1/2	-14	2	0	5
-14	0	0	-14	0	0	5
-14	0	0	-14	2	0	5
-14	2	0	-14	2	0	5
-12	0	0	-15	0	1/2	5
-12	0	0	-15	1	1/2	4
-12	0	0	-15	3	1/2	5
-13	1	1/2	-15	0	1/2	4
-13	1	1/2	-15	1	1/2	5
-13	1	1/2	-15	3	1/2	5
-14	0	0	-15	0	1/2	5
-14	0	0	-15	1	1/2	5
-14	0	0	-15	3	1/2	5
-14	2	0	-15	0	1/2	4
-14	2	0	-15	1	1/2	5
-14	2	0	-15	3	1/2	5
-15	0	1/2	-15	0	1/2	6
-15	0	1/2	-15	1	1/2	4
-15	0	1/2	-15	3	1/2	5
-15	1	1/2	-15	1	1/2	5
-15	1	1/2	-15	3	1/2	5
-15	3	1/2	-15	3	1/2	6
-12	0	0	-16	0	0	5
-12	0	0	-16	1	0	4
-12	0	0	-16	2	0	5
-12	0	0	-16	4	0	5
-13	1	1/2	-16	0	0	5
-13	1	1/2	-16	1	0	4
-13	1	1/2	-16	2	0	5
-13	1	1/2	-16	4	0	5
-14	0	0	-16	0	0	6
-14	0	0	-16	1	0	4
-14	0	0	-16	2	0	5
-14	0	0	-16	4	0	5
-14	2	0	-16	0	0	5
-14	2	0	-16	1	0	5
-14	2	0	-16	2	0	5
-14	2	0	-16	4	0	6
-15	0	1/2	-16	0	0	5
-15	0	1/2	-16	1	0	5
-15	0	1/2	-16	2	0	5
-15	0	1/2	-16	4	0	5
-15	1	1/2	-16	0	0	5
-15	1	1/2	-16	1	0	5
-15	1	1/2	-16	2	0	5
-15	1	1/2	-16	4	0	6
-15	3	1/2	-16	0	0	5
-15	3	1/2	-16	1	0	5
-15	3	1/2	-16	2	0	6
-15	3	1/2	-16	4	0	5
-16	0	0	-16	0	0	6
-16	0	0	-16	1	0	5
-16	0	0	-16	2	0	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-16	0	0	-16	4	0	6
-16	1	0	-16	1	0	5
-16	1	0	-16	2	0	5
-16	1	0	-16	4	0	5
-16	2	0	-16	2	0	6
-16	2	0	-16	4	0	6
-16	4	0	-16	4	0	5
-12	0	0	-17	0	1/2	5
-12	0	0	-17	1	1/2	5
-12	0	0	-17	2	1/2	4
-12	0	0	-17	3	1/2	5
-12	0	0	-17	5	1/2	5
-13	1	1/2	-17	0	1/2	4
-13	1	1/2	-17	1	1/2	5
-13	1	1/2	-17	2	1/2	5
-13	1	1/2	-17	3	1/2	5
-13	1	1/2	-17	5	1/2	6
-14	0	0	-17	0	1/2	5
-14	0	0	-17	1	1/2	5
-14	0	0	-17	2	1/2	5
-14	0	0	-17	3	1/2	5
-14	0	0	-17	5	1/2	6
-14	2	0	-17	0	1/2	5
-14	2	0	-17	1	1/2	5
-14	2	0	-17	2	1/2	5
-14	2	0	-17	3	1/2	6
-14	2	0	-17	5	1/2	5
-15	0	1/2	-17	0	1/2	6
-15	0	1/2	-17	1	1/2	5
-15	0	1/2	-17	2	1/2	5
-15	0	1/2	-17	3	1/2	5
-15	0	1/2	-17	5	1/2	5
-15	1	1/2	-17	0	1/2	5
-15	1	1/2	-17	1	1/2	5
-15	1	1/2	-17	2	1/2	5
-15	1	1/2	-17	3	1/2	6
-15	1	1/2	-17	5	1/2	6
-15	3	1/2	-17	0	1/2	5
-15	3	1/2	-17	1	1/2	6
-15	3	1/2	-17	2	1/2	5
-15	3	1/2	-17	3	1/2	6
-15	3	1/2	-17	5	1/2	5
-16	0	0	-17	0	1/2	6
-16	0	0	-17	1	1/2	5
-16	0	0	-17	2	1/2	5
-16	0	0	-17	3	1/2	6
-16	0	0	-17	5	1/2	6
-16	1	0	-17	0	1/2	5
-16	1	0	-17	1	1/2	5
-16	1	0	-17	2	1/2	6
-16	1	0	-17	3	1/2	5
-16	1	0	-17	5	1/2	6
-16	2	0	-17	0	1/2	5
-16	2	0	-17	1	1/2	6
-16	2	0	-17	2	1/2	5
-16	2	0	-17	3	1/2	6
-16	2	0	-17	5	1/2	5
-16	4	0	-17	0	1/2	5
-16	4	0	-17	1	1/2	6
-16	4	0	-17	2	1/2	6
-16	4	0	-17	3	1/2	5
-16	4	0	-17	5	1/2	6
-17	0	1/2	-17	0	1/2	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-17	0	1/2	-17	1	1/2	5
-17	0	1/2	-17	2	1/2	6
-17	0	1/2	-17	3	1/2	5
-17	0	1/2	-17	5	1/2	6
-17	1	1/2	-17	1	1/2	6
-17	1	1/2	-17	2	1/2	5
-17	1	1/2	-17	3	1/2	6
-17	1	1/2	-17	5	1/2	6
-17	2	1/2	-17	2	1/2	6
-17	2	1/2	-17	3	1/2	6
-17	2	1/2	-17	5	1/2	5
-17	3	1/2	-17	3	1/2	6
-17	3	1/2	-17	5	1/2	6
-17	5	1/2	-17	5	1/2	6
-12	0	0	-18	1	0	4
-12	0	0	-18	2	0	5
-12	0	0	-18	3	0	5
-12	0	0	-18	4	0	5
-13	1	1/2	-18	1	0	5
-13	1	1/2	-18	2	0	5
-13	1	1/2	-18	3	0	5
-13	1	1/2	-18	4	0	6
-14	0	0	-18	1	0	5
-14	0	0	-18	2	0	5
-14	0	0	-18	3	0	5
-14	0	0	-18	4	0	6
-14	2	0	-18	1	0	5
-14	2	0	-18	2	0	6
-14	2	0	-18	3	0	5
-14	2	0	-18	4	0	6
-15	0	1/2	-18	1	0	5
-15	0	1/2	-18	2	0	5
-15	0	1/2	-18	3	0	6
-15	0	1/2	-18	4	0	5
-15	1	1/2	-18	1	0	5
-15	1	1/2	-18	2	0	6
-15	1	1/2	-18	3	0	5
-15	1	1/2	-18	4	0	6
-15	3	1/2	-18	1	0	5
-15	3	1/2	-18	2	0	6
-15	3	1/2	-18	3	0	6
-15	3	1/2	-18	4	0	5
-16	0	0	-18	1	0	5
-16	0	0	-18	2	0	6
-16	0	0	-18	3	0	5
-16	0	0	-18	4	0	6
-16	1	0	-18	1	0	6
-16	1	0	-18	2	0	5
-16	1	0	-18	3	0	6
-16	1	0	-18	4	0	6
-16	2	0	-18	1	0	5
-16	2	0	-18	2	0	6
-16	2	0	-18	3	0	6
-16	2	0	-18	4	0	6
-16	4	0	-18	1	0	6
-16	4	0	-18	2	0	6
-16	4	0	-18	3	0	5
-16	4	0	-18	4	0	6
-17	0	1/2	-18	1	0	6
-17	0	1/2	-18	2	0	5
-17	0	1/2	-18	3	0	6
-17	0	1/2	-18	4	0	6
-17	1	1/2	-18	1	0	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-17	1	1/2	-18	2	0	6
-17	1	1/2	-18	3	0	6
-17	1	1/2	-18	4	0	6
-17	2	1/2	-18	1	0	6
-17	2	1/2	-18	2	0	6
-17	2	1/2	-18	3	0	6
-17	2	1/2	-18	4	0	6
-17	3	1/2	-18	1	0	6
-17	3	1/2	-18	2	0	6
-17	3	1/2	-18	3	0	6
-17	3	1/2	-18	4	0	6
-17	5	1/2	-18	1	0	6
-17	5	1/2	-18	2	0	6
-17	5	1/2	-18	3	0	5
-17	5	1/2	-18	4	0	6
-18	1	0	-18	1	0	6
-18	1	0	-18	2	0	6
-18	1	0	-18	3	0	6
-18	1	0	-18	4	0	6
-18	2	0	-18	2	0	6
-18	2	0	-18	3	0	6
-18	2	0	-18	4	0	7
-18	3	0	-18	3	0	7
-18	3	0	-18	4	0	5
-18	4	0	-18	4	0	6
-12	0	0	-19	0	1/2	5
-12	0	0	-19	2	1/2	5
-12	0	0	-19	3	1/2	5
-12	0	0	-19	4	1/2	5
-12	0	0	-19	5	1/2	6
-13	1	1/2	-19	0	1/2	5
-13	1	1/2	-19	2	1/2	5
-13	1	1/2	-19	3	1/2	6
-13	1	1/2	-19	4	1/2	5
-13	1	1/2	-19	5	1/2	6
-14	0	0	-19	0	1/2	6
-14	0	0	-19	2	1/2	5
-14	0	0	-19	3	1/2	6
-14	0	0	-19	4	1/2	5
-14	0	0	-19	5	1/2	6
-14	2	0	-19	0	1/2	5
-14	2	0	-19	2	1/2	5
-14	2	0	-19	3	1/2	6
-14	2	0	-19	4	1/2	6
-14	2	0	-19	5	1/2	5
-15	0	1/2	-19	0	1/2	6
-15	0	1/2	-19	2	1/2	6
-15	0	1/2	-19	3	1/2	5
-15	0	1/2	-19	4	1/2	6
-15	0	1/2	-19	5	1/2	6
-15	1	1/2	-19	0	1/2	5
-15	1	1/2	-19	2	1/2	5
-15	1	1/2	-19	3	1/2	6
-15	1	1/2	-19	4	1/2	6
-15	1	1/2	-19	5	1/2	6
-15	3	1/2	-19	0	1/2	5
-15	3	1/2	-19	2	1/2	6
-15	3	1/2	-19	3	1/2	6
-15	3	1/2	-19	4	1/2	5
-15	3	1/2	-19	5	1/2	6
-16	0	0	-19	0	1/2	6
-16	0	0	-19	2	1/2	5
-16	0	0	-19	3	1/2	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-16	0	0	-19	4	1/2	6
-16	0	0	-19	5	1/2	6
-16	1	0	-19	0	1/2	6
-16	1	0	-19	2	1/2	6
-16	1	0	-19	3	1/2	6
-16	1	0	-19	4	1/2	6
-16	1	0	-19	5	1/2	6
-16	2	0	-19	0	1/2	5
-16	2	0	-19	2	1/2	6
-16	2	0	-19	3	1/2	6
-16	2	0	-19	4	1/2	6
-16	2	0	-19	5	1/2	6
-16	4	0	-19	0	1/2	6
-16	4	0	-19	2	1/2	6
-16	4	0	-19	3	1/2	6
-16	4	0	-19	4	1/2	5
-16	4	0	-19	5	1/2	6
-17	0	1/2	-19	0	1/2	7
-17	0	1/2	-19	2	1/2	6
-17	0	1/2	-19	3	1/2	6
-17	0	1/2	-19	4	1/2	6
-17	0	1/2	-19	5	1/2	6
-17	1	1/2	-19	0	1/2	5
-17	1	1/2	-19	2	1/2	6
-17	1	1/2	-19	3	1/2	6
-17	1	1/2	-19	4	1/2	6
-17	1	1/2	-19	5	1/2	7
-17	2	1/2	-19	0	1/2	6
-17	2	1/2	-19	2	1/2	6
-17	2	1/2	-19	3	1/2	6
-17	2	1/2	-19	4	1/2	7
-17	2	1/2	-19	5	1/2	5
-17	3	1/2	-19	0	1/2	6
-17	3	1/2	-19	2	1/2	6
-17	3	1/2	-19	3	1/2	7
-17	3	1/2	-19	4	1/2	5
-17	3	1/2	-19	5	1/2	6
-17	5	1/2	-19	0	1/2	6
-17	5	1/2	-19	2	1/2	5
-17	5	1/2	-19	3	1/2	6
-17	5	1/2	-19	4	1/2	6
-17	5	1/2	-19	5	1/2	6
-18	1	0	-19	0	1/2	6
-18	1	0	-19	2	1/2	6
-18	1	0	-19	3	1/2	6
-18	1	0	-19	4	1/2	7
-18	1	0	-19	5	1/2	6
-18	2	0	-19	0	1/2	6
-18	2	0	-19	2	1/2	6
-18	2	0	-19	3	1/2	7
-18	2	0	-19	4	1/2	6
-18	2	0	-19	5	1/2	6
-18	3	0	-19	0	1/2	6
-18	3	0	-19	2	1/2	7
-18	3	0	-19	3	1/2	6
-18	3	0	-19	4	1/2	6
-18	3	0	-19	5	1/2	6
-18	4	0	-19	0	1/2	6
-18	4	0	-19	2	1/2	6
-18	4	0	-19	3	1/2	6
-18	4	0	-19	4	1/2	6
-18	4	0	-19	5	1/2	6
-19	0	1/2	-19	0	1/2	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-19	0	1/2	-19	2	1/2	6
-19	0	1/2	-19	3	1/2	6
-19	0	1/2	-19	4	1/2	7
-19	0	1/2	-19	5	1/2	6
-19	2	1/2	-19	2	1/2	7
-19	2	1/2	-19	3	1/2	6
-19	2	1/2	-19	4	1/2	7
-19	2	1/2	-19	5	1/2	6
-19	3	1/2	-19	3	1/2	7
-19	3	1/2	-19	4	1/2	6
-19	3	1/2	-19	5	1/2	6
-19	4	1/2	-19	4	1/2	6
-19	4	1/2	-19	5	1/2	6
-19	5	1/2	-19	5	1/2	7
-12	0	0	-20	1	0	5
-12	0	0	-20	3	0	5
-12	0	0	-20	4	0	6
-12	0	0	-20	5	0	5
-13	1	1/2	-20	1	0	5
-13	1	1/2	-20	3	0	5
-13	1	1/2	-20	4	0	6
-13	1	1/2	-20	5	0	6
-14	0	0	-20	1	0	5
-14	0	0	-20	3	0	5
-14	0	0	-20	4	0	6
-14	0	0	-20	5	0	6
-14	2	0	-20	1	0	5
-14	2	0	-20	3	0	6
-14	2	0	-20	4	0	6
-14	2	0	-20	5	0	5
-15	0	1/2	-20	1	0	6
-15	0	1/2	-20	3	0	6
-15	0	1/2	-20	4	0	6
-15	0	1/2	-20	5	0	6
-15	1	1/2	-20	1	0	5
-15	1	1/2	-20	3	0	6
-15	1	1/2	-20	4	0	6
-15	1	1/2	-20	5	0	6
-15	3	1/2	-20	1	0	6
-15	3	1/2	-20	3	0	6
-15	3	1/2	-20	4	0	6
-15	3	1/2	-20	5	0	5
-16	0	0	-20	1	0	5
-16	0	0	-20	3	0	6
-16	0	0	-20	4	0	6
-16	0	0	-20	5	0	6
-16	1	0	-20	1	0	6
-16	1	0	-20	3	0	6
-16	1	0	-20	4	0	6
-16	1	0	-20	5	0	7
-16	2	0	-20	1	0	6
-16	2	0	-20	3	0	6
-16	2	0	-20	4	0	7
-16	2	0	-20	5	0	5
-16	4	0	-20	1	0	6
-16	4	0	-20	3	0	5
-16	4	0	-20	4	0	6
-16	4	0	-20	5	0	6
-17	0	1/2	-20	1	0	6
-17	0	1/2	-20	3	0	6
-17	0	1/2	-20	4	0	6
-17	0	1/2	-20	5	0	7
-17	1	1/2	-20	1	0	6

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-17	1	1/2	-20	3	0	6
-17	1	1/2	-20	4	0	7
-17	1	1/2	-20	5	0	6
-17	2	1/2	-20	1	0	6
-17	2	1/2	-20	3	0	7
-17	2	1/2	-20	4	0	6
-17	2	1/2	-20	5	0	6
-17	3	1/2	-20	1	0	6
-17	3	1/2	-20	3	0	6
-17	3	1/2	-20	4	0	6
-17	3	1/2	-20	5	0	6
-17	5	1/2	-20	1	0	6
-17	5	1/2	-20	3	0	6
-17	5	1/2	-20	4	0	6
-17	5	1/2	-20	5	0	6
-18	1	0	-20	1	0	6
-18	1	0	-20	3	0	7
-18	1	0	-20	4	0	6
-18	1	0	-20	5	0	7
-18	2	0	-20	1	0	6
-18	2	0	-20	3	0	6
-18	2	0	-20	4	0	7
-18	2	0	-20	5	0	6
-18	3	0	-20	1	0	7
-18	3	0	-20	3	0	7
-18	3	0	-20	4	0	6
-18	3	0	-20	5	0	6
-18	4	0	-20	1	0	6
-18	4	0	-20	3	0	6
-18	4	0	-20	4	0	6
-18	4	0	-20	5	0	6
-19	0	1/2	-20	1	0	6
-19	0	1/2	-20	3	0	7
-19	0	1/2	-20	4	0	6
-19	0	1/2	-20	5	0	7
-19	2	1/2	-20	1	0	7
-19	2	1/2	-20	3	0	7
-19	2	1/2	-20	4	0	7
-19	2	1/2	-20	5	0	6
-19	3	1/2	-20	1	0	6
-19	3	1/2	-20	3	0	7
-19	3	1/2	-20	4	0	6
-19	3	1/2	-20	5	0	6
-19	4	1/2	-20	1	0	7
-19	4	1/2	-20	3	0	6
-19	4	1/2	-20	4	0	6
-19	4	1/2	-20	5	0	7
-19	5	1/2	-20	1	0	7
-19	5	1/2	-20	3	0	6
-19	5	1/2	-20	4	0	7
-19	5	1/2	-20	5	0	6
-20	1	0	-20	1	0	7
-20	1	0	-20	3	0	7
-20	1	0	-20	4	0	7
-20	1	0	-20	5	0	7
-20	3	0	-20	3	0	7
-20	3	0	-20	4	0	6
-20	3	0	-20	5	0	7
-20	4	0	-20	4	0	7
-20	4	0	-20	5	0	6
-20	5	0	-20	5	0	7
-12	0	0	-21	2	1/2	5
-12	0	0	-21	4	1/2	5

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-12	0	0	-21	5	1/2	6
-13	1	1/2	-21	2	1/2	5
-13	1	1/2	-21	4	1/2	6
-13	1	1/2	-21	5	1/2	6
-14	0	0	-21	2	1/2	5
-14	0	0	-21	4	1/2	6
-14	0	0	-21	5	1/2	6
-14	2	0	-21	2	1/2	6
-14	2	0	-21	4	1/2	6
-14	2	0	-21	5	1/2	6
-15	0	1/2	-21	2	1/2	6
-15	0	1/2	-21	4	1/2	6
-15	0	1/2	-21	5	1/2	6
-15	1	1/2	-21	2	1/2	6
-15	1	1/2	-21	4	1/2	6
-15	1	1/2	-21	5	1/2	7
-15	3	1/2	-21	2	1/2	6
-15	3	1/2	-21	4	1/2	5
-15	3	1/2	-21	5	1/2	6
-16	0	0	-21	2	1/2	6
-16	0	0	-21	4	1/2	6
-16	0	0	-21	5	1/2	7
-16	1	0	-21	2	1/2	6
-16	1	0	-21	4	1/2	7
-16	1	0	-21	5	1/2	6
-16	2	0	-21	2	1/2	6
-16	2	0	-21	4	1/2	6
-16	2	0	-21	5	1/2	6
-16	4	0	-21	2	1/2	6
-16	4	0	-21	4	1/2	6
-16	4	0	-21	5	1/2	6
-17	0	1/2	-21	2	1/2	6
-17	0	1/2	-21	4	1/2	7
-17	0	1/2	-21	5	1/2	6
-17	1	1/2	-21	2	1/2	6
-17	1	1/2	-21	4	1/2	6
-17	1	1/2	-21	5	1/2	7
-17	2	1/2	-21	2	1/2	7
-17	2	1/2	-21	4	1/2	7
-17	2	1/2	-21	5	1/2	6
-17	3	1/2	-21	2	1/2	6
-17	3	1/2	-21	4	1/2	6
-17	3	1/2	-21	5	1/2	6
-17	5	1/2	-21	2	1/2	6
-17	5	1/2	-21	4	1/2	6
-17	5	1/2	-21	5	1/2	7
-18	1	0	-21	2	1/2	7
-18	1	0	-21	4	1/2	7
-18	1	0	-21	5	1/2	7
-18	2	0	-21	2	1/2	6
-18	2	0	-21	4	1/2	7
-18	2	0	-21	5	1/2	6
-18	3	0	-21	2	1/2	7
-18	3	0	-21	4	1/2	6
-18	3	0	-21	5	1/2	6
-18	4	0	-21	2	1/2	7
-18	4	0	-21	4	1/2	6
-18	4	0	-21	5	1/2	7
-19	0	1/2	-21	2	1/2	7
-19	0	1/2	-21	4	1/2	7
-19	0	1/2	-21	5	1/2	7
-19	2	1/2	-21	2	1/2	7
-19	2	1/2	-21	4	1/2	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-19	2	1/2	-21	5	1/2	6
-19	3	1/2	-21	2	1/2	7
-19	3	1/2	-21	4	1/2	6
-19	3	1/2	-21	5	1/2	7
-19	4	1/2	-21	2	1/2	7
-19	4	1/2	-21	4	1/2	7
-19	4	1/2	-21	5	1/2	6
-19	5	1/2	-21	2	1/2	6
-19	5	1/2	-21	4	1/2	6
-19	5	1/2	-21	5	1/2	7
-20	1	0	-21	2	1/2	7
-20	1	0	-21	4	1/2	7
-20	1	0	-21	5	1/2	7
-20	3	0	-21	2	1/2	7
-20	3	0	-21	4	1/2	7
-20	3	0	-21	5	1/2	6
-20	4	0	-21	2	1/2	7
-20	4	0	-21	4	1/2	6
-20	4	0	-21	5	1/2	7
-20	5	0	-21	2	1/2	7
-20	5	0	-21	4	1/2	7
-20	5	0	-21	5	1/2	7
-21	2	1/2	-21	2	1/2	7
-21	2	1/2	-21	4	1/2	8
-21	2	1/2	-21	5	1/2	6
-21	4	1/2	-21	4	1/2	7
-21	4	1/2	-21	5	1/2	7
-21	5	1/2	-21	5	1/2	7
-12	0	0	-22	3	0	5
-12	0	0	-22	5	0	6
-13	1	1/2	-22	3	0	6
-13	1	1/2	-22	5	0	6
-14	0	0	-22	3	0	6
-14	0	0	-22	5	0	6
-14	2	0	-22	3	0	6
-14	2	0	-22	5	0	5
-15	0	1/2	-22	3	0	6
-15	0	1/2	-22	5	0	7
-15	1	1/2	-22	3	0	6
-15	1	1/2	-22	5	0	6
-15	3	1/2	-22	3	0	6
-15	3	1/2	-22	5	0	6
-16	0	0	-22	3	0	6
-16	0	0	-22	5	0	6
-16	1	0	-22	3	0	7
-16	1	0	-22	5	0	7
-16	2	0	-22	3	0	6
-16	2	0	-22	5	0	6
-16	4	0	-22	3	0	6
-16	4	0	-22	5	0	6
-17	0	1/2	-22	3	0	7
-17	0	1/2	-22	5	0	7
-17	1	1/2	-22	3	0	6
-17	1	1/2	-22	5	0	7
-17	2	1/2	-22	3	0	7
-17	2	1/2	-22	5	0	6
-17	3	1/2	-22	3	0	7
-17	3	1/2	-22	5	0	6
-17	5	1/2	-22	3	0	6
-17	5	1/2	-22	5	0	6
-18	1	0	-22	3	0	7
-18	1	0	-22	5	0	7
-18	2	0	-22	3	0	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-18	2	0	-22	5	0	6
-18	3	0	-22	3	0	7
-18	3	0	-22	5	0	7
-18	4	0	-22	3	0	6
-18	4	0	-22	5	0	6
-19	0	1/2	-22	3	0	7
-19	0	1/2	-22	5	0	7
-19	2	1/2	-22	3	0	7
-19	2	1/2	-22	5	0	7
-19	3	1/2	-22	3	0	7
-19	3	1/2	-22	5	0	6
-19	4	1/2	-22	3	0	7
-19	4	1/2	-22	5	0	7
-19	5	1/2	-22	3	0	6
-19	5	1/2	-22	5	0	7
-20	1	0	-22	3	0	7
-20	1	0	-22	5	0	8
-20	3	0	-22	3	0	8
-20	3	0	-22	5	0	7
-20	4	0	-22	3	0	6
-20	4	0	-22	5	0	7
-20	5	0	-22	3	0	7
-20	5	0	-22	5	0	7
-21	2	1/2	-22	3	0	8
-21	2	1/2	-22	5	0	7
-21	4	1/2	-22	3	0	7
-21	4	1/2	-22	5	0	7
-21	5	1/2	-22	3	0	7
-21	5	1/2	-22	5	0	7
-22	3	0	-22	3	0	8
-22	3	0	-22	5	0	7
-22	5	0	-22	5	0	8
-12	0	0	-23	4	1/2	6
-13	1	1/2	-23	4	1/2	6
-14	0	0	-23	4	1/2	6
-14	2	0	-23	4	1/2	6
-15	0	1/2	-23	4	1/2	7
-15	1	1/2	-23	4	1/2	6
-15	3	1/2	-23	4	1/2	6
-16	0	0	-23	4	1/2	6
-16	1	0	-23	4	1/2	7
-16	2	0	-23	4	1/2	7
-16	4	0	-23	4	1/2	6
-17	0	1/2	-23	4	1/2	7
-17	1	1/2	-23	4	1/2	7
-17	2	1/2	-23	4	1/2	7
-17	3	1/2	-23	4	1/2	6
-17	5	1/2	-23	4	1/2	6
-18	1	0	-23	4	1/2	7
-18	2	0	-23	4	1/2	7
-18	3	0	-23	4	1/2	7
-18	4	0	-23	4	1/2	6
-19	0	1/2	-23	4	1/2	7
-19	2	1/2	-23	4	1/2	8
-19	3	1/2	-23	4	1/2	6
-19	4	1/2	-23	4	1/2	7
-19	5	1/2	-23	4	1/2	7
-20	1	0	-23	4	1/2	8
-20	3	0	-23	4	1/2	7
-20	4	0	-23	4	1/2	7
-20	5	0	-23	4	1/2	7
-21	2	1/2	-23	4	1/2	8
-21	4	1/2	-23	4	1/2	7

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-21	5	1/2	-23	4	1/2	7
-22	3	0	-23	4	1/2	7
-22	5	0	-23	4	1/2	8
-23	4	1/2	-23	4	1/2	8
-12	0	0	-24	5	0	6
-13	1	1/2	-24	5	0	6
-14	0	0	-24	5	0	6
-14	2	0	-24	5	0	6
-15	0	1/2	-24	5	0	7
-15	1	1/2	-24	5	0	7
-15	3	1/2	-24	5	0	6
-16	0	0	-24	5	0	7
-16	1	0	-24	5	0	7
-16	2	0	-24	5	0	6
-16	4	0	-24	5	0	6
-17	0	1/2	-24	5	0	7
-17	1	1/2	-24	5	0	7
-17	2	1/2	-24	5	0	7
-17	3	1/2	-24	5	0	6
-17	5	1/2	-24	5	0	7
-18	1	0	-24	5	0	8
-18	2	0	-24	5	0	6
-18	3	0	-24	5	0	7
-18	4	0	-24	5	0	7
-19	0	1/2	-24	5	0	8
-19	2	1/2	-24	5	0	7
-19	3	1/2	-24	5	0	7
-19	4	1/2	-24	5	0	7
-19	5	1/2	-24	5	0	7
-20	1	0	-24	5	0	8
-20	3	0	-24	5	0	7
-20	4	0	-24	5	0	7
-20	5	0	-24	5	0	8
-21	2	1/2	-24	5	0	7
-21	4	1/2	-24	5	0	8
-21	5	1/2	-24	5	0	7
-22	3	0	-24	5	0	8
-22	5	0	-24	5	0	8
-23	4	1/2	-24	5	0	8
-24	5	0	-24	5	0	8

Table D.8: Possible parameter configurations on $\mathbb{T}^2/\mathbb{Z}_6$ with $M_L < 0$, $M_R > 0$.

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-12	0	0	12	0	0	1	
-12	0	0	14	0	0	1	
-14	0	0	14	0	0	1	
-12	0	0	15	0	1/2	1	
-12	0	0	15	3	1/2	1	
-13	1	1/2	15	3	1/2	1	
-14	2	0	15	3	1/2	1	
-15	0	1/2	15	0	1/2	1	
-15	3	1/2	15	3	1/2	1	
-12	0	0	16	0	0	1	
-12	0	0	16	2	0	1	
-12	0	0	16	4	0	1	
-13	1	1/2	16	2	0	1	
-13	1	1/2	16	4	0	1	
-14	0	0	16	0	0	1	
-14	0	0	16	4	0	1	
-14	2	0	16	2	0	1	
-14	2	0	16	4	0	1	
-15	1	1/2	16	4	0	1	
-15	3	1/2	16	2	0	1	
-16	0	0	16	0	0	1	
-16	2	0	16	4	0	1	
-16	4	0	16	2	0	1	
-12	0	0	17	0	1/2	1	
-12	0	0	17	1	1/2	1	
-12	0	0	17	3	1/2	1	
-12	0	0	17	5	1/2	1	
-13	1	1/2	17	1	1/2	1	
-13	1	1/2	17	3	1/2	1	
-13	1	1/2	17	5	1/2	1	
-14	0	0	17	0	1/2	1	
-14	0	0	17	3	1/2	1	
-14	0	0	17	5	1/2	1	
-14	2	0	17	1	1/2	1	
-14	2	0	17	3	1/2	1	
-15	0	1/2	17	0	1/2	1	
-15	1	1/2	17	3	1/2	1	
-15	1	1/2	17	5	1/2	1	
-15	3	1/2	17	1	1/2	1	
-15	3	1/2	17	3	1/2	1	
-16	0	0	17	5	1/2	1	
-16	2	0	17	3	1/2	1	
-16	4	0	17	1	1/2	1	
-17	0	1/2	17	0	1/2	1	
-17	1	1/2	17	5	1/2	1	
-17	3	1/2	17	3	1/2	1	
-17	5	1/2	17	1	1/2	1	
-12	0	0	18	2	0	1	
-12	0	0	18	3	0	1	
-12	0	0	18	4	0	1	
-13	1	1/2	18	2	0	1	
-13	1	1/2	18	3	0	1	
-13	1	1/2	18	4	0	1	
-14	0	0	18	2	0	1	
-14	0	0	18	4	0	1	
-14	2	0	18	2	0	1	
-14	2	0	18	3	0	1	
-14	2	0	18	4	0	1	
-15	0	1/2	18	3	0	1	
-15	1	1/2	18	2	0	1	
-15	1	1/2	18	4	0	1	
-15	3	1/2	18	2	0	1	

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-15	3	1/2	18	3	0	1
-16	0	0	18	4	0	1
-16	1	0	18	3	0	1
-16	2	0	18	2	0	1
-16	2	0	18	4	0	1
-16	4	0	18	2	0	1
-17	1	1/2	18	4	0	1
-17	2	1/2	18	3	0	1
-17	3	1/2	18	2	0	1
-18	2	0	18	4	0	1
-18	3	0	18	3	0	1
-18	4	0	18	2	0	1
-12	0	0	19	0	1/2	1
-12	0	0	19	2	1/2	1
-12	0	0	19	3	1/2	1
-12	0	0	19	4	1/2	1
-12	0	0	19	5	1/2	2
-13	1	1/2	19	2	1/2	1
-13	1	1/2	19	3	1/2	1
-13	1	1/2	19	4	1/2	1
-13	1	1/2	19	5	1/2	2
-14	0	0	19	0	1/2	1
-14	0	0	19	3	1/2	1
-14	0	0	19	4	1/2	1
-14	0	0	19	5	1/2	1
-14	2	0	19	2	1/2	1
-14	2	0	19	3	1/2	1
-14	2	0	19	4	1/2	1
-14	2	0	19	5	1/2	1
-15	0	1/2	19	0	1/2	1
-15	0	1/2	19	2	1/2	1
-15	0	1/2	19	4	1/2	1
-15	0	1/2	19	5	1/2	1
-15	1	1/2	19	3	1/2	1
-15	1	1/2	19	4	1/2	1
-15	1	1/2	19	5	1/2	1
-15	3	1/2	19	2	1/2	1
-15	3	1/2	19	3	1/2	1
-15	3	1/2	19	5	1/2	1
-16	0	0	19	0	1/2	1
-16	0	0	19	3	1/2	1
-16	0	0	19	5	1/2	1
-16	1	0	19	2	1/2	1
-16	1	0	19	4	1/2	1
-16	1	0	19	5	1/2	1
-16	2	0	19	3	1/2	1
-16	2	0	19	4	1/2	1
-16	4	0	19	2	1/2	1
-16	4	0	19	5	1/2	1
-17	0	1/2	19	0	1/2	1
-17	0	1/2	19	4	1/2	1
-17	1	1/2	19	3	1/2	1
-17	1	1/2	19	5	1/2	1
-17	2	1/2	19	2	1/2	1
-17	2	1/2	19	4	1/2	1
-17	3	1/2	19	3	1/2	1
-17	5	1/2	19	5	1/2	1
-18	1	0	19	4	1/2	1
-18	2	0	19	3	1/2	1
-18	3	0	19	2	1/2	1
-19	0	1/2	19	0	1/2	1
-19	2	1/2	19	4	1/2	1
-19	3	1/2	19	3	1/2	1

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-19	4	1/2	19	2	1/2	1	
-12	0	0	20	1	0	1	
-12	0	0	20	3	0	1	
-12	0	0	20	4	0	2	
-12	0	0	20	5	0	1	
-13	1	1/2	20	1	0	1	
-13	1	1/2	20	3	0	1	
-13	1	1/2	20	4	0	2	
-13	1	1/2	20	5	0	1	
-14	0	0	20	3	0	1	
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-13	1	1/2	21	2	1/2	1	
-13	1	1/2	21	4	1/2	1	
-13	1	1/2	21	5	1/2	2	
-14	0	0	21	2	1/2	1	

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-14	0	0	21	4	1/2	1
-14	0	0	21	5	1/2	2
-14	2	0	21	2	1/2	1
-14	2	0	21	4	1/2	1
-14	2	0	21	5	1/2	1
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-18	4	0	21	2	1/2	1
-18	4	0	21	5	1/2	1
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-15	0	1/2	22	5	0	2
-15	1	1/2	22	3	0	1
-15	1	1/2	22	5	0	1
-15	3	1/2	22	3	0	1
-15	3	1/2	22	5	0	1

L-sector			R-sector			H-sector	
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs	
-16	0	0	22	3	0	1	
-16	0	0	22	5	0	1	
-16	1	0	22	3	0	1	
-16	1	0	22	5	0	2	
-16	2	0	22	3	0	1	
-16	4	0	22	5	0	1	
-17	0	1/2	22	3	0	1	
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-15	0	1/2	24	5	0	2	
-15	1	1/2	24	5	0	2	
-15	3	1/2	24	5	0	1	
-16	0	0	24	5	0	1	
-16	1	0	24	5	0	2	

L-sector			R-sector			H-sector
M_L	x_L	α_L	M_R	x_R	α_R	# of Higgs
-16	2	0	24	5	0	1
-16	4	0	24	5	0	1
-17	0	1/2	24	5	0	2
-17	1	1/2	24	5	0	1
-17	2	1/2	24	5	0	1
-17	3	1/2	24	5	0	1
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-18	3	0	24	5	0	1
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-19	2	1/2	24	5	0	1
-19	4	1/2	24	5	0	1
-19	5	1/2	24	5	0	1
-20	1	0	24	5	0	1
-20	3	0	24	5	0	1
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-22	5	0	24	5	0	1

Bibliography

- [1] Y. Nambu and G. Jona-Lasinio, “DYNAMICAL MODEL OF ELEMENTARY PARTICLES BASED ON AN ANALOGY WITH SUPERCONDUCTIVITY. II,” *Phys.Rev.* **124** (1961) 246–254.
- [2] P. W. Higgs, “Broken Symmetries and the Masses of Gauge Bosons,” *Phys.Rev.Lett.* **13** (1964) 508–509.
- [3] S. Weinberg, “A Model of Leptons,” *Phys.Rev.Lett.* **19** (1967) 1264–1266.
- [4] A. Salam, “Weak and Electromagnetic Interactions,” *Conf.Proc.* **C680519** (1968) 367–377.
- [5] S. Glashow, J. Iliopoulos, and L. Maiani, “Weak Interactions with Lepton-Hadron Symmetry,” *Phys.Rev.* **D2** (1970) 1285–1292.
- [6] C. Bouchiat, J. Iliopoulos, and P. Meyer, “An Anomaly Free Version of Weinberg’s Model,” *Phys.Lett.* **B38** (1972) 519–523.
- [7] M. Kobayashi and T. Maskawa, “CP Violation in the Renormalizable Theory of Weak Interaction,” *Prog.Theor.Phys.* **49** (1973) 652–657.
- [8] **Particle Data Group** Collaboration, K. Olive *et al.*, “Review of Particle Physics,” *Chin.Phys.* **C38** (2014) 090001.
- [9] **ATLAS** Collaboration, G. Aad *et al.*, “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” *Phys.Lett.* **B716** (2012) 1–29, [arXiv:1207.7214 \[hep-ex\]](#).
- [10] **CMS** Collaboration, S. Chatrchyan *et al.*, “Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC,” *Phys.Lett.* **B716** (2012) 30–61, [arXiv:1207.7235 \[hep-ex\]](#).
- [11] T. Kaluza, “On the Problem of Unity in Physics,” *Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.)* **1921** (1921) 966–972.
- [12] O. Klein, “Quantum Theory and Five-Dimensional Theory of Relativity. (In German and English),” *Z.Phys.* **37** (1926) 895–906.
- [13] A. Einstein and P. Bergmann, “On a Generalization of Kaluza’s Theory of Electricity,” *Annals Math.* **39** (1938) 683–701.
- [14] A. Einstein, V. Bargmann, and P. Bergmann, “On the Five-Dimensional Representation of Gravitation and Electricity,”.
- [15] N. Manton, “A New Six-Dimensional Approach to the Weinberg-Salam Model,” *Nucl.Phys.* **B158** (1979) 141.
- [16] Y. Hosotani, “Dynamical Mass Generation by Compact Extra Dimensions,” *Phys.Lett.* **B126** (1983) 309.

- [17] H. Hatanaka, T. Inami, and C. Lim, “The Gauge hierarchy problem and higher dimensional gauge theories,” *Mod.Phys.Lett.* **A13** (1998) 2601–2612, [arXiv:hep-th/9805067](#) [[hep-th](#)].
- [18] I. Antoniadis, K. Benakli, and M. Quiros, “Finite Higgs mass without supersymmetry,” *New J.Phys.* **3** (2001) 20, [arXiv:hep-th/0108005](#) [[hep-th](#)].
- [19] Y. Fujimoto, T. Nagasawa, K. Nishiwaki, and M. Sakamoto, “Quark mass hierarchy and mixing via geometry of extra dimension with point interactions,” *PTEP* **2013** (2013) 023B07, [arXiv:1209.5150](#) [[hep-ph](#)].
- [20] Y. Fujimoto, K. Nishiwaki, and M. Sakamoto, “CP phase from twisted Higgs VEV in extra dimension,” *Phys.Rev.* **D88** (2013) 115007, [arXiv:1301.7253](#) [[hep-ph](#)].
- [21] Y. Fujimoto, K. Nishiwaki, M. Sakamoto, and R. Takahashi, “Realization of lepton masses and mixing angles from point interactions in an extra dimension,” [arXiv:1405.5872](#) [[hep-ph](#)].
- [22] C. Bachas, “A Way to break supersymmetry,” [arXiv:hep-th/9503030](#) [[hep-th](#)].
- [23] R. Blumenhagen, L. Goerlich, B. Kors, and D. Lust, “Noncommutative compactifications of type I strings on tori with magnetic background flux,” *JHEP* **0010** (2000) 006, [arXiv:hep-th/0007024](#) [[hep-th](#)].
- [24] R. Blumenhagen, B. Kors, and D. Lust, “Type I strings with F flux and B flux,” *JHEP* **0102** (2001) 030, [arXiv:hep-th/0012156](#) [[hep-th](#)].
- [25] C. Angelantonj, I. Antoniadis, E. Dudas, and A. Sagnotti, “Type I strings on magnetized orbifolds and brane transmutation,” *Phys.Lett.* **B489** (2000) 223–232, [arXiv:hep-th/0007090](#) [[hep-th](#)].
- [26] D. Cremades, L. Ibanez, and F. Marchesano, “Yukawa couplings in intersecting D-brane models,” *JHEP* **0307** (2003) 038, [arXiv:hep-th/0302105](#) [[hep-th](#)].
- [27] N. Manton, “Fermions and Parity Violation in Dimensional Reduction Schemes,” *Nucl.Phys.* **B193** (1981) 502.
- [28] C. Wetterich, “Dimensional Reduction of Weyl, Majorana and Majorana-weyl Spinors,” *Nucl.Phys.* **B222** (1983) 20.
- [29] G. Chapline and R. Slansky, “Dimensional Reduction and Flavor Chirality,” *Nucl.Phys.* **B209** (1982) 461.
- [30] M. Sakamoto and S. Tanimura, “An Extension of Fourier analysis for the n torus in the magnetic field and its application to spectral analysis of the magnetic Laplacian,” *J.Math.Phys.* **44** (2003) 5042–5069, [arXiv:hep-th/0306006](#) [[hep-th](#)].
- [31] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, “Strings on Orbifolds,” *Nucl.Phys.* **B261** (1985) 678–686.
- [32] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, “Strings on Orbifolds. 2.,” *Nucl.Phys.* **B274** (1986) 285–314.
- [33] Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, “Shifted orbifold models with magnetic flux,” *Phys.Rev.* **D87** (2013) 086001, [arXiv:1302.5768](#) [[hep-th](#)].
- [34] K.-S. Choi and J. E. Kim, “Quarks and leptons from orbifolded superstring,” *Lect.Notes Phys.* **696** (2006) 1–406.

- [35] L. E. Ibanez, H. P. Nilles, and F. Quevedo, “Orbifolds and Wilson Lines,” *Phys.Lett.* **B187** (1987) 25–32.
- [36] T. Kobayashi and N. Ohtsubo, “Analysis on the Wilson lines of $Z(N)$ orbifold models,” *Phys.Lett.* **B257** (1991) 56–62.
- [37] T. Kobayashi and N. Ohtsubo, “Geometrical aspects of $Z(N)$ orbifold phenomenology,” *Int.J.Mod.Phys.* **A9** (1994) 87–126.
- [38] J. Scherk and J. H. Schwarz, “Spontaneous Breaking of Supersymmetry Through Dimensional Reduction,” *Phys.Lett.* **B82** (1979) 60.
- [39] J. Scherk and J. H. Schwarz, “How to Get Masses from Extra Dimensions,” *Nucl.Phys.* **B153** (1979) 61–88.
- [40] H. Abe, T. Kobayashi, and H. Ohki, “Magnetized orbifold models,” *JHEP* **0809** (2008) 043, [arXiv:0806.4748 \[hep-th\]](#).
- [41] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, “Three generation magnetized orbifold models,” *Nucl.Phys.* **B814** (2009) 265–292, [arXiv:0812.3534 \[hep-th\]](#).
- [42] T.-h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, “ Z_N twisted orbifold models with magnetic flux,” *JHEP* **1401** (2014) 065, [arXiv:1309.4925 \[hep-th\]](#).
- [43] T.-h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, and M. Sakamoto, “Operator analysis of physical states on magnetized T^2/Z_N orbifolds,” *Nucl.Phys.* **B890** (2014) 442–480, [arXiv:1409.5421 \[hep-th\]](#).
- [44] I. Antoniadis, A. Kumar, and B. Panda, “Fermion Wavefunctions in Magnetized branes: Theta identities and Yukawa couplings,” *Nucl.Phys.* **B823** (2009) 116–173, [arXiv:0904.0910 \[hep-th\]](#).
- [45] D. Cremades, L. Ibanez, and F. Marchesano, “Computing Yukawa couplings from magnetized extra dimensions,” *JHEP* **0405** (2004) 079, [arXiv:hep-th/0404229 \[hep-th\]](#).
- [46] H. Abe, T. Kobayashi, H. Ohki, and K. Sumita, “Superfield description of 10D SYM theory with magnetized extra dimensions,” *Nucl.Phys.* **B863** (2012) 1–18, [arXiv:1204.5327 \[hep-th\]](#).
- [47] H. Abe, T. Kobayashi, K. Sumita, and Y. Tatsuta, “Gaussian Froggatt-Nielsen mechanism on magnetized orbifolds,” [arXiv:1405.5012 \[hep-ph\]](#).
- [48] H. Abe, T. Kobayashi, H. Ohki, K. Sumita, and Y. Tatsuta, “Non-Abelian discrete flavor symmetries of 10D SYM theory with magnetized extra dimensions,” *JHEP* **1406** (2014) 017, [arXiv:1404.0137 \[hep-th\]](#).
- [49] T.-h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto, and Y. Tatsuta, “Classification of three-generation models on magnetized orbifolds,” *Nucl.Phys.* **B894** (2015) 374–406, [arXiv:1501.02787 \[hep-ph\]](#).
- [50] H. Abe, K.-S. Choi, T. Kobayashi, and H. Ohki, “Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models,” *Nucl.Phys.* **B820** (2009) 317–333, [arXiv:0904.2631 \[hep-ph\]](#).
- [51] M. Berasaluce-Gonzalez, P. Camara, F. Marchesano, D. Regalado, and A. Uranga, “Non-Abelian discrete gauge symmetries in 4d string models,” *JHEP* **1209** (2012) 059, [arXiv:1206.2383 \[hep-th\]](#).

- [52] F. Marchesano, D. Regalado, and L. Vazquez-Mercado, “Discrete flavor symmetries in D-brane models,” *JHEP* **1309** (2013) 028, [arXiv:1306.1284 \[hep-th\]](#).
- [53] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby, and M. Ratz, “Stringy origin of non-Abelian discrete flavor symmetries,” *Nucl.Phys.* **B768** (2007) 135–156, [arXiv:hep-ph/0611020 \[hep-ph\]](#).
- [54] P. Ko, T. Kobayashi, J.-h. Park, and S. Raby, “String-derived D(4) flavor symmetry and phenomenological implications,” *Phys.Rev.* **D76** (2007) 035005, [arXiv:0704.2807 \[hep-ph\]](#).
- [55] Y. Hamada, T. Kobayashi, and S. Uemura, “Flavor structure in D-brane models: Majorana neutrino masses,” [arXiv:1402.2052 \[hep-th\]](#).
- [56] H. Abe, T. Kobayashi, H. Ohki, A. Oikawa, and K. Sumita, “Phenomenological aspects of 10D SYM theory with magnetized extra dimensions,” *Nucl.Phys.* **B870** (2013) 30–54, [arXiv:1211.4317 \[hep-ph\]](#).
- [57] A. Djouadi, “The Anatomy of electro-weak symmetry breaking. II. The Higgs bosons in the minimal supersymmetric model,” *Phys.Rept.* **459** (2008) 1–241, [arXiv:hep-ph/0503173 \[hep-ph\]](#).
- [58] Y. Hamada and T. Kobayashi, “Massive Modes in Magnetized Brane Models,” *Prog.Theor.Phys.* **128** (2012) , 903–923, [arXiv:1207.6867 \[hep-th\]](#).