

Kyoto University
Graduate School of Science

Revisiting the Dictionary of Celestial Holography
and Bridging with (A)dS/CFT

Doctoral Thesis
2025

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Abstract

Celestial holography is a holographic correspondence proposed for asymptotically flat spacetimes. It asserts that quantum gravity in $D+2$ dimensional asymptotically flat spacetime is equivalent to a D dimensional conformal field theory (CFT) living on the celestial sphere, with flat-space scattering amplitudes identified with CFT correlation functions. This proposal is motivated by the infrared structure of gauge and gravitational theories: in particular, for $D = 2$, the asymptotic symmetry of asymptotically flat spacetime implying that the S -matrix enjoys the symmetries of a two-dimensional CFT, the Virasoro symmetry. Nevertheless, many aspects of celestial holography remain obscure, and no fully explicit CFT model has yet been established.

In this thesis, we first uncover a serious inconsistency in the conventional construction of the celestial dictionary. Using the standard Mellin-transform prescription, the celestial two-point functions for massless external states exhibit distributional (delta-function-like) behavior, which is incompatible with the power-law behavior expected from generic CFT two-point functions and with previously computed higher-point correlators. We trace this problem to the dictionary of the celestial operator and resolve it by redefining the dictionary: outgoing particles are mapped to non-shadow celestial operators, while incoming particles are mapped to their shadow counterparts. We refer to this revised prescription as the rectified dictionary. With this modification, celestial two-point functions acquire the standard CFT form and the dictionary for massive case smoothly connect to the massless case in the massless limit, allowing celestial holography to be treated as a consistent CFT framework.

After we construct the rectified dictionary, we then pursue two complementary reinterpretations of celestial holography in terms of more familiar holographic dualities. The first approach employs a Milne slicing of $D+2$ dimensional Minkowski spacetime to decompose it into three wedges, to which we apply wedge holography, a codimension-two holographic correspondence originally developed in the AdS/BCFT context. This yields a D dimensional CFT whose central charge and scalar spectrum are found to be consistent with expectations from celestial holography. The second approach starts from a Euclidean flat-space action and, by combining a Weyl transformation, a Fourier transform, and an analytic continuation, constructs a map between theories on $D+2$ dimensional Minkowski spacetime and theories on $D+1$ dimensional de Sitter spacetime. Within this framework, massless field theories in Minkowski spacetime are mapped to theories in de Sitter spacetime with a continuous mass spectrum, and one can relate celestial operators defined by Mellin transforms to cosmological operators defined from the early/late-time behavior of de Sitter fields. This “cosmological–celestial dictionary” allows us to reinterpret conformal soft theorems as Ward–Takahashi identities of cosmological correlators, thereby providing a dS/CFT-inspired perspective on celestial holography.

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1 Introduction

In the recent particle physics theory, one of the central open problems is to construct a consistent theory of quantum gravity. Among the various approaches to this problem, the holographic principle [1–3] has attracted particular attention. It proposes that a quantum theory of gravity defined on a given spacetime can be equivalently described by a non-gravitational quantum field theory living on a lower-dimensional spacetime. The most thoroughly understood realization of this idea is the AdS/CFT correspondence [4–7]. This conjecture states that a quantum gravity theory on a $d+1$ dimensional asymptotically anti de-Sitter (AdS) spacetime is equivalent to a d dimensional conformal field theory (CFT) defined on the boundary of the asymptotically AdS spacetime. A major challenge in holography is to extend such correspondences beyond asymptotically AdS geometries. For de-Sitter (dS) spacetimes, the dS/CFT correspondence has been proposed [8–10]. Moreover, for asymptotically flat spacetimes, a different holographic framework known as celestial holography has been put forward. This thesis is devoted primarily to the study of this celestial holography.

The central claim of celestial holography is that scattering amplitudes of particles in a $d+1$ dimensional asymptotically flat spacetime are dual to correlation functions of a conformal field theory defined on the $d-1$ dimensional celestial sphere $\mathcal{CS}^{d-1}$ [11–16]. The validity of celestial holography is supported by the structure of infrared physics in gauge theories, encapsulated in the so-called infrared triangle [17–19]. In particular, in order to interpret scattering amplitudes as correlators of a two-dimensional CFT on the celestial sphere, the amplitudes must exhibit the symmetries of such a CFT; in the two-dimensional case, this requires the presence of a Virasoro symmetry. This Virasoro symmetry of the S -matrix can be demonstrated by deriving Ward–Takahashi identities associated with the extended BMS group, which is a natural enlargement of the asymptotic symmetry group of asymptotically flat spacetimes [20]. This observation provides a key piece of evidence in favor of the celestial holography proposal. This construction realizes a codimension-two holography and, at first sight, differs conceptually from more conventional holographic correspondences such as AdS/CFT. As a consequence, many fundamental aspects of the celestial CFT, including its detailed structure and the mechanism of bulk reconstruction, remain poorly understood. In this thesis, we propose the clue for these open problems by attempting to reinterpret celestial holography in terms of more familiar holographic dualities such as AdS/CFT and dS/CFT.

Before discussing the interpretation of celestial holography, let us first point out a serious structural issue in its current formulation, namely the form of the celestial dictionary [21]. For massless external particles, the celestial correlators are defined via a Mellin transform of the scattering amplitude. In particular, the n -point celestial correlator is given in terms of the scattering amplitude $\mathcal{A}$ by [12, 14]

$$\left\langle \prod_{i=1}^n \mathcal{O}_{\Delta_i} \right\rangle = \left[\prod_{j=1}^n \int d\omega_j \omega_j^{\Delta_j-1} \right] \mathcal{A}(p_1, \dots, p_n). \quad (1.1)$$

With this definition, however, the resulting two-point function exhibits a delta-function-like behavior, rather than the familiar power-law behavior characteristic of

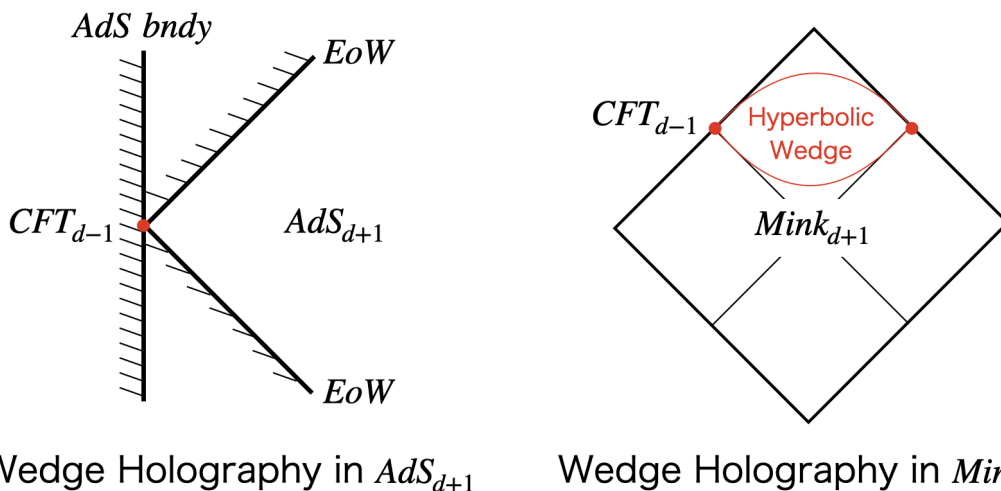


Figure 1: A schematic comparison between the original wedge holography setup in AdS_{d+1} (left) [22] and our flat-space wedge holography construction in $Mink_{d+1}$ (right) [23].

generic CFT two-point functions. By contrast, the higher point function obtained from the same dictionary reproduces the standard CFT form. This mismatch implies that the ordinary dictionary is not compatible with the operator product expansion and, therefore, is conceptually unsatisfactory. Motivated by this problem, we revisit the celestial dictionary and propose a rectified prescription. For massless particles, we define outgoing states to correspond to ordinary celestial operators, while incoming states are identified with shadow celestial operators. With this assignment, the two-point function acquires the standard power-law form expected in a CFT and becomes consistent with the three-point function. Moreover, we find that the massless limit of the massive conformal wave function is also compatible with massless ones, providing further evidence for its consistency. These observations indicate that, in order to treat celestial correlators as genuine CFT correlators in a natural way, it is necessary to adopt our rectified dictionary.

As a first strategy to reinterpret celestial holography in terms of more familiar holographic dualities, we make use of the so-called Milne slicing, which foliates Minkowski spacetime by Euclidean AdS (EAdS) and dS slices [23]. Using this foliation, a $d+1$ dimensional Minkowski spacetime can be decomposed into three regions, $EAdS_+ + dS + EAdS_-$ [23–27]. For each of these three regions, we then apply the idea of wedge holography [22, 28, 29], a variant of the AdS/BCFT correspondence, to relate them to a $d-1$ dimensional CFT. Wedge holography, like celestial holography, realizes a codimension-two holographic duality. It is implemented by inserting two end-of-the-world (EoW) branes into AdS_{d+1} such that they intersect on the AdS boundary. This configuration leads to a correspondence between the bulk AdS_{d+1} spacetime and a CFT_{d-1} living on the corner defined by the intersection of the two EoW branes. The aim of this work is to apply this construction to flat spacetime in order to identify the properties of the CFT arising in celestial holography, in particular its central charge (see Fig. 1).

In the second strategy, we reinterpret celestial holography through the lens of

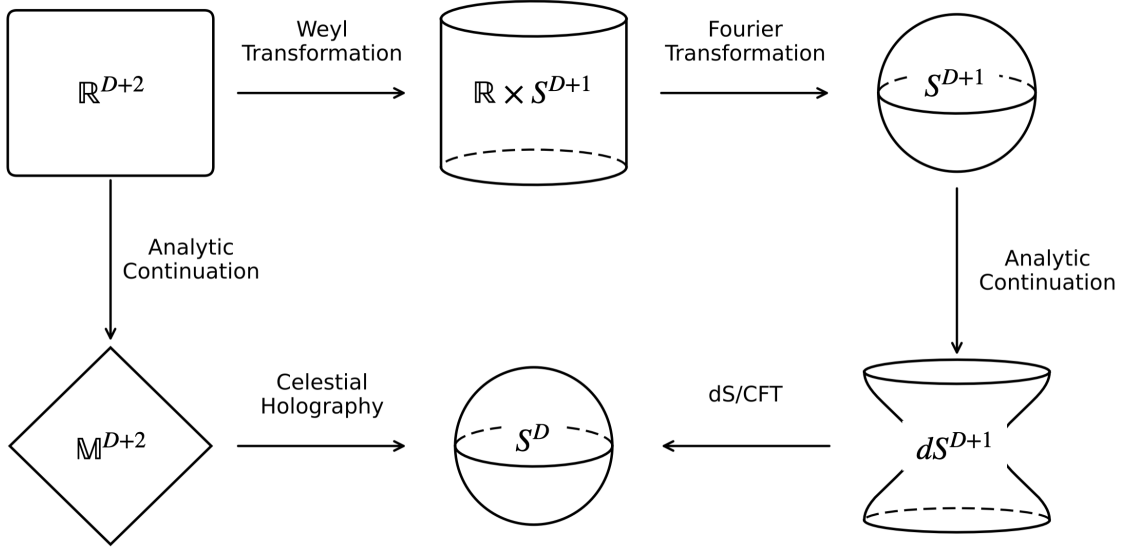


Figure 2: A schematic illustration of the intersection between celestial holography and the dS/CFT correspondence [30, 31].

the dS/CFT correspondence via Weyl transformation and analytic extension [30, 31]. In [30], we start from a massless scalar field theory on flat space $\mathbb{R}^{D+2}$. Likely in [32], we first perform a Weyl transformation to rewrite the geometry as $S^{D+1} \times \mathbb{R}$ and then apply a Fourier transformation along the $\mathbb{R}$ direction. This procedure recasts the theory as one defined on S^{D+1} with a continuous mass spectrum, and, upon analytic continuation, establishes an equivalence between a massless scalar field on Minkowski space $\mathbb{M}_{D+2}$ and a theory on de Sitter space dS_{D+1} in the Bunch–Davies vacuum [33–38] (See Fig. 2). Within this framework, celestial scalar operators $\mathcal{O}_\Delta$ defined as

$$\mathcal{O}_\Delta(p) = \int_0^\infty \frac{d\omega}{\omega} \omega^\Delta a(\omega p), \quad (1.2)$$

are naturally identified with asymptotic cosmological operators V_Δ and W_Δ on S^D defined by early/late time limit of bulk field

$$\psi_\lambda \sim e^{\mp(D-\Delta)t} V_{D-\Delta} + e^{\mp\Delta t} W_\Delta \quad (t \rightarrow \pm\infty), \quad (1.3)$$

related as

$$\begin{aligned} V_\Delta(p) &= N_\lambda i^{-\Delta} \widetilde{\mathcal{O}_{D-\Delta}}(p), \\ W_\Delta(p) &= N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p), \end{aligned} \quad (1.4)$$

where $N_\lambda = \frac{\Gamma(-i\lambda)}{2^{5/2}\pi^{D/2+3/2}}$. In [31], we extend this construction from scalars to gauge fields on $\mathbb{M}_{D+2}$. We show that a $U(1)$ gauge theory is mapped to a Proca field on dS_{D+1} with a continuous mass spectrum, while linearized gravity is mapped to a Fierz–Pauli field with a continuous mass spectrum. The gauge symmetries and asymptotic symmetries in $\mathbb{M}_{D+2}$ are then realized on dS_{D+1} as the gauge symmetry of a massless spin-1 field and as symmetries of partially massless and massless spin-2

fields. Consequently, the Ward–Takahashi identities associated with supertranslation and superrotation symmetries, which in celestial holography are encoded in conformal soft operators with dimensions $\Delta = 1$ and $\Delta = 0$, are shown to coincide with the corresponding Ward–Takahashi identities of cosmological correlators.

This thesis is organized as follows.

Section 2

We review the infrared (IR) structure of gauge and gravitational theories, with particular emphasis on the physics in the low-energy regime. In particular, we focus on the Ward identities that relate soft theorems to asymptotic symmetries and thereby establish their equivalence.

Section 3

In this chapter, we review celestial holography, a proposed holographic framework for asymptotically flat spacetimes. In the final part of the section, we also discuss an attempt to reinterpret celestial holography from an (A)dS/CFT perspective, following the construction developed in our work presented below.

N. Ogawa, T. Takayanagi, T. Tsuda, and T. Waki, “Wedge holography in flat space and celestial holography,” *Phys. Rev. D* 107 no. 2, (2023) 026001

Section 4

We have identified an incompleteness in the existing formulation of celestial holography, which in turn reveals the need for a redefine of the operator dictionary. The discussion in this section is based on our accompanying work presented in the papers below.

H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, “Celestial two-point functions and rectified dictionary,” *JHEP* 02 (2024) 063

Section 5

By performing an appropriate change of coordinates on fields in flat spacetime and subsequently analytically continuing them, we reinterpret these fields as living on de Sitter space. This procedure allows us to construct an correspondence between operators in the celestial CFT and cosmological operators, thereby providing a bridge between celestial holography and the dS/CFT correspondence. The material presented in this section is based on [30] and the following paper.

H. Furugori, N. Ogawa, S. Sugishita, and T. Waki, “Bridging dS/CFT toward Celestial Holography via Ward-Takahashi Identities,” arXiv.2512.10367

Section 6

We summarize our study and comment on its future direction.

2 Review : Infrared Structure of Gravity

In this section we review the infrared (IR) structure of gravitational theories. By “IR structure” we mean the organization of physics in the low-energy sector of gauge and gravity theories. A central organizing principle is the infrared triangle, which asserts an equivalence among three manifestations of infrared physics : soft theorems, asymptotic symmetries, and memory effects [17].

Soft theorems characterize the universal low-energy behavior of scattering processes involving massless quanta. When the energy ω of an external massless particle is taken to zero, the corresponding scattering amplitude admits an expansion in ω ; the leading term exhibits a ω^{-1} divergence—the Weinberg pole. This universality renders soft theorems both conceptually and observationally significant.

Asymptotic symmetries refer to the nontrivial symmetry group and associated conserved charges acting at the boundaries of spacetime. In asymptotically flat gravity, Bondi, van der Burg, Metzner, and Sachs (BMS) demonstrated that the symmetry group at null infinity extends the Poincaré group by an infinite-dimensional set of transformations, with concrete implications for the structure of gravitational radiation and its boundary data [39–41].

Memory effects are non-oscillatory, persistent imprints left in the asymptotic region by transient radiation. In electromagnetism and gravity, a burst of waves can induce a permanent change in the relative separation of test particles once the radiation has passed—an effect referred to as (electromagnetic or gravitational) memory [42, 43]. The gravitational memory effect is believed to be observable with current and near-future facilities [44–46].

The infrared triangle links these three facets by showing, in particular, that the soft theorems are equivalent to Ward identities of the corresponding asymptotic symmetries, and that these symmetries control the memory observables. In what follows we concentrate on the Ward identity formulation of the equivalence between soft theorems and asymptotic symmetries, as this will play a central role in the subsequent discussion of celestial holography.

2.1 Soft Graviton Theorem

Soft theorems constrain scattering amplitudes whenever massless quanta can be emitted at very low energy [47]. In gauge theory, accelerated charges radiate – bremsstrahlung in QED and gravitational waves in gravity – and these radiative quanta can be treated as additional external legs carrying four-momentum $q^\mu = (\omega, \vec{q})$ with $\omega \rightarrow 0$. The central statement is that the $(n+1)$ -point amplitude with one extra soft quantum factorizes onto the n -point hard amplitude, with a universal expansion in powers of the soft energy:

$$\lim_{\omega \rightarrow 0} \mathcal{A}_{n+1}(q) = \left[S_n^{(0)} + S_n^{(1)} + O(\omega) \right] \mathcal{A}_n. \quad (2.1)$$

soft photon theorem

For QED the soft particle is the photon. Below we review the leading (Weinberg) and subleading photon factors and their gauge properties, following [48–50] and

the classic derivations [47, 51]. Throughout, the n -point hard amplitude $\mathcal{A}_n$ carries external momenta p_k^μ and charges Q_k , and we take $\eta_k = +1$ for incoming and $\eta_k = -1$ for outgoing hard legs.

At leading order in ω the only contributions arise from attaching the soft photon to the external charged legs. For emission from leg k , the diagram factorizes into the product of the soft photon propagator/vertex and the hard amplitude,

$$\mathcal{A}_{n+1,k} = (\text{propagator}) \times (\text{vertex}) \times \mathcal{A}_n.$$

With scalar QED interaction $L_{\text{int}} = -A^\mu j_\mu$ and current $j_\mu = ie(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi)$, the contributions are

$$\text{propagator : } \frac{-i}{(p_k + \eta_k q)^2 - m^2} \xrightarrow{\text{on shell, } q^2=0} \frac{-i}{2\eta_k p_k \cdot q}, \quad (2.2)$$

$$\text{vertex : } ie(2p_{k\mu} + \eta_k q_\mu) \varepsilon^\mu(q) = 2ie p_k \cdot \varepsilon(q) + \mathcal{O}(\omega), \quad (2.3)$$

so that

$$\begin{aligned} \mathcal{A}_{n+1,k} &= \left[\frac{-i}{2\eta_k p_k \cdot q} \right] [2ie p_k \cdot \varepsilon(q) + \mathcal{O}(\omega)] \mathcal{A}_n \\ &= \left[e\eta_k \frac{p_k \cdot \varepsilon(q)}{p_k \cdot q} + \mathcal{O}(\omega^0) \right] \mathcal{A}_n. \end{aligned} \quad (2.4)$$

Summing over legs yields the universal leading soft factor

$$S_n^{(0)} = \sum_{k=1}^n \eta_k e Q_k \frac{p_k \cdot \varepsilon(q)}{p_k \cdot q} \propto \omega^{-1} \quad (2.5)$$

which scales as ω^{-1} , called Weinberg pole.

At order ω^0 one must include two classes of diagrams [50]: (a) external emissions, upgraded by next-to-soft Feynman rules, and (b) internal emissions, where the soft photon is attached to internal lines. Their sum organizes into a compact operator acting on the hard amplitude:

$$S_n^{(1)} = -i \sum_{k=1}^n \eta_k e Q_k \frac{q_\mu \mathcal{J}_k^{\mu\nu} \varepsilon_\nu(q)}{p_k \cdot q}, \quad (2.6)$$

where $\mathcal{J}_k^{\mu\nu} = L_k^{\mu\nu} + S_k^{\mu\nu}$ is the total angular momentum operator for leg k .

The polarization has a gauge redundancy $\varepsilon^\mu(q) \rightarrow \varepsilon^\mu(q) + f(q)q^\mu$. For the leading factor, the shift changes

$$\delta S_n^{(0)} = f(q) \sum_k \eta_k e Q_k = 0, \quad (2.7)$$

which vanishes by charge conservation, so the leading factor is gauge invariant. For the subleading factor, antisymmetry of $\mathcal{J}_k^{\mu\nu}$ implies $q_\mu q_\nu \mathcal{J}_k^{\mu\nu} = 0$, so the subleading factor is also gauge invariant. Thus the soft expansion (2.1) respects gauge symmetry at each displayed order.

soft graviton theorem

In gravity, the leading and subleading universal soft factors for the emission of a graviton with momentum q and helicity $\pm$ are [47, 52]

$$S_n^{(0)\pm} = \frac{\kappa}{2} \sum_{k=1}^n \eta_k \frac{(p_k \cdot \varepsilon^\pm(q))^2}{p_k \cdot q}, \quad S_n^{(1)\pm} = -\frac{i\kappa}{2} \sum_{k=1}^n \eta_k \frac{(p_k \cdot \varepsilon^\pm(q)) \cdot (q \cdot \mathcal{J}_k \cdot \varepsilon^\pm(q))}{p_k \cdot q}, \quad (2.8)$$

where $\varepsilon_{\mu\nu}^\pm(q)$ is the graviton polarization tensor and $\mathcal{J}_k^{\mu\nu}$ is the total angular momentum operator for leg k . The soft factors are invariant under the gauge shift of the polarization tensor

$$\varepsilon_{\mu\nu}(q) \rightarrow \varepsilon_{\mu\nu}(q) + \Lambda_\mu(q)q_\nu + \Lambda_\nu(q)q_\mu. \quad (2.9)$$

Indeed,

$$\begin{aligned} (p_k \cdot \varepsilon^\pm(q))^\mu &\rightarrow (p_k \cdot \varepsilon^\pm(q))^\mu + (p_k \cdot q)\Lambda^\mu(q) + (p_k \cdot \Lambda(q))q^\mu, \\ (q \cdot \mathcal{J}_k \cdot \varepsilon^\pm(q))^\mu &\rightarrow (q \cdot \mathcal{J}_k \cdot \varepsilon^\pm(q))^\mu + (q \cdot \mathcal{J}_k \cdot \Lambda(q))q^\mu, \end{aligned} \quad (2.10)$$

where antisymmetry of $\mathcal{J}_k^{\mu\nu}$ implies $q \cdot \mathcal{J}_k \cdot q = 0$. For the leading factor, all gauge-variant pieces cancel using total momentum conservation $\sum_k \eta_k p_k = 0$, whence $S_n^{(0)\pm}$ is gauge invariant. For the subleading factor, the remaining variation is proportional to $\sum_k \eta_k (q \cdot \mathcal{J}_k)$ and therefore cancels by total angular momentum conservation $\sum_k \eta_k \mathcal{J}_k^{\mu\nu} = 0$ acting on the hard amplitude. Thus $S_n^{(1)\pm}$ is gauge invariant as well [17, 52].

2.2 Asymptotically Flat Spacetimes

We work in retarded Bondi coordinates (u, r, x^A) and impose the Bondi gauge [39, 40]

$$g_{rr} = 0, \quad g_{rA} = 0, \quad \partial_r \det \left(\frac{g_{AB}}{r^2} \right) = 0, \quad (2.11)$$

so that the line element takes the form

$$ds^2 = g_{uu}du^2 + 2g_{ur}dudr + 2g_{uA}dudx^A + g_{AB}dx^A dx^B. \quad (2.12)$$

“Asymptotically flat” means that the causal structure at null infinity matches that of Minkowski space and that suitable fall-off conditions are imposed so that conserved charges are finite and the asymptotic structure is stable. In four-dimensional Minkowski spacetime, the Bondi metric reads

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad \gamma_{z\bar{z}} \equiv \frac{2}{(1+z\bar{z})^2}, \quad (2.13)$$

and we impose the standard fall-offs [53, 54]

$$\begin{aligned} g_{uu} &= -1 + \mathcal{O}(r^{-1}), & g_{ur} &= -1 + \mathcal{O}(r^{-2}), & g_{uz} &= \mathcal{O}(1), \\ g_{zz} &= \mathcal{O}(r), & g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} + \mathcal{O}(1). \end{aligned} \quad (2.14)$$

Here, we use stereographic coordinate $(z, \bar{z})$, see App.A. Near $\mathcal{I}^+$, a general asymptotically flat metric in Bondi gauge admits the expansion [17, 55]

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ & + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + \left[D^z C_{zz} + \frac{1}{r} \left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}D_z(C_{zz}C^{zz}) \right) \right] dudz + \text{c.c.} \\ & + (\text{sub-subleading terms}), \end{aligned} \quad (2.15)$$

where D_z is the covariant derivative with respect to the round metric $\gamma_{z\bar{z}}$ on S^2 , m_B is the Bondi mass aspect, and N_z is the angular momentum aspect. The News tensor is

$$N_{zz} \equiv \partial_u C_{zz}, \quad (2.16)$$

and $N_{AB}N^{AB}$ controls the gravitational-wave energy flux through null infinity.

2.3 Asymptotic Symmetry Group

In this part, we introduce the concept of asymptotic symmetries, which plays a crucial role in defining physical quantities in the asymptotic regions of spacetime. This refers to a symmetry that acts non-trivially at the boundary of spacetime while preserving the prescribed boundary conditions there. In general, a symmetry transformation is defined as a transformation of the fields that leaves the action invariant up to a boundary term. Among such symmetries, those whose generators depend arbitrarily on spacetime coordinates are called gauge symmetries.

Gauge transformations are naturally divided into two classes, usually referred to as “small” and “large” gauge transformations. A small gauge transformation is one whose gauge parameter vanishes at the boundary; such transformations are regarded as pure redundancies, as they do not change the physical state. By contrast, a large gauge transformation has a gauge parameter that approaches a non-vanishing value at the boundary. These transformations act non-trivially on the physical state and are therefore associated with non-zero conserved charges. Equivalently, small gauge transformations can be viewed as those with vanishing charge, while large ones carry non-zero charge.

In what follows, we regard large gauge transformations as genuine physical symmetries and treat small gauge transformations as redundancies. The asymptotic symmetry group (ASG) is then defined as [56]

$$\text{ASG} = \frac{\text{Allowed gauge symmetry}}{\text{Trivial gauge symmetry}}, \quad (2.17)$$

that is, as the quotient of all gauge transformations compatible with the boundary conditions by those that act trivially on the physical states. Taking this quotient is conceptually equivalent to partially fixing the gauge:

$$\text{ASG} = \frac{\text{Allowed gauge symmetry}}{\text{Trivial gauge symmetry}} \xrightarrow{\text{partial gauge fixing}} \frac{\text{Allowed gauge symmetry}}{\text{Trivial gauge symmetry}}. \quad (2.18)$$

The gauge transformations that remain unfixed after imposing such a partial gauge choice are referred to as the residual gauge symmetries; these furnish a representative description of the ASG (See also [57]).

Here, “allowed gauge symmetry” denotes the set of all gauge transformations that preserve the chosen boundary conditions. When we work at infinity, these boundary conditions include assumptions about how fast the fields approach a reference background. Such requirements are known as fall-off conditions. In general there is no unique, canonical choice of fall-off conditions: they must be imposed by hand in a way that is appropriate to the physical situation one wishes to describe, and different choices may lead to different asymptotic symmetry groups.

The asymptotic symmetry group of four-dimensional asymptotically flat space-times is obtained by quotienting diffeomorphisms that preserve a Bondi gauge representative and its fall-offs by those that act trivially on the radiative data at null infinity. Concretely, start from retarded Bondi coordinates $(u, r, z, \bar{z})$ and impose the gauge conditions (2.11) together with the large- r fall-offs (2.14). An infinitesimal vector ξ^μ is an allowed asymptotic symmetry if it preserves both the gauge and the fall-offs,

$$\begin{aligned} \mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad \mathcal{L}_\xi \det \frac{g_{AB}}{r^2} = 0, \\ \mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = \mathcal{O}(1), \quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r). \end{aligned} \quad (2.19)$$

Solving these conditions yields the general form [20, 55]

$$\begin{aligned} \xi = & \left(1 + \frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^z D_z Y^z \partial_z - \frac{1}{2}(u+r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + \text{c.c.} \\ & + f \partial_u - \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r + \dots, \end{aligned} \quad (2.20)$$

where $f(z, \bar{z})$ is an arbitrary function on the celestial sphere and $Y^A = (Y^z, Y^{\bar{z}})$ is a vector field on S^2 . The f -sector generates supertranslations,

$$\xi_f = f \partial_u - \frac{1}{r} (D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}) + D^z D_z f \partial_r + \dots, \quad (2.21)$$

which shift retarded time by an angle-dependent amount at $\mathcal{I}^+$. The Y -sector generates superrotations,

$$\xi_Y = \left(1 + \frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^z \partial_z - \frac{1}{2}(u+r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + \text{c.c.} + \dots, \quad (2.22)$$

and preserving the fall-offs requires Y^A to be a conformal Killing vector (CKV) of the round metric γ_{AB} on S^2 ,

$$D_A Y_B + D_B Y_A = \gamma_{AB} D_C Y^C \iff \partial_{\bar{z}} Y^z = 0, \quad \partial_z Y^{\bar{z}} = 0. \quad (2.23)$$

The globally well-defined CKVs form a finite six-dimensional subalgebra generated by $Y^z = z^{k+1}$ with $k = -1, 0, 1$, which reproduces the Lorentz transformations.

Similarly, the subset of supertranslations that reproduces the four global translations is spanned by

$$f_0 = 1, \quad f_1 = \frac{z + \bar{z}}{1 + z\bar{z}}, \quad f_2 = i \frac{\bar{z} - z}{1 + z\bar{z}}, \quad f_3 = \frac{1 - z\bar{z}}{1 + z\bar{z}}. \quad (2.24)$$

These facts imply that the ASG of asymptotically flat space contains the Poincaré group and enhances translations to angle-dependent shifts:

$$\text{BMS} = \text{Lorentz} \ltimes \text{Supertranslation}. \quad (2.25)$$

In Bondi gauge the Killing fields ξ obeying $\mathcal{L}_\xi \eta_{\mu\nu} = 0$ take the same functional form as (2.20) but with f and Y restricted to the four translations and six global CKVs, respectively. Writing, in stereographic coordinates on the sphere, the six CKVs as

$$\begin{aligned} Y_{12}^z &= iz, & Y_{13}^z &= -\frac{1}{2}(1 + z^2), & Y_{23}^z &= -\frac{i}{2}(1 - z^2), \\ Y_{03}^z &= z, & Y_{01}^z &= -\frac{1}{2}(1 - z^2), & Y_{02}^z &= -\frac{i}{2}(1 + z^2), \end{aligned}$$

one recovers the Poincaré algebra with $[\xi_f, \xi_{f'}] = 0$ and $[\xi_Y, \xi_{Y'}] = \xi_{[Y, Y']}$, i.e. Poincaré = Lorentz $\ltimes$ Translation.

Two enlargements of the BMS group are frequently discussed in the literature. Allowing locally holomorphic solutions of (2.23) (meromorphic vector fields with isolated poles) extends global CKVs to a pair of Virasoro algebras and gives the “extended BMS” group [55, 58–60],

$$\text{extended BMS} = \text{Vir} \times \text{Vir} \ltimes \text{Supertranslation}, \quad (2.26)$$

central to celestial holography. Alternatively, one may relax (2.23) to accept all smooth diffeomorphisms of the sphere, yielding the “generalized BMS” group with superrotations $Y^A \in \text{Diff}(S^2)$ [61]. In both cases, the generators are still of the form (2.20), but the admissible Y^A are enlarged, and one must correspondingly refine fall-offs and charge definitions to ensure finite, integrable charges in the covariant phase space [55].

2.4 Superrotation Charge

The superrotation charge on future null infinity $\mathcal{I}^+$ splits into a hard part and a soft part,

$$Q^+(Y) = Q_H^+(Y) + Q_S^+(Y), \quad (2.27)$$

with the standard Bondi-gauge expressions (indices on the sphere are raised and lowered with $\gamma_{z\bar{z}}$)

$$\begin{aligned} Q_H^+(Y) &= \frac{1}{4} \int_{\mathcal{I}^+} du d^2 z \gamma_{z\bar{z}} \left[u D_z Y^z N_{zz} N^{zz} - Y^z D_z (C_{zz} N^{zz}) - 2 Y^z C_{zz} D_z N^{zz} + (\text{matter}) \right] + \text{c.c.}, \\ Q_S^+(Y) &= -\frac{1}{2} \int_{\mathcal{I}^+} du d^2 z D_z^3 Y^z u N^z_{\bar{z}} + \text{c.c.} \end{aligned} \quad (2.28)$$

Here $C_{zz}(u, z, \bar{z})$ is the shear, $N_{zz} = \partial_u C_{zz}$ is the News tensor, and $N^z_{\bar{z}} := \gamma^{z\bar{z}} N_{\bar{z}\bar{z}}$. By construction, the “soft part” is linear in the News. Here the charge Q associated with a symmetry generated by ξ is defined such that

$$[Q(\xi), f] = \delta_\xi f. \quad (2.29)$$

This charge is constructed within the covariant phase space formalism [62–64]. For further details, see [65, 66].

Soft part in creation/annihilation modes.

To make the meaning of Q_S^+ explicit, consider a perturbation about Minkowski space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with the standard mode expansion

$$h_{\mu\nu}(x) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega} \left[\varepsilon_{\mu\nu}^{\alpha*}(\vec{q}) a_\alpha(\vec{q}) e^{iq \cdot x} + \varepsilon_{\mu\nu}^\alpha(\vec{q}) a_\alpha^\dagger(\vec{q}) e^{-iq \cdot x} \right], \quad (2.30)$$

with $[a_\alpha(\vec{q}), a_\beta^\dagger(\vec{q}')] = 2\omega \delta_{\alpha\beta} (2\pi)^3 \delta^{(3)}(\vec{q} - \vec{q}')$ and $\omega = |\vec{q}|$. In retarded Bondi coordinates $(u, r, z, \bar{z})$ one has

$$\begin{aligned} C_{\bar{z}\bar{z}}(u, z, \bar{z}) &= 2 \lim_{r \rightarrow \infty} \frac{1}{r} h_{\bar{z}\bar{z}}(r, u, z, \bar{z}) \\ &= -\frac{i}{4\pi^2} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \int_0^\infty d\omega \left[a_-(\omega \hat{x}) e^{-i\omega u} - a_+^\dagger(\omega \hat{x}) e^{+i\omega u} \right], \end{aligned} \quad (2.31)$$

where $\hat{x} = \hat{x}(z, \bar{z})$ is the null direction, and

$$\hat{\varepsilon}_{\bar{z}\bar{z}}^+ = \frac{\partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu}{r^2} \varepsilon_{\mu\nu}^+ = \frac{2}{(1 + z\bar{z})^2}. \quad (2.32)$$

Define the Fourier modes of the News by

$$N_{\bar{z}\bar{z}}^\omega \equiv \int_{-\infty}^{+\infty} du e^{i\omega u} \partial_u C_{\bar{z}\bar{z}}(u, z, \bar{z}). \quad (2.33)$$

Then

$$N_{\bar{z}\bar{z}}^\omega = -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_-(\omega \hat{x}), \quad N_{\bar{z}\bar{z}}^{-\omega} = -\frac{1}{2\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \omega a_+^\dagger(\omega \hat{x}), \quad (2.34)$$

and the u -moment of the News is

$$\begin{aligned} N_{\bar{z}\bar{z}}^{(1)} &\equiv \int du u N_{\bar{z}\bar{z}} = -\lim_{\omega \rightarrow 0} \frac{i}{2} (\partial_\omega N_{\bar{z}\bar{z}}^\omega + \partial_{-\omega} N_{\bar{z}\bar{z}}^{-\omega}) \\ &= \frac{i}{4\pi} \hat{\varepsilon}_{\bar{z}\bar{z}}^+ \lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \left[a_-(\omega \hat{x}) - a_+^\dagger(\omega \hat{x}) \right]. \end{aligned} \quad (2.35)$$

Hence the soft part Q_S^+ creates/annihilates a soft graviton mode [20].

2.5 Virasoro Symmetry of Scattering Amplitude via Ward Identity

Extending BMS by local conformal transformations on the celestial sphere is non-trivial; one must show that the associated superrotation charges act as symmetries of the $\mathcal{S}$ -matrix [20, 53]. Symmetry requires

$$\langle \text{out} | [Q(Y), \mathcal{S}] | \text{in} \rangle = \langle \text{out} | (Q^+(Y)\mathcal{S} - \mathcal{S}Q^-(Y)) | \text{in} \rangle = 0. \quad (2.36)$$

Decomposing into soft and hard pieces,

$$\langle \text{out} | (Q_S^+(Y)\mathcal{S} - \mathcal{S}Q_S^-(Y)) | \text{in} \rangle = - \langle \text{out} | (Q_H^+(Y)\mathcal{S} - \mathcal{S}Q_H^-(Y)) | \text{in} \rangle. \quad (2.37)$$

Using the subleading soft graviton theorem and crossing symmetry, one finds that

$$\lim_{\omega \rightarrow 0} (1 + \omega \partial_\omega) \langle \text{out} | a_{\text{out}}(q) \mathcal{S} | \text{in} \rangle = S^{(1)} \langle \text{out} | \mathcal{S} | \text{in} \rangle, \quad (2.38)$$

with

$$S^{(1)} = -i \sum_k \frac{p_{k\mu} \varepsilon^{\mu\nu} q^\lambda \mathcal{J}_{k\lambda\nu}}{p_k \cdot q}. \quad (2.39)$$

In stereographic coordinates, for massless hard legs of energies E_k and the soft graviton of energy ω ,

$$\begin{aligned} p_k^\mu &= \frac{E_k}{1 + z_k \bar{z}_k} (1 + z_k \bar{z}_k, \bar{z}_k + z_k, i(\bar{z}_k - z_k), 1 - z_k \bar{z}_k), \\ q^\mu &= \frac{\omega}{1 + z \bar{z}} (1 + z \bar{z}, \bar{z} + z, i(\bar{z} - z), 1 - z \bar{z}), \\ \varepsilon^{-\mu} &= \frac{1}{\sqrt{2}} (z, 1, i, -z). \end{aligned} \quad (2.40)$$

Substituting yields the familiar differential operator form [20]

$$S^{(1)} = \sum_k \left[\frac{E_k(z - z_k)(1 + z \bar{z}_k)}{(\bar{z}_k - \bar{z})(1 + z_k \bar{z}_k)} \partial_{E_k} + \frac{(z - z_k)^2}{\bar{z}_k - \bar{z}} \partial_{z_k} \right]. \quad (2.41)$$

Acting with $\gamma^{z\bar{z}} D_z^3 (\hat{\varepsilon}_{\bar{z}\bar{z}}^+ \cdot)$ gives the local insertion

$$\gamma^{z\bar{z}} D_z^3 (\hat{\varepsilon}_{\bar{z}\bar{z}}^+ S^{(1)-}) = -2\pi \sum_k [D_z \delta^{(2)}(z - z_k) E_k \partial_{E_k} + 2\delta^{(2)}(z - z_k) \partial_{z_k}]. \quad (2.42)$$

Inserting this into the left-hand side of (2.37) reproduces the hard action

$$\langle \text{out} | (Q_S^+(Y)\mathcal{S} - \mathcal{S}Q_S^-(Y)) | \text{in} \rangle = -i \sum_k \left[Y^z(z_k) \partial_{z_k} - \frac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} \right] \langle \text{out} | \mathcal{S} | \text{in} \rangle, \quad (2.43)$$

which equals $-\langle \text{out} | (Q_H^+(Y)\mathcal{S} - \mathcal{S}Q_H^-(Y)) | \text{in} \rangle$ and thus establishes the Virasoro Ward identity for the $\mathcal{S}$ -matrix [20].

3 Review : Celestial Holography and Celestial Amplitude

Celestial holography posits a duality between scattering amplitudes in a $d+1$ dimensional asymptotically flat spacetime and correlation functions of a conformal field theory (CFT) living on the $d-1$ dimensional celestial sphere $\mathcal{CS}^{d-1} \cong S^{d-1}$ [11–16]. Schematically,

$$\langle \text{out} | \mathcal{S} | \text{in} \rangle \iff \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{\mathcal{CS}^{d-1}}. \quad (3.1)$$

In the previous section we examined, from the perspective of the IR structure, how the symmetry of scattering in asymptotically flat gravity is enlarged to Virasoro symmetries. Motivated by these symmetries, we now review the celestial amplitude: a reorganization of ordinary momentum-space amplitudes into a basis that transforms as correlators of conformal primaries on $\mathcal{CS}^{d-1}$. Concretely, one introduces a Mellin transform over the energies of external massless states, which maps plane waves to boost eigenstates and yields [12, 14]

$$\langle \mathcal{O}_{\Delta_1, J_1}(x_1) \cdots \mathcal{O}_{\Delta_n, J_n}(x_n) \rangle_{\mathcal{CS}^{d-1}} = \left[\prod_{i=1}^n \int_0^\infty d\omega_i \omega_i^{\Delta_i-1} \right] \mathcal{A}_n(\{\omega_i, \hat{p}_i(x_i), J_i\}), \quad (3.2)$$

where $\mathcal{A}_n$ is the momentum-space amplitude, ω_i are energies, $\hat{p}_i(x_i)$ are null directions labeled by points $x_i \in \mathcal{CS}^{d-1}$, and (Δ_i, J_i) denote the conformal weights (dimension and spin/helicity) of the associated celestial primary operators.

The celestial correlators (3.2) furnish a natural stage on which the enlarged asymptotic symmetry acts linearly, with Ward identities reproducing the soft-theorem constraints on $\mathcal{A}_n$. In what follows, we first review the construction and symmetry properties of celestial amplitudes, then develop the operator dictionary (dimensions, spins/helicities, shadow transforms, and basic OPE data). Furthermore, we also discuss operators and its interpretations from AdS/CFT in celestial holography based on our work [23].

3.1 Conformal Primary Wavefunctions and Celestial Amplitudes

In the previous section, we established that scattering amplitudes exhibit the symmetries of a two-dimensional CFT. Motivated by this observation, it is natural to express scattering amplitudes in a basis that makes these symmetries manifest. Such a basis is referred to as the conformal basis. As a preparation for constructing this basis, we now introduce the conformal primary wave functions, which are solutions to the wave equation that respect conformal symmetry. We begin with a scalar field obeying the Klein–Gordon equation

$$(\nabla^2 - m^2)\Psi = 0. \quad (3.3)$$

We seek solutions that transform as "highest-weight states" under the Lorentz group $SO(1, 3)$. Writing the generators as

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \quad L^{\mu\nu} = -(x^\mu \partial^\nu - x^\nu \partial^\mu), \quad (3.4)$$

and specializing to scalars ($S^{\mu\nu} = 0$), the six generators split into rotations

$$J_1 = -(x^2\partial^3 - x^3\partial^2), \quad J_2 = x^1\partial^3 - x^3\partial^1, \quad J_3 = -(x^1\partial^2 - x^2\partial^1), \quad (3.5)$$

and boosts

$$K_1 = -(x^0\partial^1 - x^1\partial^0), \quad K_2 = -(x^0\partial^2 - x^2\partial^0), \quad K_3 = -(x^0\partial^3 - x^3\partial^0). \quad (3.6)$$

Introduce the standard $SL(2, \mathbb{C})$ combinations

$$\begin{aligned} L_0 &= -\frac{i}{2}(J_3 + iK_3), & \bar{L}_0 &= \frac{i}{2}(J_3 - iK_3), \\ L_1 &= -\frac{i}{2}(J_1 + iK_1 + i(J_2 + iK_2)), & \bar{L}_1 &= \frac{i}{2}(J_1 - iK_1 - i(J_2 - iK_2)), \\ L_{-1} &= \frac{i}{2}(J_1 + iK_1 - i(J_2 + iK_2)), & \bar{L}_{-1} &= -\frac{i}{2}(J_1 - iK_1 + i(J_2 - iK_2)), \end{aligned} \quad (3.7)$$

which satisfy the global conformal algebra

$$[L_m, L_n] = (m - n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n}. \quad (3.8)$$

We define a scalar conformal primary Ψ_Δ by

$$(L_0 + \bar{L}_0)\Psi_\Delta = \Delta\Psi_\Delta \quad (\text{conformal dimension } \Delta), \quad (3.9)$$

$$(L_0 - \bar{L}_0)\Psi_\Delta = 0 \quad (\text{spin } 0), \quad (3.10)$$

and impose the highest-weight conditions

$$L_1\Psi_\Delta = \bar{L}_1\Psi_\Delta = 0. \quad (3.11)$$

Solving (3.9)–(3.11) and imposing decay at large timelike separation, a massive scalar conformal primary wavefunction can be written as

$$\Psi_\Delta(\hat{q}; x) \propto \frac{(\sqrt{-x^2})^{\Delta-1}}{(\hat{q} \cdot x)^\Delta} K_{\Delta-1}(m\sqrt{x^2}), \quad \hat{q} = (1, \hat{n}_3) = \hat{q}(z, \bar{z}). \quad (3.12)$$

Taking $m \rightarrow 0$ in the massive primary using the small-argument expansion

$$\begin{aligned} K_{\Delta-1}(x) &= x^{-\Delta} (2^{\Delta-2}\Gamma(\Delta-1)x + \mathcal{O}(x^2)) \\ &+ x^\Delta \left(2^{-\Delta}\Gamma(1-\Delta)x^{-1} + 2^{-2-\Delta}\frac{\Gamma(1-\Delta)}{\Delta}x + \mathcal{O}(x^2) \right), \end{aligned} \quad (3.13)$$

one finds the massless scalar primary

$$\varphi_\Delta(\hat{q}; x) \equiv \lim_{m \rightarrow 0} \Psi_\Delta(\hat{q}; x) \propto \frac{1}{(\hat{q} \cdot x)^\Delta}. \quad (3.14)$$

The massive scalar primary may be represented as a superposition of plane waves over the unit hyperboloid $\mathbb{H}_3$ [12],

$$\Psi_\Delta(x; z, \bar{z}) = \int_{\mathbb{H}_3} d^3\hat{p} \, G_\Delta(\hat{p}; z, \bar{z}) e^{im\hat{p} \cdot x}, \quad (3.15)$$

where G_Δ is the AdS₃/CFT₂ bulk-to-boundary propagator [6],

$$G_\Delta(y, w, \bar{w}; z, \bar{z}) = \left(\frac{y}{y^2 + |z - w|^2} \right)^\Delta. \quad (3.16)$$

Using this transformation, we change basis in scattering amplitudes. Let $\mathcal{A}$ denote the momentum-space amplitude and $\tilde{\mathcal{A}}$ the amplitude in the conformal basis, the celestial amplitude. Then

$$\tilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \prod_{i=1}^n \int_{\mathbb{H}_3} \frac{d^3 \hat{p}_i}{p_i^0} G_{\Delta_i}(\hat{p}_i; z_i, \bar{z}_i) \mathcal{A}(\epsilon_i m_i \hat{p}_i). \quad (3.17)$$

In the massless case this reduces to a Mellin transform in the energies,¹

$$\tilde{\mathcal{A}}(\Delta_i, z_i, \bar{z}_i) = \prod_{i=1}^n \left(\int_0^\infty d\omega_i \omega_i^{\Delta_i-1} \right) \mathcal{A}(\epsilon_i \omega_i \hat{q}_i), \quad (3.19)$$

i.e. a Mellin transform from energies to conformal dimensions. The massless plane-wave Mellin transform is

$$\Phi_\Delta^\pm(X; z, \bar{z}) \equiv \int_0^\infty d\omega \omega^{\Delta-1} e^{\pm i\omega q \cdot X - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-q(z) \cdot X \mp i\epsilon)^\Delta}, \quad (3.20)$$

which furnishes the conformal basis for massless scalars. For spinning fields one similarly defines $\Phi_{\Delta,J}^s(X^\mu; z, \bar{z})$.

3.2 Dictionary (Ordinary)

In AdS/CFT, bulk excitations and boundary operators are related by the GKP–Witten prescription and the extrapolate dictionary. An analogous dictionary exists for celestial holography [67]:

$$\mathcal{O}_{\Delta,J}^{s,\pm}(w, \bar{w}) \equiv i(\hat{O}^s(X^\mu), \Phi_{\Delta^*,-J}^s(X_\mp^\mu; w, \bar{w}))_\Sigma, \quad (3.21)$$

where $(\cdot, \cdot)_\Sigma$ is the Klein-Gordon inner product,

$$(f_1, f_2) = -i \int_\Sigma d^3 X \left(f_1 \partial_{X^0} f_2^* - f_2^* \partial_{X^0} f_1 \right). \quad (3.22)$$

With these operators, celestial amplitudes are boundary correlators:

$$\langle \mathcal{O}_{\Delta_1}^\pm(z_1, \bar{z}_1) \cdots \mathcal{O}_{\Delta_n}^\pm(z_n, \bar{z}_n) \rangle = \prod_{i=1}^k \int_0^\infty d\omega_i \omega_i^{\Delta_i-1} \prod_{j=k}^n \int_{\mathbb{H}_3} \frac{d^3 \hat{p}_j}{p_j^0} G_{\Delta_j}(\hat{p}_j; z_j, \bar{z}_j) \langle \text{out} | \mathcal{S} | \text{in} \rangle. \quad (3.23)$$

¹Fixing $\omega \equiv m/(2y)$ and taking $m \rightarrow 0$ one has

$$\lim_{y \rightarrow 0} G_\Delta(y, z, \bar{z}; w, \bar{w}) = \frac{\pi}{\Delta-1} y^{2-\Delta} \delta^{(2)}(z-w) + \frac{y^\Delta}{(w-z)^{2\Delta}} + \mathcal{O}(y^{4-\Delta}). \quad (3.18)$$

Principal series The conformal dimensions consistent with a positive-definite KG inner product form the principal series [12]. For massless scalars, take $\Delta = c + i\nu$. Using

$$\partial_{X^0}\Phi_{\Delta}^{\pm}(X; z, \bar{z}) = \pm i \int_0^{\infty} d\omega \omega^{\Delta} q^0(z) e^{\pm i\omega q \cdot X}, \quad (3.24)$$

one finds

$$\begin{aligned} & (\Phi_{c+i\nu_1}^{\pm}(X; z_1), \Phi_{c+i\nu_2}^{\pm}(X; z_2)) \\ &= \pm (2\pi)^3 \int_0^{\infty} d\omega_1 \omega_1^{c-1+i\nu_1} \int_0^{\infty} d\omega_2 \omega_2^{c-1-i\nu_2} \omega_1 (\omega_1 q^0(z_1) + \omega_2 q^0(z_2)) \delta^{(3)}(\omega_1 q(z_1) - \omega_2 q(z_2)) \\ &= \pm 4\pi^3 \delta^{(2)}(z_1 - z_2) \int_0^{\infty} d\omega \omega^{2c-2+i(\nu_1-\nu_2)-1}. \end{aligned} \quad (3.25)$$

The remaining integral is precisely when it reduces to

$$\int_0^{\infty} d\omega \omega^{i\nu-1} = \int_{-\infty}^{+\infty} dx e^{i\nu x} = 2\pi \delta(\nu) \quad (x = \log \omega), \quad (3.26)$$

which requires $c = 1$. Thus the allowed dimensions are

$$\Delta = 1 + i\nu, \quad \nu \in \mathbb{R}, \quad (3.27)$$

the principal series. With this choice,

$$(\Phi_{1+i\nu_1}^{\pm}(X; z_1), \Phi_{1+i\nu_2}^{\pm}(X; z_2)) = \pm 8\pi^4 \delta(\nu_1 - \nu_2) \delta^{(2)}(z_1 - z_2). \quad (3.28)$$

3.3 Shadow Transformation and Shadow Operator

The shadow transform is the canonical conformally covariant intertwiner between a primary operator and its “shadow” representation [68–72]. Let $\mathcal{O}_{\Delta}(x)$ be a scalar primary of scaling dimension Δ on $\mathbb{R}^d$. Its shadow $\tilde{\mathcal{O}}_{\Delta}$ is defined by the $SO(d+1, 1)$ -equivariant integral transform

$$\tilde{\mathcal{O}}_{\Delta}(x) = \kappa_{\Delta} \int d^d y \frac{\mathcal{O}_{\Delta}(y)}{|x - y|^{2(d-\Delta)}}, \quad d \geq 2, \quad (3.29)$$

where the normalization κ_{Δ} is chosen so that the transform is an involution on the appropriate function space,

$$\widetilde{(\tilde{\mathcal{O}}_{\Delta})} = \mathcal{O}_{\Delta}. \quad (3.30)$$

The shadow carries conformal dimension $d - \Delta$ and transforms as a scalar primary. In two dimensions it is convenient to parametrize representations by holomorphic weights $(h, \bar{h})$, with total dimension and helicity

$$\Delta = h + \bar{h}, \quad J = h - \bar{h}. \quad (3.31)$$

The shadow transform is $SL(2, \mathbb{C})$ -equivariant and exchanges a primary of weights $(h, \bar{h})$ with one of weights

$$(1 - \bar{h}, 1 - h), \quad (3.32)$$

equivalently $\Delta \mapsto 2 - \Delta$ and $J \mapsto -J$. Working on the Riemann sphere with complex coordinates $(z, \bar{z})$ and measure $d^2w \equiv \frac{i}{2}dw \wedge d\bar{w}$, the scalar shadow takes the factorized form

$$\tilde{\mathcal{O}}_{h, \bar{h}}(z, \bar{z}) = \kappa_{h, \bar{h}} \int d^2w \frac{\mathcal{O}_{h, \bar{h}}(w, \bar{w})}{(z - w)^{2(1-h)}(\bar{z} - \bar{w})^{2(1-\bar{h})}}, \quad (3.33)$$

which produces a primary of weights $(1 - h, 1 - \bar{h})$.

Embedding formalism Following [68], we introduce the conformal integration measure $D^D p$ by

$$\int D^D p f(p) \equiv \frac{2}{\text{VolGL}(1, \mathbb{R})^+} \int_{p^0 > 0} d^{D+2} p \delta(p^2) f(p), \quad (3.34)$$

where the test function $f(p)$ is required to scale with degree $-D$ under the rescaling $p \rightarrow ap$. Choosing the “gauge” representative $p = (1, \mathbf{p})$ with $\mathbf{p}$ a unit vector in $\mathbb{R}^{D+1}$, this reduces to

$$\int D^D p f(p) = \int_{S^D} d^D \mathbf{p} f(p) \Big|_{p=(1, \mathbf{p})}. \quad (3.35)$$

For a scalar operator, the shadow transform $\widetilde{\mathcal{O}}_\Delta$ is defined as

$$\widetilde{\mathcal{O}}_\Delta(p) = \frac{1}{S_\Delta} \int d^D k (-2p \cdot k)^{\Delta-D} \mathcal{O}_\Delta(k). \quad (3.36)$$

The normalization constant S_Δ is taken to be

$$S_\Delta = 2^{2\Delta-D} \pi^h \frac{\Gamma(\Delta - h)}{\Gamma(D - \Delta)}, \quad h = \frac{D}{2}, \quad (3.37)$$

so that the transformation is involutive:²

$$\widetilde{(\widetilde{\mathcal{O}}_\Delta)} = \mathcal{O}_\Delta. \quad (3.38)$$

For fields with spin, the shadow transform is generalized to

$$\widetilde{\mathcal{O}_{I_1, \dots, I_l, \Delta}}(p) = \frac{1}{S_{\Delta, l}} \int d^D k (-2p \cdot k)^{\Delta-D} \prod_{n=1}^l \text{Pr}^{J_n}_{I_n}(p, k) \mathcal{O}_{J_1, \dots, J_l, \Delta}(k), \quad (3.39)$$

where the projection tensor Pr is defined by

$$\text{Pr}^J_I(X, Y) = \delta^J_I - \frac{(Y \cdot Y)X^J X_I - (X \cdot Y)(X^J Y_I + Y^J X_I) + (X \cdot X)Y^J Y_I}{(X \cdot X)(Y \cdot Y) - (X \cdot Y)^2}. \quad (3.40)$$

²The requirement (3.38) does not uniquely determine S_Δ . One could, for example, choose $S_\Delta = \pi^h \frac{\Gamma(\Delta - h)}{\Gamma(D - \Delta)}$. The additional factor $2^{2\Delta-D}$ is inserted here for later convenience.

3.4 Celestial Symmetry

In this part, we examine how bulk spacetime symmetries act on celestial amplitudes [73–76]. Starting from the Lorentz generators L_I ($I = -1, 0, 1$) defined in (3.7) and including the spin action, one finds the standard global conformal action on a celestial primary $\mathcal{O}_{h,\bar{h}}(z, \bar{z})$:

$$\begin{aligned} L_0 \mathcal{O}_{h,\bar{h}} &= 2(z\partial_z + h) \mathcal{O}_{h,\bar{h}}, & L_{-1} \mathcal{O}_{h,\bar{h}} &= \partial_z \mathcal{O}_{h,\bar{h}}, & L_1 \mathcal{O}_{h,\bar{h}} &= (z^2\partial_z + 2zh) \mathcal{O}_{h,\bar{h}}, \\ \bar{L}_0 \mathcal{O}_{h,\bar{h}} &= 2(\bar{z}\partial_{\bar{z}} + \bar{h}) \mathcal{O}_{h,\bar{h}}, & \bar{L}_{-1} \mathcal{O}_{h,\bar{h}} &= \partial_{\bar{z}} \mathcal{O}_{h,\bar{h}}, & \bar{L}_1 \mathcal{O}_{h,\bar{h}} &= (\bar{z}^2\partial_{\bar{z}} + 2\bar{z}\bar{h}) \mathcal{O}_{h,\bar{h}}. \end{aligned} \quad (3.41)$$

Here $h, \bar{h}$ are the holomorphic and antiholomorphic weights.

Since Lorentz transformations are bulk symmetries, an n -point celestial amplitude $\tilde{\mathcal{A}}_n$ obeys the global Ward identities [73]

$$\mathcal{L}_I \tilde{\mathcal{A}}_n = \bar{\mathcal{L}}_I \tilde{\mathcal{A}}_n = 0, \quad \mathcal{L}_I \equiv \sum_{k=1}^n L_{I,k}, \quad \bar{\mathcal{L}}_I \equiv \sum_{k=1}^n \bar{L}_{I,k}. \quad (3.42)$$

Physically, a common global Lorentz action on all external legs leaves the amplitude invariant. Analogous statements hold for translations [74]:

$$\mathcal{P}_\mu \tilde{\mathcal{A}}_n = 0, \quad \mathcal{P}_\mu \equiv \sum_{k=1}^n P_{\mu,k}. \quad (3.43)$$

Action of translations in the conformal basis Consider first massless external legs. In momentum space the translation generator acting on the k -th leg gives

$$\hat{P}_\mu^{(k)} \mathcal{A}(q_1, \dots, q_n) = \epsilon_k \omega_k \hat{q}_{k\mu} \mathcal{A}(q_1, \dots, q_n), \quad (3.44)$$

with $\epsilon_k = \pm 1$ for outgoing/incoming and $q_k^\mu = \omega_k \hat{q}_k^\mu$. Passing to the celestial (Mellin) basis, one finds

$$\begin{aligned} P_{\mu,k} \tilde{\mathcal{A}}(\Delta_1, \dots, \Delta_n) &= \prod_{j \neq k} \left(\int_0^\infty d\omega_j \omega_j^{\Delta_j-1} \right) \int_0^\infty d\omega_k \omega_k^{\Delta_k} \epsilon_k \hat{q}_{k\mu} \mathcal{A}(\dots, \omega_k \hat{q}_k, \dots) \\ &= \epsilon_k \hat{q}_{k\mu} \tilde{\mathcal{A}}(\Delta_1, \dots, \Delta_k+1, \dots, \Delta_n). \end{aligned} \quad (3.45)$$

Hence, in the conformal basis translations act as weight-shifting operators:

$$P_{\mu,k} = \epsilon_k \hat{q}_{k\mu}(z_k, \bar{z}_k) e^{\partial_{\Delta_k}}. \quad (3.46)$$

The zeroth component of (3.43) then yields the discrete Ward identity

$$\tilde{\mathcal{A}}_n(\Delta_1+1, \Delta_2, \dots, \Delta_n) + \tilde{\mathcal{A}}_n(\Delta_1, \Delta_2+1, \dots, \Delta_n) + \dots + \tilde{\mathcal{A}}_n(\Delta_1, \dots, \Delta_n+1) = 0. \quad (3.47)$$

These operator representations make manifest that bulk Poincaré symmetry becomes a combination of global conformal transformations on the celestial sphere and weight-shifting operations in the conformal dimensions of external insertions. They provide the starting point for deriving celestial Ward identities and for organizing amplitudes into celestial conformal multiplets [73–76].

3.5 Conformal Soft Theorem

Consider four-dimensional scattering with massless external states. Let a soft leg carry momentum $q^\mu = \omega \hat{q}^\mu(z, \bar{z})$ with $\omega \rightarrow 0$ and helicity data encoded in a polarization tensor $\varepsilon_{\mu_1 \dots \mu_\ell}^{(\ell)}(z, \bar{z})$ for spin ℓ . We assume tree level throughout. For a hard configuration $\{p_k\}_{k=1}^n$, the $(n+1)$ -point amplitude admits an energetic soft expansion

$$\mathcal{A}_{n+1}(\{p_i\}; q, \varepsilon) = [S^{(0)}(q, \varepsilon) + S^{(1)}(q, \varepsilon) + O(\omega)] \mathcal{A}(\{p_i\}) \quad (3.48)$$

where $S^{(0)} \propto \omega^{-1}$ is the leading term, and $S^{(1)} \propto \omega^0$ is the subleading term.

For each massless external leg we define the celestial primary by a Mellin transform over its energy,

$$\mathcal{O}_\Delta(z, \bar{z}) := \int_0^\infty d\omega \, \omega^{\Delta-1} a(\omega, \hat{q}(z, \bar{z})), \quad (3.49)$$

so that the n -point celestial amplitude is

$$\tilde{\mathcal{A}}_n(\{\Delta_i, J_i; z_i, \bar{z}_i\}) = \prod_{i=1}^n \int_0^\infty d\omega_i \, \omega_i^{\Delta_i-1} \mathcal{A}_n(\{\omega_i, z_i, \bar{z}_i, J_i\}). \quad (3.50)$$

This basis diagonalizes boosts and realizes the Lorentz group as the global $SL(2, \mathbb{C})$ acting on $(z, \bar{z})$ [18, 19]. If a given soft term in (3.48) carries ω^{m-1} ($m \in \mathbb{Z}$), its Mellin integral equals

$$\int_0^\infty d\omega \, \omega^{\Delta-1} \omega^{m-1} e^{-\varepsilon \omega} = \Gamma(\Delta + m - 1) \varepsilon^{-(\Delta+m-1)}, \quad (3.51)$$

which has a simple pole at $\Delta = 1 - m$. Hence each energetic soft power ω^{m-1} maps to a simple pole of the celestial correlator at $\Delta = 1 - m$.

Therefore, we define the conformally soft operator at the integer point $\Delta = n \in \mathbb{Z}$ by [77]³

$$\mathcal{O}_n(z, \bar{z}) \equiv \lim_{\Delta \rightarrow n} (\Delta - n) \mathcal{O}_{\Delta, J}(z, \bar{z}). \quad (3.56)$$

³Let $F(\Delta)$ denote the Mellin transform of the function $f(\omega) = \sum_k f_k \omega^k$. Then we can derive the following relation:

$$\begin{aligned} \lim_{\Delta \rightarrow -n} (\Delta + n) F(\Delta) &= \lim_{\Delta \rightarrow -n} (\Delta + n) \int_0^\infty d\omega \, \omega^{\Delta-1} f(\omega) \\ &= \lim_{\Delta \rightarrow -n} (\Delta + n) \int_0^\infty d\omega \, \omega^{\Delta-1} \sum_k \omega^k f_k \\ &= \lim_{\Delta \rightarrow -n} (\Delta + n) \sum_k \int_0^\infty d\omega \, \omega^{\Delta+k-1} f_k \\ &= f_n. \end{aligned} \quad (3.52)$$

Here, we use

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2} |x|^{\epsilon-1} = \delta(x). \quad (3.53)$$

In the case where the summation index k starts from -1 , the coefficients f_{-1} and f_0 are obtained

Spin 1 (photon): Kac–Moody current from $\Delta = 1$

For $U(1)$ gauge theory, the energetic leading soft theorem states

$$\mathcal{A}_{n+1}(\{p_i\}; q, \varepsilon) = \omega^{-1} \left[\sum_{k=1}^n Q_k \frac{\varepsilon \cdot p_k}{\hat{q} \cdot p_k} \right] \mathcal{A}_n(\{p_i\}) + \mathcal{O}(\omega^0). \quad (3.57)$$

So,

$$\begin{aligned} \lim_{\omega \rightarrow 0} \omega \mathcal{A}_{n+1}(\{p_i\}; q, \varepsilon) &= \left[\sum_{k=1}^n Q_k \frac{\varepsilon \cdot p_k}{\hat{q} \cdot p_k} \right] \mathcal{A}_n(\{p_i\}) \\ \iff \lim_{\omega \rightarrow 0} \omega \left\langle a(q, \varepsilon) \prod_{i=1}^n a(p_i) \right\rangle &= \left[\sum_{k=1}^n Q_k \frac{\varepsilon \cdot p_k}{\hat{q} \cdot p_k} \right] \left\langle \prod_{i=1}^n a(p_i) \right\rangle \end{aligned} \quad (3.58)$$

Mellin-transforming the soft leg and taking the residue at $\Delta = 1$ defines the holomorphic conformally soft photon

$$J(z) := \lim_{\Delta \rightarrow 1} (\Delta - 1) \mathcal{O}_\Delta(z, \bar{z}), \quad (3.59)$$

whose insertion produces the celestial Ward identity [11]

$$\left\langle J(z) \prod_{i=1}^n \mathcal{O}_i(z_i, \bar{z}_i) \right\rangle = \sum_{k=1}^n \frac{Q_k}{z - z_k} \left\langle \prod_{i=1}^n \mathcal{O}_i(z_i, \bar{z}_i) \right\rangle \quad (3.60)$$

The rational kernel $1/(z - z_k)$ follows from the standard choice of reference polarization, for which $\frac{\varepsilon \cdot p_k}{\hat{q} \cdot p_k} = \frac{1}{z - z_k}$ [19].

Spin 2 (graviton): supertranslations and Virasoro

For gravity, the leading soft theorem implies that the $\Delta = 1$ residue of the positive-helicity conformal primary defines a supertranslation current $\mathcal{P}(z, \bar{z})$ whose insertion acts as

$$\langle \mathcal{P}(z, \bar{z}) \prod_{i=1}^n \mathcal{O}_i \rangle = \sum_{k=1}^n \frac{\varepsilon_{\mu\nu}(z, \bar{z}) p_k^\mu p_k^\nu}{\hat{q}(z, \bar{z}) \cdot p_k} \langle \cdots \mathcal{O}_{\Delta_k+1, J_k}(z_k, \bar{z}_k) \cdots \rangle. \quad (3.61)$$

At subleading order, the soft theorem yields a celestial operator whose shadow is the holomorphic stress tensor $T(z)$ obeying the Virasoro Ward identity

$$\langle T(z) \prod_{i=1}^n \mathcal{O}_i(z_i, \bar{z}_i) \rangle = \sum_{k=1}^n \left(\frac{h_k}{(z - z_k)^2} + \frac{1}{z - z_k} \partial_{z_k} \right) \langle \prod_{i=1}^n \mathcal{O}_i(z_i, \bar{z}_i) \rangle \quad (3.62)$$

with h_k the holomorphic weights of the inserted primaries [78]. The appearance of $T(z)$ ties the subleading soft graviton to the Virasoro enhancement of the Lorentz $SL(2, \mathbb{C})$ acting on the celestial sphere [19, 78, 79].

as follows.

$$f_{-1} = \lim_{\omega \rightarrow 0} \omega f(\omega) \quad (3.54)$$

$$f_0 = \lim_{\omega \rightarrow 0} \frac{1}{2} (1 + \omega \partial_\omega) f(\omega) \quad (3.55)$$

3.6 Connection with (A)dS/CFT via Wedge Holography

In this part, we review an approach that seeks to interpret celestial holography through alternative holographic dualities: Minkowski spacetime is partitioned into three regions, $\text{EAdS}_+ + \text{EAdS}_- + \text{dS}$, and the wedge holography [22] is applied to each component to account for the celestial holographic dictionary. This section is based on our work [23].

3.6.1 Milne Slicing

Before starting this section, we have to introduce Milne coordinates (η, ρ) via

$$t = \eta \cosh \rho, \quad r = \eta \sinh \rho. \quad (3.63)$$

The Minkowski metric becomes

$$ds^2 = -d\eta^2 + \eta^2 (d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2), \quad (3.64)$$

so $\eta = \text{const.}$ slices are hyperbolic spaces H_d of radius η (“hyperbolic slices”).

Alternatively, set

$$t = r' \sinh \tau, \quad r = r' \cosh \tau, \quad (3.65)$$

giving

$$ds^2 = dr'^2 + r'^2 (-d\tau^2 + \cosh^2 \tau d\Omega_{d-1}^2), \quad (3.66)$$

so $r' = \text{const.}$ slices are de Sitter spaces (“de Sitter slices”). This family of foliations is referred to as Milne slicing (See Fig.3) [80, 81].

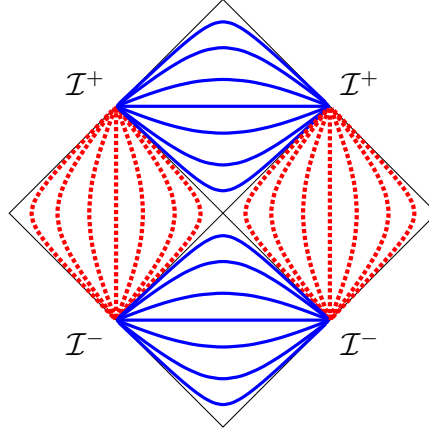


Figure 3: Penrose diagram of Minkowski spacetime with Milne slicing. Blue curves represent hyperbolic slices, and red dotted curves represent dS slices.

3.6.2 Hyperbolic and de Sitter Slices of Flat Spacetime

To exhibit a bridge between flat holography and (A)dS/CFT in a wedge-like setup, it is convenient to foliate $d+1$ dimensional Minkowski space $\mathbb{R}^{1,d}$,

$$ds^2 = -dT^2 + dR^2 + R^2 d\Omega_{d-1}^2, \quad (3.67)$$

by two families of codimension-one slices that are individually maximally symmetric: hyperbolic slices H_d and de Sitter slices dS_d [80] (see also [32, 82–84]).

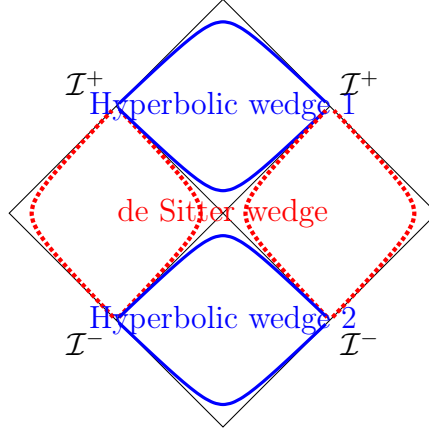


Figure 4: Two hyperbolic wedges and de Sitter wedge. Note that, although there appear to be two de Sitter wedges, they are in fact connected and should therefore be regarded as a single region.

Hyperbolic patch. Introduce

$$T = \eta \cosh \rho, \quad R = \eta \sinh \rho, \quad (3.68)$$

which brings the metric to

$$ds^2 = -d\eta^2 + \eta^2 (d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2), \quad [\text{hyperbolic patch}], \quad (3.69)$$

so that constant- η hypersurfaces are copies of H_d with radius η . The range is $0 \leq \eta < \infty$.

de Sitter patch. Alternatively, set

$$T = r \sinh t, \quad R = r \cosh t, \quad (3.70)$$

to obtain

$$ds^2 = dr^2 + r^2 (-dt^2 + \cosh^2 t d\Omega_{d-1}^2), \quad [\text{de Sitter patch}], \quad (3.71)$$

with $0 \leq r < \infty$. Gluing the two patches across $\eta = 0$ and $r = 0$ reconstructs the full Minkowski space.

Wedge regularization and UV cutoffs. Motivated by wedge holography [22] as a doubled cousin of AdS/BCFT [85, 86], we regulate the radial coordinates by

$$0 \leq \eta \leq \eta_\infty, \quad 0 \leq r \leq r_\infty, \quad (3.72)$$

and, in analogy with AdS/CFT [4] and dS/CFT [9], introduce boundary UV cutoffs

$$\rho \leq \rho_\infty, \quad t \leq t_\infty. \quad (3.73)$$

In the sequel we analyze the two slicings in turn and then interpret the resulting holographic data from the vantage point of a CFT living on S^{d-1} at $\rho \rightarrow \infty$.

3.6.3 Free Scalar Field in Wedges

hyperbolic patch Consider a massive real scalar Ψ with action

$$I_{\text{scalar}} = \frac{1}{2} \int d^4x \sqrt{-g} \left[-g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi - m^2 \Psi^2 \right], \quad (3.74)$$

whose equation of motion is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) - m^2 \Psi = 0. \quad (3.75)$$

In coordinates (3.69) (we take $d+1 = 4$ for concreteness and $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$), the equation becomes [19]

$$-\partial_\eta^2 \Psi - \frac{3}{\eta} \partial_\eta \Psi + \frac{1}{\eta^2} (\partial_\rho^2 \Psi + 2 \coth \rho \partial_\rho \Psi) + \frac{1}{\eta^2 \sinh^2 \rho} \Delta_2 \Psi - m^2 \Psi = 0, \quad (3.76)$$

and separates under

$$\Psi(\eta, \rho, \theta, \phi) = f_p(\eta) g_{p,l}(\rho) Y_{lm}(\theta, \phi), \quad (3.77)$$

with

$$\begin{aligned} \left(-\partial_\eta^2 - \frac{3}{\eta} \partial_\eta + \frac{p^2}{\eta^2} - m^2 \right) f_p &= 0, \\ \left(\partial_\rho^2 + 2 \coth \rho \partial_\rho - \frac{l(l+1)}{\sinh^2 \rho} - p^2 \right) g_{p,l} &= 0, \\ \Delta_2 Y_{lm} &= -l(l+1) Y_{lm}. \end{aligned} \quad (3.78)$$

Writing $\nu \equiv \sqrt{1+p^2}$, a convenient basis is

$$f_p(\eta) = \alpha \frac{I_\nu(m\eta)}{\eta} + \beta \frac{K_\nu(m\eta)}{\eta}, \quad (3.79)$$

and

$$g_{p,l}(\rho) = \frac{1}{\sinh \rho} Q_l^\nu(\coth \rho), \quad (3.80)$$

where regularity at $\rho = 0$ selects the associated Legendre Q over P .

de Sitter patch. The de Sitter family of solutions follows by analytic continuation

$$\eta = -ir, \quad \rho = t - \frac{\pi}{2}i, \quad (3.81)$$

which yields

$$\Psi(r, t, \theta, \phi) = \tilde{f}_p(r) \tilde{g}_{p,l}(t) Y_{lm}(\theta, \phi), \quad (3.82)$$

with

$$\tilde{f}_p(r) = -\alpha e^{\pi i \nu} \frac{J_\nu(mr)}{r} - \beta \frac{\pi i}{2} e^{-\pi i \nu} \frac{H_\nu^{(1)}(mr)}{r}, \quad (3.83)$$

$$\tilde{g}_{p,l}(t) = \frac{1}{\cosh t} Q_l^\nu(\tanh t). \quad (3.84)$$

Because $Q_l^\nu(z)$ is $z^{-\nu-\ell-1}$ times an even function of z , transporting from $t = +\infty$ to $t = -\infty$ multiplies the mode by $(-1)^l$, implementing the standard antipodal identification between future and past celestial spheres.

Delta source on S^2 . A pointlike source at (θ_0, ϕ_0) is represented by the harmonic expansion

$$\Psi_0^h(\eta, \rho, \theta, \phi) = f_p(\eta) \frac{\mathcal{N}}{\sinh \rho} \sum_{l=0}^{\infty} \left(\frac{2l+1}{4\pi} \right) P_l(\cos \gamma) Q_l^\nu(\coth \rho), \quad (3.85)$$

with $\cos \gamma = \cos \theta_0 \cos \theta + \sin \theta \sin \theta_0 \cos(\phi - \phi_0)$ and $\mathcal{N} = e^{-\pi i \nu} / \Gamma(\nu)$. For $\rho \rightarrow \infty$ one finds the characteristic near-boundary split into a delta-supported non-normalizable piece plus a normalizable tail,

$$\Psi_0^h \rightarrow f_p(\eta) \left[e^{(\nu-1)\rho} \delta^2(\Omega - \Omega_0) + e^{-(\nu+1)\rho} \frac{\nu}{4\pi} \left(\frac{1 - \cos \gamma}{2} \right)^\Delta \right], \quad (3.86)$$

with the scaling dimension of the dual scalar operator identified as

$$\Delta = 1 + \nu = 1 + \sqrt{1 + p^2}. \quad (3.87)$$

The sum (3.85) can be performed in closed form [80],

$$\Psi_0^h(\eta, \rho, \theta, \phi) = \frac{\Delta - 1}{4\pi} f_p(\eta) \frac{1}{(\cosh \rho - \cos \gamma \sinh \rho)^\Delta}, \quad (3.88)$$

and the de Sitter counterpart follows by (3.81),

$$\Psi_0^{ds}(r, t, \theta, \phi) = \frac{\Delta - 1}{4\pi} e^{\frac{\pi i}{2}(1-\Delta)} \tilde{f}_p(r) \frac{1}{(\sinh t - \cos \gamma \cosh t + i\tilde{\epsilon})^\Delta}, \quad (3.89)$$

with the usual $i\epsilon$ prescription [80].

3.6.4 Wedge Holography for Hyperbolic Slices

We now define the hyperbolic wedge W^h by restricting the time-like radius to

$$\eta_1 \leq \eta \leq \eta_2, \quad (3.90)$$

and apply Neumann boundary conditions on the two end-of-the-world (EOW) branes $Q^{h(1)}$ and $Q^{h(2)}$ at $\eta = \eta_1, \eta_2$:

$$K_{ab} - h_{ab}K = -Th_{ab}. \quad (3.91)$$

For the metric (3.69), this is solved by the tensions

$$T^{h(i)} = \frac{d-1}{d} K^{h(i)} = \frac{d-1}{\eta_i}, \quad i = 1, 2. \quad (3.92)$$

By analogy with wedge holography in AdS [22], we are led to the duality: gravity on W^h is holographically dual to a $(d-1)$ -dimensional Euclidean CFT on the celestial sphere S^{d-1} at $\rho \rightarrow \infty$. Each H_d slice enjoys $SO(1, d)$, matching the conformal group of the boundary CFT; in $d = 3$ this is naturally enlarged by superrotations to a pair of Virasoro algebras when one includes the asymptotic symmetry enhancement of flat space [32, 84]. The dual theory is non-unitary, as signaled by purely imaginary central charge below, in a manner reminiscent of dS/CFT [10, 87–90].

Partition function and central charge The gravitational action on W^h is

$$I_G = \frac{1}{16\pi G_N} \int_{W^h} \sqrt{-g} R + \frac{1}{8\pi G_N} \left[\int_{Q^{h(1)}} \sqrt{|\gamma|} (K^{h(1)} - T^{h(1)}) - \int_{Q^{h(2)}} \sqrt{|\gamma|} (K^{h(2)} - T^{h(2)}) \right]. \quad (3.93)$$

Since $R = 0$ for Minkowski, only the boundary terms contribute. Denoting

$$\omega_{d-1} = \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}, \quad J_d(\rho_\infty) = \int_0^{\rho_\infty} d\rho \sinh^d \rho, \quad (3.94)$$

one finds

$$I_G = -\frac{1}{8\pi G_N} (\eta_2^{d-1} - \eta_1^{d-1}) \omega_{d-1} J_{d-1}(\rho_\infty). \quad (3.95)$$

Evaluating for $d = 3, 4, 5$ and identifying the UV cutoff via $\epsilon = e^{-\rho_\infty}$, we obtain:

$$\begin{aligned} d = 3: \quad I_G &= -\frac{\eta_2^2 - \eta_1^2}{16G_N \epsilon^2} - \frac{\eta_2^2 - \eta_1^2}{4G_N} \log \epsilon, \\ d = 4: \quad I_G &= -\frac{\eta_2^3 - \eta_1^3}{48G_N \epsilon^3} + \frac{3(\eta_2^3 - \eta_1^3)}{16G_N \epsilon}, \\ d = 5: \quad I_G &= \frac{\pi(\eta_2^4 - \eta_1^4)}{2G_N} \left[-\frac{1}{96\epsilon^4} + \frac{1}{12\epsilon^2} + \frac{1}{4} \log \epsilon \right]. \end{aligned} \quad (3.96)$$

Comparing with the S^2 and S^4 partition functions of CFTs [91–93]⁴, and setting $Z_{\text{CFT}} \sim e^{iI_G}$, we read off

$$d = 3 : \quad c = i \frac{3(\eta_2^2 - \eta_1^2)}{4G_N}, \quad (3.99)$$

$$d = 5 : \quad a = i \frac{\pi(\eta_2^4 - \eta_1^4)}{32G_N}. \quad (3.100)$$

The absence of a logarithmic term for $d = 4$ is consistent with the lack of conformal anomaly in odd-dimensional CFTs.

Holographic entanglement entropy. The extremal surface $\Gamma_A \subset W^h$ anchored on an entangling region $A \subset S^{d-1}$ is a one-parameter family (labelled by η) of minimal hypersurfaces $\gamma_A^H \subset H_d$; the area is purely imaginary, as in dS/CFT [7, 90, 94–97]:

$$S_A = \frac{\text{Area}(\Gamma_A)}{4G_N} = \frac{i(\eta_2^{d-1} - \eta_1^{d-1})}{4(d-1)G_N} \text{Area}(\gamma_A^H). \quad (3.101)$$

For a symmetric interval $-\theta_0 \leq \theta \leq \theta_0$ on S^{d-1} , with the H_d metric

$$ds^2 = d\rho^2 + \sinh^2 \rho (d\theta_1^2 + \sin^2 \theta_1 d\Omega_{d-2}^2), \quad (3.102)$$

the minimal surface spanning the interval $\theta \in [-\theta_0, \theta_0]$ has area

$$A(\gamma_A^H) = \omega_{d-3} \cdot \int_{\frac{\delta}{L}}^1 dy \frac{(1-y^2)^{\frac{d-4}{2}}}{y^{d-2}}, \quad (3.103)$$

with the UV regulator δ and the minimal-surface scale L related to the cutoff ρ_∞ and the opening angle θ_0 where

$$L = \frac{\sin \theta_0}{1 + \cos \theta_0}, \quad (3.104)$$

$$\frac{1}{\delta} = \left(\frac{1 + \cos \theta_0}{2} \right) e^{\rho_\infty} + \left(\frac{1 - \cos \theta_0}{2} \right) e^{-\rho_\infty}. \quad (3.105)$$

The minimal areas evaluate to

$$\begin{aligned} d = 3 : \quad \text{Area}(\gamma_A^H) &= 2 \log \left(\frac{2L}{\delta} \right) = 2 \log(e^{\rho_\infty} \sin \theta_0), \\ d = 4 : \quad \text{Area}(\gamma_A^H) &= \pi \sin \theta_0 e^{\rho_\infty} - 2\pi, \\ d = 5 : \quad \text{Area}(\gamma_A^H) &= \frac{\pi}{2} \sin^2 \theta_0 e^{2\rho_\infty} - 2\pi \log \left(\frac{\sin \theta_0}{4} e^{\rho_\infty} \right) - \pi \cos \theta_0 (2 - \cos \theta_0). \end{aligned} \quad (3.106)$$

⁴Sphere partition function of two-dimensional CFT with the central charge c can be written as [91, 92]

$$Z_{\text{CFT } S^2} \sim e^{\frac{A}{\epsilon^2} - \frac{c}{3} \log \epsilon}, \quad (3.97)$$

$$Z_{\text{CFT } S^4} \sim e^{4a \log \epsilon + (\text{other parts})}. \quad (3.98)$$

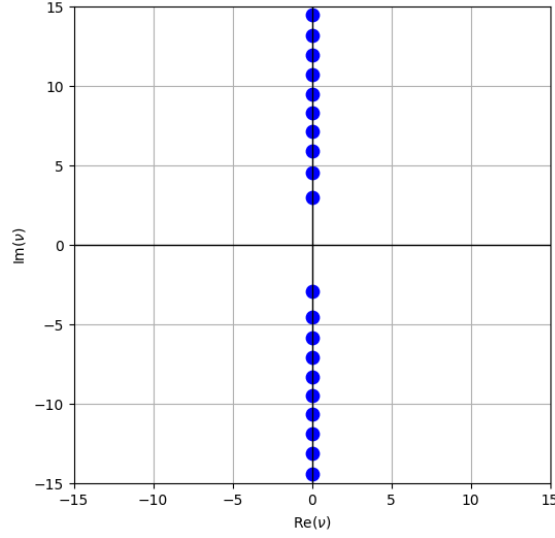


Figure 5: Solution of (3.109), in the case of $m\eta_1 = 1$, $m\eta_2 = 10$.

Thus, for $d = 3$,

$$S_A = \frac{i(\eta_2^2 - \eta_1^2)}{8G_N} \log\left(\frac{\sin^2 \theta_0}{\epsilon^2}\right), \quad (3.107)$$

which matches the S^2 CFT result $S_A = (c/6) \log(\sin^2(\Delta\theta/2)/\epsilon^2)$ [98, 99] upon using (3.99). For $d = 5$ we similarly extract the a -anomaly coefficient from the universal $\log \epsilon$ term [94, 100], reproducing (3.100).

Scalar spectra and two-point functions. On W^h the separated time-radial mode (3.79) must obey boundary conditions at $\eta = \eta_{1,2}$.

Dirichlet:

$$f_p(\eta_1) = f_p(\eta_2) = 0. \quad (3.108)$$

This quantizes $\nu = \sqrt{1 + p^2}$ by the condition

$$D^h(\nu, x_1, x_2) \equiv I_\nu(x_1)K_\nu(x_2) - I_\nu(x_2)K_\nu(x_1) = 0, \quad x_a = m\eta_a, \quad (3.109)$$

whose zeros lie on the imaginary axis, $\nu = i\lambda_k$ ($\lambda_k \in \mathbb{R}$), forming an infinite discrete set (See Fig. 5).

Hence a bulk scalar of mass m maps to an infinite tower of boundary primaries with

$$\Delta_k = 1 + i\lambda_k. \quad (3.110)$$

This complex spectrum anticipates non-unitarity, as also familiar in the celestial literature [14]. The imaginary nature of ν is transparent in the limits $\eta_2 \rightarrow \infty$, $\eta_1 \rightarrow 0$, using the small-argument form of K_ν ,

$$K_\nu(z) \simeq \frac{\pi}{2 \sin \pi \nu} \left[\frac{(z/2)^{-\nu}}{\Gamma(1 - \nu)} - \frac{(z/2)^\nu}{\Gamma(1 + \nu)} \right], \quad z \ll 1, \quad (3.111)$$

which diverges if $\text{Re}\nu \neq 0$.

Neumann:

$$\partial_\eta f_p(\eta_1) = \partial_\eta f_p(\eta_2) = 0. \quad (3.112)$$

Using standard recurrence relations for I_ν and K_ν , this reduces to

$$\begin{aligned} N^h(\nu, x_1, x_2) &= \{(\nu+1)I_{\nu+1}(x_1) + (\nu-1)I_{\nu-1}(x_1)\} \{(\nu+1)K_{\nu+1}(x_2) + (\nu-1)K_{\nu-1}(x_2)\} \\ &\quad - \{(\nu+1)I_{\nu+1}(x_2) + (\nu-1)I_{\nu-1}(x_2)\} \{(\nu+1)K_{\nu+1}(x_1) + (\nu-1)K_{\nu-1}(x_1)\} \\ &= 0. \end{aligned} \quad (3.113)$$

Once again $\nu = i\lambda_k$ with an infinite discrete set of real λ_k .

Finally, the on-shell scalar action evaluated on (3.88) produces the standard two-point function on the celestial sphere at fixed p ,

$$\langle O_p(\theta_1, \phi_1) O_p(\theta_2, \phi_2) \rangle \propto (1 - \cos \gamma_{12})^{-\Delta}, \quad (3.114)$$

with $\cos \gamma_{12}$ defined by

$$\cos \gamma_{12} = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2), \quad (3.115)$$

and Δ related to p as in (3.87), in agreement with S^{d-1} CFT expectations [6, 101].

3.6.5 Wedge Holography for de Sitter Slices

We now turn to the second wedge construction, based on the de Sitter foliation (3.71). Define the $(d+1)$ -dimensional wedge

$$r_1 \leq r \leq r_2, \quad (3.116)$$

whose boundaries at $r = r_1, r_2$ are end-of-the-world (EOW) branes $Q^{\text{ds}(1)}$ and $Q^{\text{ds}(2)}$. Imposing the Neumann condition (3.91) on each brane fixes their tensions to

$$T^{\text{ds}(i)} = \frac{d-1}{d} K^{\text{ds}(i)} = \frac{d-1}{r_i}, \quad i = 1, 2. \quad (3.117)$$

We will argue that gravity on this wedge W^{ds} is holographically dual to a $(d-1)$ -dimensional Euclidean CFT living on a single S^{d-1} obtained by identifying the two spherical boundaries at $t = \pm\infty$ via the antipodal map. We regulate the time coordinate by $t = \pm t_\infty$. Each fixed- r slice is dS_d with isometry $SO(1, d)$, which matches the conformal group of the boundary theory; for $d = 3$, the asymptotic symmetry enhancement in flat space implies a pair of Virasoro algebras (superrotations) [58, 60]. As in the AdS wedge [22], the present construction adds a finite spatial width to a dS_d slice, so non-unitary boundary data are again expected, in line with dS/CFT [9, 10, 87–90]. We substantiate this by computing the partition function, holographic entanglement entropy, and scalar spectra.

Partition function and central charge The gravitational action on W^{ds} is

$$I_G = \frac{1}{16\pi G_N} \int_{W^{\text{ds}}} \sqrt{-g} R - \frac{1}{8\pi G_N} \left[\int_{Q^{\text{ds}(1)}} \sqrt{-h} (K^{\text{ds}(1)} - T^{\text{ds}(1)}) - \int_{Q^{\text{ds}(2)}} \sqrt{-h} (K^{\text{ds}(2)} - T^{\text{ds}(2)}) \right]. \quad (3.118)$$

We evaluate the on-shell action on the half-spacetime $0 \leq t \leq t_\infty$ (antipodal identification implements the single S^{d-1} ; without it one simply doubles the result). Using

$$\int_{Q^{\text{ds}(i)}} \sqrt{-h} = r_i^d \omega_{d-1} \int_0^{t_\infty} dt \cosh^{d-1} t, \quad I_d \equiv \int_0^{t_\infty} dt \cosh^d t, \quad (3.119)$$

one finds

$$I_G = \frac{1}{8\pi G_N} (r_2^{d-1} - r_1^{d-1}) \omega_{d-1} I_{d-1}, \quad (3.120)$$

with the recursion

$$I_d = \frac{1}{d} \cosh^{d-1} t_\infty \sinh t_\infty + \frac{d-1}{d} I_{d-2}. \quad (3.121)$$

Introducing the CFT cutoff $\epsilon = e^{-t_\infty}$, the explicit results for $d = 3, 4, 5$ are

$$\begin{aligned} d=3: \quad I_G &= \frac{r_2^2 - r_1^2}{16G_N} (e^{2t_\infty} + 4t_\infty) = \frac{r_2^2 - r_1^2}{16G_N \epsilon^2} - \frac{r_2^2 - r_1^2}{4G_N} \log \epsilon, \\ d=4: \quad I_G &= \frac{r_2^3 - r_1^3}{48G_N \epsilon^3} + \frac{3(r_2^3 - r_1^3)}{16G_N \epsilon}, \\ d=5: \quad I_G &= \frac{\pi(r_2^4 - r_1^4)}{2G_N} \left[\frac{1}{96\epsilon^4} + \frac{1}{12\epsilon^2} - \frac{1}{4} \log \epsilon \right]. \end{aligned} \quad (3.122)$$

Comparing $Z_{\text{CFT}} \sim e^{I_G}$ to the S^2 and S^4 partition functions, yields the (imaginary) central charge

$$d=3: \quad c = i \frac{3(r_2^2 - r_1^2)}{4G_N}, \quad (3.123)$$

$$d=5: \quad a = -i \frac{\pi(r_2^4 - r_1^4)}{32G_N}. \quad (3.124)$$

Holographic entanglement entropy Extremal surfaces anchored on a boundary region $A \subset S^{d-1}$ at $t = t_\infty$ are time-like and sweep a family of extremal surfaces $\gamma_A^{\text{ds}} \subset \text{dS}_d$ parametrized by r :

$$ds^2|_{\text{dS}_d} = -dt^2 + \cosh^2 t (d\theta_1^2 + \sin^2 \theta_1 d\Omega_{d-2}^2). \quad (3.125)$$

For a symmetric interval $-\theta_0 \leq \theta \leq \theta_0$ at $t = t_\infty$, the area of γ_A^{ds} takes the form

$$A(\gamma_A^{\text{ds}}) = i\omega_{d-3} \int_{\epsilon/L}^\infty dy \frac{(1+y^2)^{\frac{d-4}{2}}}{y^{d-2}}, \quad (3.126)$$

with the standard relations between (ϵ, L) and (t_∞, θ_0) given in (3.104), (3.105). As usual in dS/CFT, the surface crosses to the opposite boundary at $t = -t_\infty$; evaluating only the $t \geq 0$ half and doubling reproduces the full Lorentzian result (alternatively, one may attach the Euclidean cap in a Hartle–Hawking construction [89, 90]). For $d = 3, 4, 5$:

$$\begin{aligned} d = 3 : \quad A(\gamma_A^{\text{dS}}) &= 2i \log\left(\frac{2L}{\delta}\right) = 2i \log(e^{t_\infty} \sin \theta_0), \\ d = 4 : \quad A(\gamma_A^{\text{dS}}) &= 2\pi i \frac{L}{\delta} = \pi i \sin \theta_0 e^{t_\infty}, \\ d = 5 : \quad A(\gamma_A^{\text{dS}}) &= \frac{\pi i}{2} \sin^2 \theta_0 e^{2t_\infty} + 2\pi i \log\left(\frac{\sin \theta_0}{4} e^{t_\infty}\right) - \pi i \cos \theta_0 (2 + \cos \theta_0). \end{aligned} \quad (3.127)$$

Integrating over r gives

$$A(\Gamma_A^{\text{dS}}) = \int_{r_1}^{r_2} dr r^{d-1} A(\gamma_A^{\text{dS}}) = \frac{r_2^{d-1} - r_1^{d-1}}{d-1} A(\gamma_A^{\text{dS}}), \quad (3.128)$$

and thus

$$S_A = \frac{A(\Gamma_A^{\text{dS}})}{4G_N} = \frac{r_2^{d-1} - r_1^{d-1}}{4(d-1)G_N} A(\gamma_A^{\text{dS}}). \quad (3.129)$$

For $d = 3$,

$$S_A = \frac{i(r_2^2 - r_1^2)}{8G_N} \log\left(\frac{\sin^2 \theta_0}{\epsilon^2}\right), \quad (3.130)$$

matching the S^2 CFT formula⁵ with

$$c = \frac{3i}{4G_N} (r_2^2 - r_1^2), \quad (3.132)$$

in agreement with (3.124). For $d = 5$,

$$S_A = i \frac{\pi(r_2^4 - r_1^4)}{16G_N} \left[\frac{\sin^2 \theta_0}{2\epsilon^2} + 2 \log\left(\frac{\sin \theta_0}{4\epsilon}\right) + O(1) \right], \quad (3.133)$$

from which the universal $\log \epsilon$ term reproduces

$$a = -\frac{\pi i(r_2^4 - r_1^4)}{32G_N}, \quad (3.134)$$

again consistent with (3.124).

⁵Entanglement entropy in a two-dimensional CFT on S^2 with a central charge c , where the subsystem A is taken to be an interval; $\theta^{(1)} \leq \theta_1 \leq \theta^{(2)}$, given by [98, 99]:

$$S_A = \frac{c}{6} \log \left[\frac{\sin^2 \left(\frac{\theta^{(1)} - \theta^{(2)}}{2} \right)}{\epsilon^2} \right]. \quad (3.131)$$

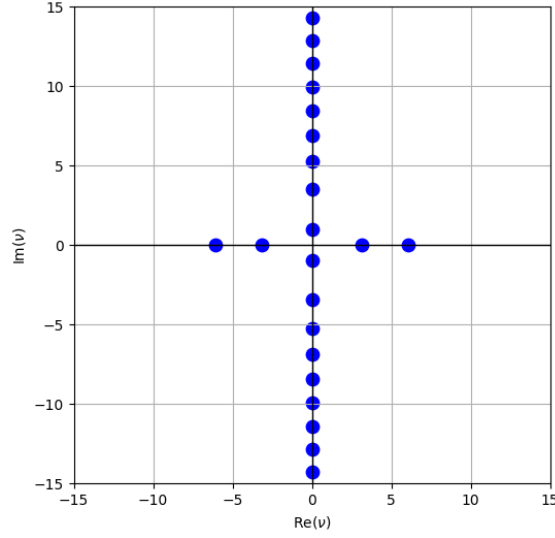


Figure 6: Solution of (3.136) in the case of $mr_1 = 1$, $mr_2 = 10$.

Scalar field propagation Consider now a massive free scalar on W^{ds} with Dirichlet or Neumann conditions at $r = r_{1,2}$. As in the hyperbolic wedge, one obtains a tower of boundary operators with dimensions $\Delta = 1 + i\lambda_k$, reflecting an infinite set of real λ_k and, in addition, a finite set of imaginary λ_k (i.e. real $\nu = \sqrt{1 + p^2}$). In the language of dS_3/CFT_2 , where $\Delta = 1 \pm \sqrt{1 - M^2}$ [9], this corresponds to a finite number of $0 < M < 1$ modes and an infinite set with $M > 1$.

Dirichlet:

Using (3.82)–(3.83), the boundary conditions

$$\tilde{f}_p(r_1) = \tilde{f}_p(r_2) = 0 \quad (3.135)$$

quantize $\nu = \sqrt{1 + p^2}$ through

$$D^{\text{ds}}(\nu, x_1, x_2) \equiv J_\nu(x_1)H_\nu^{(1)}(x_2) - J_\nu(x_2)H_\nu^{(1)}(x_1) = 0, \quad x_a = mr_a. \quad (3.136)$$

Numerically one finds an infinite sequence of purely imaginary $\nu = i\lambda_k$ and a finite number of real ν , with the latter increasing and the former becoming dense as $r_1 \rightarrow 0, r_2 \rightarrow \infty$.

Neumann:

Imposing

$$\partial_r \tilde{f}_p(r_1) = \partial_r \tilde{f}_p(r_2) = 0 \quad (3.137)$$

and using the Bessel/Hankel recurrences yields the quantization condition

$$\begin{aligned} N^{\text{ds}}(\nu, x_1, x_2) &= \{(\nu + 1)J_{\nu+1}(x_1) - (\nu - 1)J_{\nu-1}(x_1)\} \left\{ (\nu + 1)H_{\nu+1}^{(1)}(x_2) - (\nu - 1)H_{\nu-1}^{(1)}(x_2) \right\} \\ &\quad - \{(\nu + 1)J_{\nu+1}(x_2) - (\nu - 1)J_{\nu-1}(x_2)\} \left\{ (\nu + 1)H_{\nu+1}^{(1)}(x_1) - (\nu - 1)H_{\nu-1}^{(1)}(x_1) \right\} \\ &= 0. \end{aligned} \quad (3.138)$$

The root structure mirrors the Dirichlet case.

Similarly in hyperbolic case, evaluating the on-shell scalar action using the near-boundary profile (3.89) gives the standard celestial two-point structures on the future (+) and past (−) spheres:

$$\begin{aligned}\langle O_p^{(\pm)}(\theta_1, \phi_1) O_p^{(\pm)}(\theta_2, \phi_2) \rangle &\propto (1 - \cos \gamma_{12})^{-\Delta}, \\ \langle O_p^{(\pm)}(\theta_1, \phi_1) O_p^{(\mp)}(\theta_2, \phi_2) \rangle &\propto (1 + \cos \gamma_{12})^{-\Delta}.\end{aligned}\tag{3.139}$$

The cross correlator simply implements the antipodal identification, so all correlators agree with CFT expectations once the two spheres are identified.

3.6.6 Is Celestial Holography a Wedge Holography?

We now superpose the two wedges to assess to what extent celestial holography—the proposal that gravity on Minkowski is dual to a CFT on S^{d-1} at null infinity—matches the wedge picture. For vacuum observables (partition functions, entanglement) the match is clean; for dynamical excitations one must adjust the r -boundary condition on the de Sitter side.

Partition function in Minkowski spacetime Take the regulated ranges

$$\rho \leq \rho_\infty, \quad t \leq t_\infty,\tag{3.140}$$

and combine the hyperbolic and de Sitter contributions (3.95), (3.120) with

$$\eta_2 = \eta_\infty, \quad r_2 = r_\infty, \quad \eta_1 = r_1 = 0,\tag{3.141}$$

doubling to cover both $t > 0$ and $t < 0$:

$$I_G = \frac{1}{4\pi G_N} r_\infty^{d-1} \omega_{d-1} I_{d-1} - \frac{1}{4\pi G_N} \eta_\infty^{d-1} \omega_{d-1} J_{d-1}.\tag{3.142}$$

Using $\epsilon = e^{-\rho_\infty} = e^{-t_\infty}$, one finds for $d = 3, 4, 5$:

$$\begin{aligned}d = 3: \quad I_G &= \frac{r_\infty^2 - \eta_\infty^2}{8G_N \epsilon^2} - \frac{r_\infty^2 + \eta_\infty^2}{2G_N} \log \epsilon, \\ d = 4: \quad I_G &= \frac{r_\infty^3 - \eta_\infty^3}{24G_N \epsilon^3} + \frac{3(r_\infty^3 + \eta_\infty^3)}{8G_N \epsilon}, \\ d = 5: \quad I_G &= \frac{\pi(r_\infty^4 - \eta_\infty^4)}{96G_N \epsilon^4} + \frac{\pi(r_\infty^4 + \eta_\infty^4)}{12G_N \epsilon^2} - \frac{\pi(r_\infty^4 - \eta_\infty^4)}{4G_N} \log \epsilon.\end{aligned}\tag{3.143}$$

Extracting the universal logarithms gives

$$d = 3: \quad c = i \frac{3(r_\infty^2 + \eta_\infty^2)}{2G_N}, \quad d = 5: \quad a = i \frac{\pi(\eta_\infty^4 - r_\infty^4)}{16G_N},\tag{3.144}$$

again imaginary and hence non-unitary. In the continuum limit $\eta_\infty, r_\infty \rightarrow \infty$ the $d = 3$ central charge diverges, consistent with [32, 102]; interestingly, a may be tuned to zero at $\eta_\infty = r_\infty$.

Holographic entanglement entropy in Minkowski spacetime For a region $|\theta| \leq \theta_0 \subset S^{d-1}$ the total entropy is the sum of the hyperbolic and de Sitter wedge contributions (doubled to cover the full spacetime):

$$S_A = \frac{A(\Gamma_A^H) + A(\Gamma_A^{\text{dS}})}{4G_N} = \frac{i\eta_\infty^{d-1}}{2(d-1)G_N} A(\gamma_A^H) + \frac{r_\infty^{d-1}}{2(d-1)G_N} A(\gamma_A^{\text{dS}}). \quad (3.145)$$

For $d = 3$ and $d = 5$:

$$\begin{aligned} d = 3: \quad S_A &= \frac{i}{4G_N} (\eta_\infty^2 + r_\infty^2) \log \left(\frac{\sin^2 \theta_0}{\epsilon^2} \right), \\ d = 5: \quad S_A &= i \frac{\pi(\eta_\infty^4 + r_\infty^4)}{16G_N \epsilon^2} + i \frac{\pi(r_\infty^4 - \eta_\infty^4)}{4G_N} \log \left(\frac{\sin \theta_0}{4\epsilon} \right) + O(1), \end{aligned} \quad (3.146)$$

from which the same central charges (3.144) are recovered.

3.6.7 Celestial Holography versus Wedge Holography with Excitations

Celestial holography relates 4D scattering amplitudes $\mathcal{A}(k_i)$ to celestial correlators by smearing with conformal primary wavefunctions [14, 19]:

$$\langle O_{\Delta_1} \cdots O_{\Delta_N} \rangle_{S^2} = \prod_{i=1}^N \left[\int d^4 X_i \phi^{\Delta_i, (\pm)}(X_i; \theta_i, \phi_i) \int d^4 k_i e^{ik_i \cdot X_i} \right] \mathcal{A}(k_1, \dots, k_N), \quad (3.147)$$

with

$$\phi^{\Delta, \pm}(X^\mu; \theta_0, \phi_0) = \frac{(\sqrt{-X^2})^{\Delta-1}}{(q \cdot X^\pm)^\Delta} K_{\Delta-1}(m\sqrt{-X^2}), \quad (3.148)$$

where q^μ points to (θ_0, ϕ_0) and $X^{\pm\mu} = X^\mu \pm i\epsilon\{-1, 0, 0, 0\}$. In hyperbolic/de Sitter coordinates these CPWFs read (with $\epsilon \rightarrow 0$)

$$\phi^{\Delta, \pm}|_{X^2 < 0} = \frac{K_{\Delta-1}(m\eta)}{\eta} \left(\frac{1 + \cos \theta_0}{2} \right)^\Delta (\cosh \rho - \cos \gamma \sinh \rho)^{-\Delta}, \quad (3.149)$$

$$\phi^{\Delta, \pm}|_{X^2 > 0} = -\frac{\pi i H_{\Delta-1}^{(a)}(mr)}{2r} \left(\frac{1 + \cos \theta_0}{2} \right)^\Delta (\sinh t - \cos \gamma \cosh t)^{-\Delta}, \quad (3.150)$$

with $a = 1, 2$ for out-/in-going waves. These are precisely of the delta-sourced form (3.88), (3.89). On the hyperbolic side, the celestial and wedge prescriptions coincide: the large- η behavior $K_\nu(z) \sim e^{-z}$ enforces the same (Dirichlet/Neumann) boundary condition at $\eta = \eta_\infty$. On the de Sitter side, celestial holography imposes radiative (Hankel) boundary conditions at $r = r_\infty$, whereas our wedge analysis above adopted Dirichlet/Neumann; this changes only an overall normalization in correlators, not their angular dependence (3.139). Finally, the admissible spectrum in both patches is the principal series, $\Delta = 1 + i\lambda$ with $\lambda \in \mathbb{R}$, which in the celestial framework follows from normalizability rather than hard boundary conditions [12].

4 Rectified Dictionary for Celestial Holography

In celestial holography, a proposed relation connects scattering amplitudes of massless fields in Minkowski space to CCFT correlation functions (e.g. [18]):

$$\left\langle \prod_{i=1}^n \mathcal{O}_{\Delta_i} \right\rangle = \left[\prod_{j=1}^n \int d\omega_j \omega_j^{\Delta_j-1} \right] \mathcal{A}(p_1, \dots, p_n). \quad (4.1)$$

For massless particles the two-point amplitude is

$$\mathcal{A}(p_1, p_2) = (2\pi)^3 2p_1^0 \delta^{(3)}(\vec{p}_1 - \vec{p}_2). \quad (4.2)$$

Consequently, the celestial two-point correlator becomes

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1=1+i\lambda_1}(z_1) \mathcal{O}_{\Delta_2=1+i\lambda_2}(z_2) \rangle &= \prod_{i=1,2} \left(\int d\omega_i \omega_i^{i\lambda_i} \right) \mathcal{A}(p_1, p_2) \\ &= (2\pi)^4 \delta(\lambda_1 + \lambda_2) \delta^{(2)}(z_1 - z_2). \end{aligned} \quad (4.3)$$

Thus the massless scalar two-point function is distributional, supported at coincident points. While such a form is consistent with conformal covariance, it is unsatisfactory for CCFT structure. In this section we therefore refine the dictionary so that two-point functions take the standard power-law form. The modification is also motivated by taking the massless limit of the massive dictionary and by implementing crossing symmetry for scattering amplitudes. This section is based on our paper [21].

4.1 Notations in this section

Massless on-shell momenta p^μ are described in terms of the energy ω ($0 \leq \omega < \infty$) and angular variables, for which we employ the complex coordinate $z \in \mathbb{C}$, as

$$p^\mu = \omega q^\mu(z), \quad q^\mu(z) = (1 + z\bar{z}, z + \bar{z}, i(\bar{z} - z), 1 - z\bar{z}). \quad (4.4)$$

On the complex plane, we adopt the integration measure $\int d^2z := 2 \int d\text{Re}[z] d\text{Im}[z]$ and define the two-dimensional delta function by $\delta^{(2)}(z_1 - z_2) \equiv \frac{1}{2} \delta(\text{Re}z_1 - \text{Re}z_2) \delta(\text{Im}z_1 - \text{Im}z_2)$, following the convention of [103].

For massive on-shell momenta p^μ with mass m , it is convenient to introduce the rescaled vector $\hat{p}^\mu := m^{-1}p^\mu$. The vector $\hat{p}^\mu$ satisfies $\hat{p}^2 = -1$ and can be regarded as the embedding of a point in the three-dimensional unit hyperbolic space H^3 into $\mathbf{R}^{1,3}$. In terms of Poincaré coordinates (y, z) , $\hat{p}^\mu$ is parametrized as

$$\hat{p}^\mu(y, z) = \frac{1}{2y} (1 + y^2 + z\bar{z}, z + \bar{z}, i(\bar{z} - z), 1 - y^2 - z\bar{z}). \quad (4.5)$$

The integration measure on H^3 is then defined by

$$\int [d\hat{p}] := \int \frac{d^3\hat{p}^i}{\hat{p}^0} = \int \frac{dy d\text{Re}[z] d\text{Im}[z]}{y^3} = \int \frac{dy d^2z}{2y^3}. \quad (4.6)$$

4.2 Massless Scalar

The celestial dictionary translates Minkowski amplitudes into CCFT correlators. For massless scalars we consider the amplitude $\mathcal{A}(\{p_i^{\text{out}}, p_j^{\text{in}}\})$ and propose the rectified dictionary as follow:

$$\left\langle \prod_i^m \mathcal{O}_{\Delta_i}^- \prod_j^n \tilde{\mathcal{O}}_{\Delta_j}^+ \right\rangle := \prod_{i=1}^m \prod_{j=1}^n \int d\omega_i \frac{-i\omega_i^{i\lambda_i}}{\sqrt{(2\pi)^3 \lambda_i}} \int d\omega_j \frac{i\omega_j^{-i\lambda_j}}{\sqrt{(2\pi)^3 \lambda_j}} i\mathcal{A}(p_1^{\text{out}}, \dots, p_m^{\text{out}}; p_1^{\text{in}}, \dots, p_n^{\text{in}}). \quad (4.7)$$

Here $\Delta_i \in 1 + i\lambda_i$ with $\lambda_i \in \mathbb{R}$ is the principal series. This is the Mellin transform of the amplitude, but with a crucial interpretational change: operators associated with incoming particles are taken to be the shadow transforms $\tilde{\mathcal{O}}_{\Delta}$,

$$\tilde{\mathcal{O}}_{\Delta}(z) := \frac{\Delta - 1}{2\pi} \int d^2 z' \frac{1}{|z - z'|^{2(2-\Delta)}} \mathcal{O}_{\Delta}(z'). \quad (4.8)$$

This choice ensures that non-shadow operators exhibit the conventional CFT two-point structure, as we now show.

For the two-point function, the amplitude is

$$\mathcal{A}(p_1; p_2) = (2\pi)^3 2p_1^0 \delta^{(3)}(\vec{p}_1 - \vec{p}_2). \quad (4.9)$$

The delta function becomes

$$\delta^{(3)}(\vec{p}_1 - \vec{p}_2) = \frac{1}{2\omega_1^2(1 + |z_1|^2)} \delta(\omega_1 - \omega_2) \delta^{(2)}(z_1 - z_2), \quad (4.10)$$

so that

$$\mathcal{A}(p_1; p_2) = \frac{(2\pi)^3}{\omega_1} \delta(\omega_1 - \omega_2) \delta^{(2)}(z_1 - z_2). \quad (4.11)$$

Inserting this into (4.7) yields

$$\left\langle \mathcal{O}_{1+i\lambda_1}^-(z_1) \tilde{\mathcal{O}}_{1+i\lambda_2}^+(z_2) \right\rangle = \frac{2\pi i}{\lambda_1} \delta(\lambda_1 - \lambda_2) \delta^{(2)}(z_1 - z_2). \quad (4.12)$$

Applying the inverse shadow transform to $\tilde{\mathcal{O}}^+(z_2)$ gives the standard power-law form⁶

$$\langle \mathcal{O}_{1+i\lambda_1}^-(z_1) \mathcal{O}_{1+i\lambda_2}^+(z_2) \rangle = \frac{\delta(\lambda_1 - \lambda_2)}{|z_1 - z_2|^{2(1+i\lambda_1)}}. \quad (4.15)$$

⁶We use

$$\int d^2 z \frac{1}{|z - z'|^{2(1+i\lambda)} |z - z''|^{2(1-i\lambda)}} = \frac{4\pi^2}{\lambda^2} \delta^{(2)}(z' - z''), \quad (4.13)$$

leading to

$$\tilde{\mathcal{O}}_{1+i\lambda}(z) = \frac{-i\lambda}{2\pi} \frac{i\lambda}{2\pi} \int d^2 z' d^2 z'' \frac{1}{|z - z'|^{2(1+i\lambda)} |z' - z''|^{2(1-i\lambda)}} \mathcal{O}_{1+i\lambda}(z'') = \mathcal{O}_{1+i\lambda}(z). \quad (4.14)$$

Hence the rectified dictionary restores the usual CFT two-point structure. If, instead, one did not identify incoming operators with their shadows, the outcome would remain the delta-function form (4.12).

The dictionary can be rephrased as an operator mapping between Minkowski creation/annihilation operators and CCFT operators:

$$\mathcal{O}_{1+i\lambda}^-(z) = \int_0^\infty d\omega \frac{-i\omega^{i\lambda}}{\sqrt{(2\pi)^3\lambda}} a_{\text{out}}(\omega q(z)), \quad \tilde{\mathcal{O}}_{1+i\lambda}^+(z) = \int_0^\infty d\omega \frac{i\omega^{-i\lambda}}{\sqrt{(2\pi)^3\lambda}} a_{\text{in}}^\dagger(\omega q(z)). \quad (4.16)$$

The bulk field $\phi(x)$ approaches asymptotic fields in the Heisenberg picture,

$$\phi(x) \rightarrow \sqrt{Z} \phi^{\text{out}}(x) \quad (t \rightarrow +\infty), \quad (4.17)$$

$$\phi(x) \rightarrow \sqrt{Z} \phi^{\text{in}}(x) \quad (t \rightarrow -\infty), \quad (4.18)$$

with mode expansions

$$\phi^{\text{out/in}}(x) = \int \frac{d^3p}{(2\pi)^3(2p^0)} \left[a_{\text{out/in}}(p) e^{ip \cdot x} + a_{\text{out/in}}^\dagger(p) e^{-ip \cdot x} \right]. \quad (4.19)$$

For free fields $a_{\text{in}} = a_{\text{out}}$, and thus

$$(\mathcal{O}_{1+i\lambda}^-(z))^\dagger = \int_0^\infty d\omega \frac{i\omega^{-i\lambda}}{\sqrt{(2\pi)^3\lambda}} a^\dagger(\omega q(z)) = \tilde{\mathcal{O}}_{1+i\lambda}^+(z). \quad (4.20)$$

Therefore, Hermitian conjugation with respect to the Klein–Gordon inner product on Minkowski space corresponds to the shadow transform on the celestial sphere. This differs from [104], where conjugation is defined via a “shadow product” distinct from the Klein–Gordon product, and $\langle \mathcal{O} \tilde{\mathcal{O}} \rangle$ assumes a power-law form accordingly. In our setup $\tilde{\mathcal{O}}$ is not the BPZ conjugate of $\mathcal{O}$, since $\langle \mathcal{O} \tilde{\mathcal{O}} \rangle$ is distributional.

The relations (4.16) can be recast as

$$\mathcal{O}_{1+i\lambda}^-(z) = (\varphi_{1+i\lambda}^-(x; z), \phi^{\text{out}}(x))_{\text{KG}}, \quad \tilde{\mathcal{O}}_{1+i\lambda}^+(z) = (\tilde{\varphi}_{1+i\lambda}^+(x; z), \phi^{\text{in}}(x))_{\text{KG}}, \quad (4.21)$$

where $(\cdot)_{\text{KG}}$ denotes the Klein–Gordon inner product,

$$(\phi_1, \phi_2)_{\text{KG}} := -i \int d^3\Sigma^\mu (\phi_1 \partial_\mu \phi_2^* - (\partial_\mu \phi_1) \phi_2^*). \quad (4.22)$$

The massless conformal primary wavefunctions are

$$\varphi_{1+i\lambda}^-(x; z) := \int_0^\infty d\omega \frac{i\omega^{i\lambda}}{\sqrt{(2\pi)^3\lambda}} e^{-i\omega q(z) \cdot x}, \quad (4.23)$$

$$\tilde{\varphi}_{1+i\lambda}^+(x; z) := \int_0^\infty d\omega \frac{i\omega^{-i\lambda}}{\sqrt{(2\pi)^3\lambda}} e^{i\omega q(z) \cdot x}. \quad (4.24)$$

Evaluating the integrals gives

$$\varphi_{1+i\lambda}^-(x; z) = -\frac{ie^{-\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(i\lambda)}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1+i\lambda}}, \quad (4.25)$$

$$\tilde{\varphi}_{1+i\lambda}^+(x; z) = -\frac{ie^{-\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(-i\lambda)}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1-i\lambda}}. \quad (4.26)$$

This $\tilde{\varphi}^+$ is chosen to realize (4.21) and is proportional to the standard (non-shadow) wavefunctions, cf. [18, 19].

We can likewise compute $\mathcal{O}_{1+i\lambda}^+$ directly as

$$\mathcal{O}_{1+i\lambda}^+(z) = (\varphi_{1+i\lambda}^+(x; z), \phi^{\text{in}}(x))_{\text{KG}}, \quad (4.27)$$

where $\varphi_{1+i\lambda}^+$ is the inverse-shadow of $\tilde{\varphi}_{1+i\lambda}^+$:

$$\begin{aligned} \varphi_{1+i\lambda}^+(x; z) &:= \frac{-i\lambda}{2\pi} \int d^2 z' \frac{1}{|z - z'|^{2(1+i\lambda)}} \tilde{\varphi}_{1+i\lambda}^+(x; z') \\ &= -\frac{ie^{\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(-i\lambda) (x^2)^{i\lambda}}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1+i\lambda}}. \end{aligned} \quad (4.28)$$

Up to an overall factor this agrees with the shadow conformal primary wavefunction used in [12]. Employing it for incoming particles is what ultimately yields the power-law two-point form (4.15).

4.3 Massive Scalar

We now turn to massive scalars. Apart from overall normalizations, our dictionary matches the standard one in the literature, and we will show that its massless limit reproduces the massless wavefunctions appearing in the rectified dictionary of Sec. 4.2. This continuity supports our prescription.

The massive dictionary reads

$$\begin{aligned} &\left\langle \prod_i^m \mathcal{O}_{1+i\lambda_i}^-(z_i) \prod_j^n \mathcal{O}_{1+i\lambda_j}^+(z_j) \right\rangle \\ &:= \prod_{i=1}^m \prod_{j=1}^n \int [d\hat{p}_i] c(m_i, \lambda_i) G_{1+i\lambda_i}(\hat{p}_i^{\text{out}}; z_i) \int [d\hat{p}_j] c(m_j, \lambda_j)^* G_{1+i\lambda_j}(\hat{p}_j^{\text{in}}; z_j) i\mathcal{A}(p_1^{\text{out}}, \dots, p_m^{\text{out}}; p_1^{\text{in}}, \dots, p_n^{\text{in}}), \end{aligned} \quad (4.29)$$

with $c(m, \lambda) := 2^{-i\lambda} m^{1+i\lambda} \sqrt{\lambda} / (2\pi)^{\frac{5}{2}}$. Here $G_{\Delta}(\hat{p}; z)$ is the bulk-to-boundary propagator for EAdS₃:

$$G_{\Delta}(\hat{p}(y, w); z) = \frac{1}{(-\hat{p}(y, w) \cdot q(z))^{\Delta}} = \left(\frac{y}{y^2 + |z - w|^2} \right)^{\Delta}. \quad (4.30)$$

For massive scalars the principal series is $\Delta \in 1 + i\lambda$ with $\lambda > 0$. Up to the stated normalizations, this is the conventional dictionary (see, e.g., [12]).

Including shadow operators, (4.29) implies⁷

$$\begin{aligned} & \left\langle \prod_i^m \mathcal{O}_{1+i\lambda_i}^- \prod_j^n \tilde{\mathcal{O}}_{1+i\lambda_j}^+ \right\rangle \\ &= \prod_{i=1}^m \prod_{j=1}^n \int [d\hat{p}_i] c(m_j, \lambda_j) G_{1+i\lambda_i}(\hat{p}_i^{\text{out}}; z_i) \int [d\hat{p}_j] c(m_j, \lambda_j)^* G_{1-i\lambda_j}(\hat{p}_j^{\text{in}}; z_j) i\mathcal{A}(p_1^{\text{out}}, \dots, p_m^{\text{out}}; p_1^{\text{in}}, \dots, p_n^{\text{in}}). \end{aligned} \quad (4.32)$$

For the Minkowski two-point amplitude

$$\mathcal{A}(p_1; p_2) = (2\pi)^3 2p_1^0 \delta^{(3)}(\vec{p}_1 - \vec{p}_2) = \frac{(2\pi)^3}{m^2} 2\hat{p}_1^0 \delta^{(3)}(\hat{p}_1 - \hat{p}_2), \quad (4.33)$$

the dictionary produces

$$\langle \mathcal{O}_{1+i\lambda_1}^-(z_1) \mathcal{O}_{1+i\lambda_2}^+(z_2) \rangle = \frac{\delta(\lambda_1 - \lambda_2)}{|z_1 - z_2|^{2(1+i\lambda_1)}}, \quad (4.34)$$

$$\langle \mathcal{O}_{1+i\lambda_1}^-(z_1) \tilde{\mathcal{O}}_{1+i\lambda_2}^+(z_2) \rangle = \frac{2\pi i}{\lambda_1} \delta(\lambda_1 - \lambda_2) \delta^{(2)}(z_1 - z_2). \quad (4.35)$$

Thus, for massive scalars, the standard power-law two-point structure follows directly.

As in the massless case, CCFT operators are related to creation/annihilation modes through

$$\mathcal{O}_{1+i\lambda}^-(z) = \int [d\hat{p}] \frac{2^{-i\lambda} m^{1+i\lambda} \sqrt{\lambda} G_{1+i\lambda}(\hat{p}; z)}{(2\pi)^{\frac{5}{2}}} a_{\text{out}}(m\hat{p}), \quad (4.36)$$

$$\tilde{\mathcal{O}}_{1+i\lambda}^+(z) = \int [d\hat{p}] \frac{2^{i\lambda} m^{1-i\lambda} \sqrt{\lambda} G_{1-i\lambda}(\hat{p}; z)}{(2\pi)^{\frac{5}{2}}} a_{\text{in}}^\dagger(m\hat{p}). \quad (4.37)$$

For free fields $a_{\text{in}} = a_{\text{out}}$, hence

$$(\mathcal{O}_{1+i\lambda}^-(z))^\dagger = \int [d\hat{p}] \frac{m \sqrt{\lambda} G_{1-i\lambda}(\hat{p}; z)}{(2\pi)^{\frac{5}{2}}} a^\dagger(m\hat{p}) = \tilde{\mathcal{O}}_{1+i\lambda}^+(z). \quad (4.38)$$

This mirrors the massless relation.

The operator map can also be written as⁸

$$\mathcal{O}_{1+i\lambda}^-(z) = (\Phi_{1+i\lambda}^-(x; z), \phi^{\text{out}}(x))_{\text{KG}}, \quad \mathcal{O}_{1+i\lambda}^+(z) = (\Phi_{1+i\lambda}^+(x; z), \phi^{\text{in}}(x))_{\text{KG}}, \quad (4.39)$$

$$\tilde{\mathcal{O}}_{1+i\lambda}^+(z) = (\tilde{\Phi}_{1+i\lambda}^+(x; z), \phi^{\text{in}}(x))_{\text{KG}}, \quad (4.40)$$

⁷Note that

$$\frac{-i\lambda}{2\pi} \int d^2 z' \frac{1}{|z - z'|^{2(1+i\lambda)}} G_{1-i\lambda}(\hat{p}; z') = G_{1+i\lambda}(\hat{p}; z). \quad (4.31)$$

⁸One may similarly define $\tilde{\mathcal{O}}_{1+i\lambda}^-(z) = (\tilde{\Phi}_{1+i\lambda}^-(x; z), \phi^{\text{out}}(x))_{\text{KG}}$ with $\tilde{\Phi}^-$; we omit it as it will not be needed below.

with massive conformal primary wavefunctions defined by

$$\Phi_{1+i\lambda}^{\mp}(x; z) = \mp \int [d\hat{p}] \frac{2^{\mp i\lambda} m^{1 \pm i\lambda} \sqrt{\lambda} G_{1+i\lambda}(\hat{p}; z)}{(2\pi)^{\frac{5}{2}}} e^{\mp i m \hat{p} \cdot x}, \quad (4.41)$$

$$\tilde{\Phi}_{1+i\lambda}^+(x; z) = \int [d\hat{p}] \frac{2^{i\lambda} m^{1-i\lambda} \sqrt{\lambda} G_{1-i\lambda}(\hat{p}; z)}{(2\pi)^{\frac{5}{2}}} e^{+i m \hat{p} \cdot x}. \quad (4.42)$$

Using the integral identity⁹

$$\int [d\hat{p}] G_{1+i\lambda}(\hat{p}; q) e^{\pm i m \hat{p} \cdot x} = \frac{4\pi e^{\mp \frac{i\pi}{2}} e^{\pm \frac{\pi\lambda}{2}} (\sqrt{x^2})^{i\lambda}}{m(-q \cdot x \mp i\epsilon)^{1+i\lambda}} K_{i\lambda}(m\sqrt{x^2}), \quad (4.43)$$

one finds

$$\Phi_{1+i\lambda}^{\mp}(x; z) = -\frac{i 2^{1 \mp i\lambda} e^{\mp \frac{\pi}{2}\lambda} \sqrt{\lambda} m^{\pm i\lambda} (\sqrt{x^2})^{i\lambda}}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1 \pm i\lambda}} K_{i\lambda}(m\sqrt{x^2}), \quad (4.44)$$

$$\tilde{\Phi}_{1+i\lambda}^+(x; z) = -\frac{i 2^{1+i\lambda} e^{-\frac{\pi}{2}\lambda} \sqrt{\lambda} (m\sqrt{x^2})^{-i\lambda}}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1-i\lambda}} K_{-i\lambda}(m\sqrt{x^2}). \quad (4.45)$$

4.4 Massless Limit

Consider now the limit $m \rightarrow 0$. For small argument, the Bessel function expands as

$$K_{i\lambda}(m\sqrt{x^2}) = 2^{-1+i\lambda} \Gamma(i\lambda) (m\sqrt{x^2})^{-i\lambda} + 2^{-1-i\lambda} \Gamma(-i\lambda) (m\sqrt{x^2})^{i\lambda} + \mathcal{O}(m^2). \quad (4.46)$$

Hence, near $m = 0$,

$$\Phi_{1+i\lambda}^-(x; z) = -\frac{i e^{-\frac{\pi\lambda}{2}} \sqrt{\lambda}}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1+i\lambda}} \left[\Gamma(i\lambda) + 2^{-2i\lambda} \Gamma(-i\lambda) (m\sqrt{x^2})^{2i\lambda} + \mathcal{O}(m^2) \right]. \quad (4.47)$$

By the Riemann–Lebesgue lemma, $m^{\pm 2i\lambda} \rightarrow 0$ in the distributional sense, and the oscillatory term can be dropped. Therefore,

$$\lim_{m \rightarrow 0} \Phi_{1+i\lambda}^-(x; z) = -\frac{i e^{-\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(i\lambda)}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1+i\lambda}} = \varphi_{1+i\lambda}^-(x; z), \quad (4.48)$$

which exactly¹⁰ matches (4.25). Similarly,

$$\lim_{m \rightarrow 0} \Phi_{1+i\lambda}^+(x; z) = -\frac{i e^{\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(-i\lambda) (x^2)^{i\lambda}}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1-i\lambda}} = \varphi_{1+i\lambda}^+(x; z). \quad (4.49)$$

$$\lim_{m \rightarrow 0} \tilde{\Phi}_{1+i\lambda}^+(x; z) = -\frac{i e^{-\frac{\pi\lambda}{2}} \sqrt{\lambda} \Gamma(-i\lambda)}{(2\pi)^{\frac{3}{2}} (-q(z) \cdot x)^{1-i\lambda}} = \tilde{\varphi}_{1+i\lambda}^+(x; z). \quad (4.50)$$

Thus the standard massive dictionary (4.29) collapses smoothly to the rectified massless dictionary (4.7).

⁹A regulator is needed to ensure convergence.

¹⁰Phases of the massless wavefunctions are convention-dependent; we have fixed them to match the $m \rightarrow 0$ limit of the massive expressions.

4.5 Comments on Crossing Symmetry

Crossing symmetry relates an amplitude with an incoming positive-energy particle to one with an outgoing negative-energy particle. We have not yet specified a dictionary for negative-energy states, but it is natural to treat all outgoing insertions uniformly, irrespective of their energy sign:

$$\langle \mathcal{O}_1^{-,m}(z_1) \mathcal{O}_2'^{-,m}(z_2) \rangle \propto \int [dp_1][dp_2] G_{1+i\lambda_1}(p_1; z_1) G_{1+i\lambda_2}(-p_2; z_2) A(p_1, -p_2; \text{nothing}). \quad (4.51)$$

Here p_2 is future-directed, so $-p_2$ is past-directed (negative energy). We use the superscript “ $-$ ” to denote an outgoing operator and “ m ” to indicate a massive Minkowski particle. Thus $\mathcal{O}^-$ corresponds to an outgoing positive-energy particle, whereas $\mathcal{O}'^-$ denotes an outgoing negative-energy one.

For comparison, viewing the same kinematics as one incoming positive-energy and one outgoing positive-energy particle, we write

$$\langle \mathcal{O}_1^{-,m}(z_1) \mathcal{O}_2^{+,m}(z_2) \rangle \propto \int [dp_1][dp_2] G_{1+i\lambda_1}(p_1; z_1) G_{1+i\lambda_2}(p_2; z_2) A(p_1; p_2), \quad (4.52)$$

where $\mathcal{O}^+$ represents an incoming positive-energy insertion. Crossing implies

$$A(p_1, -p_2; \text{nothing}) = A(p_1; p_2). \quad (4.53)$$

Comparing (4.51) and (4.52) then yields, at the operator level,

$$\mathcal{O}_2'^{-,m} \propto \mathcal{O}_2^{+,m}. \quad (4.54)$$

The argument extends to more general choices of $\mathcal{O}_1^-$, i.e. it is an operator identity for $\mathcal{O}_2$.

In our rectified dictionary, both (4.51) and (4.52) reduce to the standard two-point form in the massive and massless theories. Consequently, the usual dictionary suffices for massive fields (up to normalization), and, by continuity of the massless limit, the same pattern holds for massless fields as well.¹¹

4.6 Operator Product Expansion

The celestial dictionary proposed in the literature leads to the distinctive feature that two-point correlators exhibit distributional support, whereas higher-point functions display power-law behavior. In this subsection we revisit the operator product expansion (OPE) in CCFT and examine its compatibility with the dictionary, emphasizing that the rectified prescription restores the standard power-law two-point structure and thereby yields an OPE consistent with three- and four-point data.

¹¹Massless scalars are subtler. A naive extension would be

$$\langle \mathcal{O}_1^-(z_1) \mathcal{O}_2^{\text{naive}}(z_2) \rangle \propto \int d\omega_1 d\omega_2 \omega^{i\lambda_1} (-\omega_2)^{i\lambda_2} A(p_1, -p_2; \text{nothing}), \quad (4.55)$$

but this does not behave properly; a more careful prescription is needed to match the massless limit.

In any two-dimensional CFT, three-point functions of primary operators take the form

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle = \frac{f_{123}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3} |z_{23}|^{\Delta_2+\Delta_3-\Delta_1} |z_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \quad (4.56)$$

with structure constant f_{123} . The OPE can be written schematically as (see, e.g., [105])

$$\mathcal{O}_1(z_1) \mathcal{O}_2(z_2) = \sum_k C_{12k}(z_{12}, \partial_1, \bar{\partial}_1) \mathcal{O}_k(z_1), \quad (4.57)$$

where, in CCFT, the “sum” is in fact an integral over the continuous principal-series spectrum. Inserting this into a three-point function gives

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle = \sum_k C_{12k}(z_{12}, \partial_1, \bar{\partial}_1) \langle \mathcal{O}_k(z_1) \mathcal{O}_3(z_3) \rangle. \quad (4.58)$$

If the two-point function entering (4.58) is of the power-law form, $\langle \mathcal{O}_k(z_1) \mathcal{O}_3(z_3) \rangle \propto |z_{13}|^{-2\Delta_3}$, one recovers

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle \propto \sum_k C_{12k}(z_{12}, \partial_1, \bar{\partial}_1) \frac{1}{|z_{13}|^{2\Delta_3}}, \quad (4.59)$$

which is consistent with (4.56). By contrast, if one employs the earlier dictionary in which two-point functions are proportional to $\delta^{(2)}(z_{13})$, then (4.58) would produce

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle \propto \sum_k C_{12k}(z_{12}, \partial_1, \bar{\partial}_1) \delta^{(2)}(z_1 - z_3), \quad (4.60)$$

which is distributional and therefore incompatible with the smooth power-law behavior in (4.56). The rectified dictionary we adopt avoids this inconsistency by restoring the standard power-law two-point structure, as we now illustrate with a concrete three-point example.

Consider the tree-level amplitude for two incoming massless and one outgoing massive scalars with coupling g [19, 106]:

$$\mathcal{A}(1, 2, 3) = -i(2\pi)^4 g \delta^{(4)}(\omega_1 \hat{q}_1 + \omega_2 \hat{q}_2 - m \hat{p}_3). \quad (4.61)$$

Under the celestial transform dictated by our dictionary, the associated three-point correlator reads

$$\begin{aligned} & \langle \tilde{\mathcal{O}}_{\Delta_1=1+i\lambda_1}^+(z_1) \tilde{\mathcal{O}}_{\Delta_2=1+i\lambda_2}^+(z_2) \mathcal{O}_{\Delta_3=1+i\lambda_3}^{-,m}(z_3) \rangle \\ &= \prod_{i=1,2} \int d\omega_i \frac{i\omega_i^{-i\lambda_i}}{\sqrt{(2\pi)^3 \lambda_i}} \int [d\hat{p}_3] c(m, \lambda_3) G_{1+i\lambda_3}(\hat{p}_3; z_3) i\mathcal{A}(1, 2, 3) \\ &= -g(2\pi)^{-\frac{3}{2}} \sqrt{\frac{\lambda_3}{\lambda_1 \lambda_2}} m^{-\Delta_1-\Delta_2+\Delta_3} \frac{B\left(\frac{\Delta_1^*-\Delta_2^*+\Delta_3}{2}, \frac{\Delta_2^*-\Delta_1^*+\Delta_3}{2}\right)}{|z_{12}|^{\Delta_1^*+\Delta_2^*-\Delta_3} |z_{13}|^{\Delta_1^*+\Delta_3-\Delta_2^*} |z_{23}|^{\Delta_2^*+\Delta_3-\Delta_1^*}}. \end{aligned} \quad (4.62)$$

Applying the inverse shadow transform on the incoming legs¹², one obtains the standard three-point form

$$\langle \mathcal{O}_{\Delta_1}^+(z_1) \mathcal{O}_{\Delta_2}^+(z_2) \mathcal{O}_{\Delta_3}^{-,m}(z_3) \rangle = \frac{C(\Delta_1, \Delta_2, \Delta_3)}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3} |z_{13}|^{\Delta_1+\Delta_3-\Delta_2} |z_{23}|^{\Delta_2+\Delta_3-\Delta_1}}, \quad (4.63)$$

with

$$\begin{aligned} C(\Delta_1, \Delta_2, \Delta_3) = & -g(2\pi)^{-\frac{3}{2}} \sqrt{\frac{\lambda_3}{\lambda_1 \lambda_2}} \frac{m^{-\Delta_1-\Delta_2+\Delta_3}}{2^{-\Delta_1-\Delta_2+\Delta_3+3}} \frac{\Delta_1^* + \Delta_2^* + \Delta_3 - 2}{\Delta_1 + \Delta_2 + \Delta_3 - 2} \\ & \times \frac{B\left(\frac{\Delta_1-\Delta_2+\Delta_3}{2}, \frac{\Delta_2-\Delta_1+\Delta_3}{2}\right) B\left(\frac{\Delta_1+\Delta_2+\Delta_3}{2}, \frac{\Delta_1+\Delta_2-\Delta_3}{2}\right) B(\Delta_1^*, \Delta_2^*)}{B\left(\frac{\Delta_1^*+\Delta_2^*+\Delta_3}{2}, \frac{\Delta_1^*+\Delta_2^*-\Delta_3}{2}\right) B(\Delta_1, \Delta_2)}. \end{aligned} \quad (4.64)$$

In the OPE channel $|z_{23}| \rightarrow 0$ one finds

$$\langle \mathcal{O}_{\Delta_1}^+(z_1) \mathcal{O}_{\Delta_2}^+(z_2) \mathcal{O}_{\Delta_3}^{-,m}(z_3) \rangle \rightarrow \frac{C(\Delta_1, \Delta_2, \Delta_3)}{|z_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \frac{1}{|z_{12}|^{2\Delta_1}}, \quad (4.65)$$

which is precisely the power-law behavior implied by (4.59). If instead one attempted to realize this limit using a delta-function two-point structure, (4.58) would force the correlator to be supported at $z_1 = z_2$ (as in (4.60)), contradicting (4.65). In the OPE decomposition

$$\langle \mathcal{O}_{\Delta_1}^+(z_1) \mathcal{O}_{\Delta_2}^+(z_2) \mathcal{O}_{\Delta_3}^{-,m}(z_3) \rangle = \sum_k C_{23k}(z_{23}, \partial_{z_2}, \bar{\partial}_{z_2}) \langle \mathcal{O}_{\Delta_1}^+(z_1) \mathcal{O}_{\Delta_k}^k(z_2) \rangle, \quad (4.66)$$

the intermediate sum runs over both massless and massive primaries; however, only operators with nonvanishing two-point overlap with the massless $\mathcal{O}_{\Delta_1}^+$ actually contribute in this channel, which again enforces a power-law rather than distributional kernel.

A similar conclusion holds for four-point functions. Using the earlier dictionary, the power-law decomposition of the four-point function is spoiled by delta-function two-point inputs, leading to distributional celestial correlators for massless external states [107, 108]. By contrast, with the rectified dictionary one recovers the expected conformal-block-like expansion,

$$\langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \mathcal{O}_4(z_4) \rangle = \sum_k C_{12k}(z_{12}, \partial_{z_1}, \bar{\partial}_{z_1}) C_{34k}(z_{34}, \partial_{z_3}, \bar{\partial}_{z_3}) \frac{1}{|z_{13}|^{2\Delta_k}}, \quad (4.67)$$

with the sum understood as an integral over the continuous principal series (and spins, as appropriate). Thus the rectified dictionary restores the standard CFT logic: two-point functions are power laws, three-point functions take the universal form (4.63), OPE limits such as (4.65) are smooth, and four-point functions admit non-distributional celestial expansions.

¹²We use the integral identity [68]:

$$\int d^2 z_0 \frac{1}{|z_{10}|^{2a} |z_{20}|^{2b} |z_{30}|^{2c}} = \frac{2\pi \Gamma(1-a) \Gamma(1-b) \Gamma(1-c)}{\Gamma(a) \Gamma(b) \Gamma(c)} \frac{1}{|z_{12}|^{2(1-c)} |z_{23}|^{2(1-a)} |z_{31}|^{2(1-b)}},$$

which underlies the inverse shadow transform manipulations quoted above.

5 Bridging dS/CFT towards Celestial Holography

In this section, we develop a strategy to reinterpret celestial holography in terms of the dS/CFT correspondence, following our previous work [30, 31]. By combining a Weyl transformation and a Fourier transform, we map massless fields on Minkowski space $\mathbb{M}_{D+2}$ to fields with a continuous mass spectrum on de Sitter space dS_{D+1} , thereby relating celestial operators to cosmological operators on S^D . From this construction, we find that a $U(1)$ gauge field on $D+2$ dimensional Minkowski spacetime is mapped to a Proca field with a continuous mass spectrum on dS_{D+1} , while linearized gravity on $\mathbb{M}_{D+2}$ is mapped to a Fierz–Pauli field with a continuous mass spectrum on dS_{D+1} . Moreover, the pure-gauge modes in Minkowski space are shown to correspond to (partially) massless modes on de Sitter space. As a consequence, the Ward–Takahashi identities on the de Sitter boundary can be identified with the conformal soft theorems in celestial holography. In particular, for linearized gravity, by focusing on the superrotation sector—namely the massless spin-2 mode on dS_{D+1} —one can interpret the celestial stress tensor as the boundary limit of the massless mode of the Fierz–Pauli field on de Sitter space. Scalar analysis in this section is review of our paper [30]. Gauge field analysis in this section based on our paper [31].

5.1 Notations in this section

Coordinate and metric We start from $D+2$ dimensional Euclidean space $\mathbb{R}^{D+2}$ with Cartesian coordinates X^I , where indices $I, J, \dots$ are raised and lowered using the flat metric $G_{IJ} = \delta_{IJ}$.

On the unit sphere S^{D+1} we introduce intrinsic coordinates x^i , with indices $i, j, \dots$ manipulated with the sphere metric γ_{ij} . In addition, we employ embedding coordinates x^I for S^{D+1} subject to the constraint $x^I x_I = 1$.

The basic line element on $\mathbb{R}^{D+2}$ is thus

$$ds^2 = G_{IJ} dX^I dX^J. \quad (5.1)$$

Introducing polar coordinates by writing $X^I = r x^I$ with $r > 0$ and $x \in S^{D+1}$, we obtain

$$ds^2 = dr^2 + r^2 \gamma_{ij} dx^i dx^j = e^{2\xi} (d\xi^2 + \gamma_{ij} dx^i dx^j), \quad r = e^\xi. \quad (5.2)$$

Hence, after a Weyl rescaling, the flat metric can be cast into a cylindrical form,

$$e^{-2\xi} G_{IJ} dX^I dX^J = d\xi^2 + \gamma_{ij} dx^i dx^j. \quad (5.3)$$

We next parametrize Minkowski space $\mathbb{M}_{D+2}$ as

$$X(t, r, \mathbf{p}) = (r \sinh t, r \cosh t \mathbf{p}), \quad (5.4)$$

where $\mathbf{p}$ is a unit vector that we identify with a point on S^D . We denote by x^α the coordinates on S^D and by $\gamma_{\alpha\beta}$ the corresponding metric.

The de Sitter space dS_{D+1} is then embedded via

$$x(t, r, \mathbf{p}) = (\sinh t, \cosh t \mathbf{p}). \quad (5.5)$$

For later use, it is convenient to decompose X into null vectors $p_{\pm} = (1, \pm \mathbf{p})$ as

$$X = \frac{1}{2} r e^t p_+ - \frac{1}{2} r e^{-t} p_-, \quad (5.6)$$

and similarly to write the de Sitter embedding coordinates x as

$$x = \frac{1}{2} e^t p_+ - \frac{1}{2} e^{-t} p_-. \quad (5.7)$$

Definition of celestial operators To streamline the notation and to make analyticity in the complex- p plane (in particular, crossing symmetry) manifest, we introduce momentum-space operators by

$$\begin{aligned} a(\omega p) &\equiv a_{\text{out}}(\omega \mathbf{p}), \\ a(-\omega p) &\equiv a_{\text{in}}^{\dagger}(\omega \mathbf{p}). \end{aligned} \quad (5.8)$$

Using these operators, we define the celestial operators as¹³

$$\mathcal{O}_{\Delta}(p) = \int_0^{\infty} \frac{d\omega}{\omega} \omega^{\Delta} a(\omega p). \quad (5.9)$$

In the context of celestial holography, we employ the rectified dictionary of [21], according to which the creation and annihilation operators of bulk fields in flat space are mapped to primary and shadow operators in the celestial CFT as

$$\begin{aligned} \mathcal{O}_{\Delta}^{+}(p) &= \mathcal{O}_{\Delta}(p), \\ \widetilde{\mathcal{O}_{\Delta}^{-}(p)} &= \mathcal{O}_{\Delta}(-p), \end{aligned} \quad (5.10)$$

where, for simplicity, we have suppressed additional index structures.

Integral measure When integrating over a manifold M , we write:

$$\int_M [d^D x] \equiv \int_M d^D x \sqrt{g_M} \quad (5.11)$$

where g_M is the determinant of the metric of the manifold M .

5.2 Dimensional Reduction

Following [30–32], we carry out a radial decomposition of free scalar, Maxwell theory, and in parallel of linearized gravity, in $D+2$ dimensions. The construction proceeds by a Weyl map from $\mathbb{R}^{D+2} \setminus \{0\}$ to the cylinder $S^{D+1} \times \mathbb{R}_{\xi}$, a convenient gauge choice along the radial direction, and a Fourier transform in ξ that organizes the dynamics into a continuum of massive modes on the unit sphere.¹⁴

¹³Here we allow the embedding vector p that labels the operator $\mathcal{O}_{\Delta}$ to be past-directed as well. This is a departure from the usual convention for embedding-space CFT operators, but it is purely a matter of notation; the standard CFT operator can be recovered using (5.10).

¹⁴The Weyl map is the standard one induced by $X^I = e^{\xi} \hat{x}^I(x)$ with $\hat{x}^2 = 1$, so that $ds_{\mathbb{R}^{D+2}}^2 = e^{2\xi} (d\xi^2 + \gamma_{ij}(x) dx^i dx^j)$. See, e.g., [32].

5.2.1 Scalar Field

We start from a real scalar field on the Euclidean space $\mathbb{R}^{D+2}$ with action

$$S = \int_{\mathbb{R}^{D+2}} [d^{D+2}X] \left(\frac{1}{2} G^{IJ} \nabla_I \Phi \nabla_J \Phi + \frac{D}{8(D+1)} R \Phi^2 \right), \quad (5.12)$$

where G_{IJ} is the flat Euclidean metric on $\mathbb{R}^{D+2}$ and R denotes its scalar curvature.

We now perform a Weyl rescaling to pass from the flat metric G_{IJ} to the cylinder metric,

$$g_{IJ} = e^{-2\xi} G_{IJ}, \quad (5.13)$$

and simultaneously redefine the scalar field as

$$\phi = e^{\frac{D}{2}\xi} \Phi. \quad (5.14)$$

With this choice of conformal factor and curvature coupling, the action can be expressed as a theory on $S^{D+1} \times \mathbb{R}$. Denoting by g_{ij}^S the metric on the unit sphere S^{D+1} and by ∇_i the corresponding covariant derivative, the action takes the form

$$S = \int_{S^{D+1}} [d^{D+1}x] \int d\xi \left(\frac{1}{2} g_S^{ij} \nabla_i \phi \nabla_j \phi + \frac{1}{2} \partial_\xi \phi \partial_\xi \phi + \frac{D^2}{8} \phi^2 \right). \quad (5.15)$$

In other words, the original flat-space theory has been recast as a scalar field on the cylinder with a constant mass term. To diagonalise the dynamics along the non-compact ξ -direction, we now perform a Fourier transform,

$$\begin{aligned} \psi_\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi e^{-i\lambda\xi} \phi(\xi, x), \\ \phi(\xi, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda e^{i\lambda\xi} \psi_\lambda(x), \end{aligned} \quad (5.16)$$

where $\lambda \in \mathbb{R}$ labels the Fourier modes. Since the original scalar Φ (and hence ϕ) is real, the Fourier modes satisfy

$$\psi_\lambda^\dagger = \psi_{-\lambda}. \quad (5.17)$$

Substituting the mode expansion (5.16) into the cylinder action and integrating over ξ , we obtain an equivalent description in terms of an infinite tower of fields ψ_λ living on S^{D+1} :

$$S = \int_{S^{D+1}} [d^{D+1}x] \int_0^\infty d\lambda \left(g_S^{ij} \nabla_i \psi_\lambda^\dagger \nabla_j \psi_\lambda + \left(\frac{D^2}{4} + \lambda^2 \right) \psi_\lambda^\dagger \psi_\lambda \right). \quad (5.18)$$

Thus, each Fourier mode ψ_λ behaves as a scalar field on S^{D+1} with mass

$$m_\lambda = \sqrt{\frac{D^2}{4} + \lambda^2}. \quad (5.19)$$

The relation between the original flat-space field Φ and the spherical modes ψ_λ can be written more explicitly by combining the Weyl rescaling with a change of

variables from ξ to $r = e^\xi$. Introducing a regulator through an exponential damping factor $e^{-\epsilon r}$, we obtain

$$\begin{aligned}\psi_\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int d\xi e^{(D-\Delta)\xi} e^{-\epsilon r} \Phi(X(\xi, x)) \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \Phi(X(r, x)),\end{aligned}\tag{5.20}$$

with $\Delta = \frac{D}{2} + i\lambda$. In this way, the free scalar field on $\mathbb{R}^{D+2}$ is equivalently described as a continuous family of massive scalar fields on S^{D+1} labelled by λ .

5.2.2 Maxwell Field

We begin with free $U(1)$ gauge theory on flat space,

$$S = \frac{1}{4} \int_{\mathbb{R}^{D+2}} [d^{D+2}X] F_{IJ} F^{IJ}, \quad F_{IJ} := \partial_I A_J - \partial_J A_I.\tag{5.21}$$

Under the Weyl map to $S^{D+1} \times \mathbb{R}_\xi$ it is convenient to rescale the field by

$$\mathcal{A}_I = e^{\frac{D-2}{2}\xi} A_I,\tag{5.22}$$

and to impose the radial gauge for the rescaled field,

$$\mathcal{A}_\xi = 0.\tag{5.23}$$

With indices raised/lowered by the round metric γ_{ij} on S^{D+1} and its Levi-Civita connection ∇_i , the action becomes

$$S = \frac{1}{4} \int_{\mathbb{R}} d\xi \int_{S^{D+1}} [d^{D+1}x] \left(\mathcal{F}_{ij} \mathcal{F}^{ij} + 2 \left(\partial_\xi \mathcal{A} - \frac{D-2}{2} \mathcal{A} \right)^2 \right),\tag{5.24}$$

where

$$\mathcal{F}_{ij} := \nabla_i \mathcal{A}_j - \nabla_j \mathcal{A}_i.\tag{5.25}$$

We now Fourier transform along the ξ -direction,

$$a_i(\lambda, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi e^{-i\lambda\xi} \mathcal{A}_i(\xi, x), \quad \mathcal{A}_i(\xi, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda e^{i\lambda\xi} a_i(\lambda, x),\tag{5.26}$$

with the standard reality condition $a_i(-\lambda, x) = a_i(\lambda, x)^\dagger$. Inserting this into (5.24) yields a decoupled continuum of complex vector fields on S^{D+1} labeled by λ :

$$S = \int_{S^{D+1}} [d^{D+1}x] \int_0^\infty d\lambda \left[\frac{1}{4} f_{ij}(\lambda) f^{ij}(-\lambda) + \frac{1}{2} \left(\lambda^2 + \frac{(D-2)^2}{4} \right) a_i(\lambda) a^i(-\lambda) \right],\tag{5.27}$$

where $f_{ij}(\lambda) := \nabla_i a_j(\lambda) - \nabla_j a_i(\lambda)$.

Each Fourier mode obeys a Proca equation on S^{D+1} with Kaluza–Klein mass

$$m_\lambda^2 = \lambda^2 + \frac{(D-2)^2}{4}. \quad (5.28)$$

These modes are organized by principal series representations with conformal weights

$$\Delta_\pm = \frac{D}{2} \pm i\lambda, \quad (5.29)$$

consistent with the spin- ℓ Casimir relation for symmetric traceless tensors [109]

$$-m^2 = (\Delta + \ell - 2)(\Delta - \ell - D + 2). \quad (5.30)$$

Thus, the free spin-1 field on $\mathbb{R}^{D+2}$ is realized, after radial reduction, as a tower of vector modes on S^{D+1} with a continuous mass spectrum.

Writing $r = e^\xi$ so that $d\xi = dr/r$ and $X^I(r, x) = r\hat{x}^I(x)$, the map (5.22) together with the Fourier transform gives

$$\begin{aligned} a_i(\lambda, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\xi e^{\frac{D-2}{2}\xi} e^{-i\lambda\xi} A_i(X(\xi, x)) \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dr}{r} r^{\Delta-1} A_i(X(r, x)), \end{aligned} \quad (5.31)$$

where, for later convenience, Δ denotes the conformal parameter used in the principal series. In the embedding formalism it is sometimes useful to treat a_i as the pullback of a $(D+2)$ -vector a_I tangent to the sphere,

$$a_i \equiv \frac{\partial X^I}{\partial x^i} a_I, \quad X^I a_I = 0, \quad (5.32)$$

so that (5.31) can be rewritten as

$$a_I(\lambda, x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dr}{r} r^{\Delta-1} A_I(X(r, x)), \quad (5.33)$$

using $A_i = (\partial_i X^I) A_I$ on the cylinder.

Let's consider a pure-gauge potential $A_I = \partial_I \Lambda$ on $\mathbb{R}^{D+2}$ that preserves the radial gauge $\mathcal{A}_\xi = 0$ for the rescaled field. Since $\delta \mathcal{A}_\xi = e^{\frac{D-2}{2}\xi} \partial_\xi \Lambda$, preservation of $\mathcal{A}_\xi = 0$ requires

$$\partial_\xi \Lambda = 0, \quad (5.34)$$

and hence

$$\mathcal{A}_i(\xi, x) = e^{\pm \frac{D-2}{2}\xi} \partial_i \Lambda(x). \quad (5.35)$$

In view of (5.24), one might naively conclude that only the transformation with the plus sign generates a symmetry. Nevertheless, the action remains invariant, up to boundary contributions, under the substitution $(\partial_\xi \mathcal{A} - \frac{D-2}{2} \mathcal{A})^2 \longrightarrow (\partial_\xi \mathcal{A} + \frac{D-2}{2} \mathcal{A})^2$,

and therefore the transformation with the minus sign also defines a legitimate symmetry. This structure can ultimately be traced back to the inversion symmetry $X^I \rightarrow \frac{X^I}{|X|^2}$ of the original action (5.21), namely $\xi \rightarrow -\xi$. Its Fourier transform is supported at a purely imaginary frequency,

$$a_i(\lambda, x) = \sqrt{2\pi} \delta^G \left(\lambda \pm \frac{i}{2}(D-2) \right) \partial_i \Lambda(x), \quad (5.36)$$

where δ^G denotes the generalized delta function (see App. C). Thus, the pure-gauge data are localized at $\lambda = \pm i(D-2)/2$, which we refer to as the “massless modes.” It is important to note that, for $D \neq 2$, these massless modes $\lambda = \pm i(D-2)/2$ do not appear in the spectrum (5.27). Nevertheless, by analytic continuation we can still meaningfully incorporate these massless modes. In the context of celestial holography, this procedure corresponds to defining the conformal soft mode via an analytic continuation of the conformal wave functions [79]. Indeed, in later section, we will see that there exists the Ward-Takahashi identity associated with these fields.

In subsequent sections, we will also see that the choice of sign is equivalent to specifying whether the cosmological operators $V_{\Delta-}$ and $W_{\Delta+}$, associated with a massless particle, are assigned conformal dimensions $D-1$ or 1 , respectively. Equivalently, this amounts to choosing which of the two modes is identified as the celestial operator and which as its shadow operator. We should note that, in the case $D=2$, these two modes become degenerate. Finally, we remark that the relation between inversion symmetry in Minkowski spacetime and the shadow transformation in CFT is discussed in detail in [110].

5.2.3 Linearized Gravity

We next consider the gravitational sector. As a starting point, we take the $D+2$ dimensional Einstein–Hilbert action with zero cosmological constant,

$$S = \frac{1}{2} \int_G [d^{D+2} X] R'[G], \quad (5.37)$$

where $R'[G]$ denotes the Ricci scalar constructed from the $D+2$ dimensional metric G_{IJ} . We impose radial gauge and parametrize the metric as

$$G_{IJ} dX^I dX^J = e^{2\xi} (d\xi^2 + g_{ij}(\xi, x) dx^i dx^j). \quad (5.38)$$

We then perform the Weyl transformation

$$g_{IJ} = e^{-2\xi} G_{IJ}, \quad (5.39)$$

so that $g_{IJ} dX^I dX^J = d\xi^2 + g_{ij}(\xi, x) dx^i dx^j$. Using the standard ADM decomposition of the scalar curvature, the action (5.37) can be rewritten as

$$S = \frac{1}{2} \int d\xi e^{D\xi} \int_g [d^{D+1} x] \left(R[g] - D(D-1) + K^2 - K_{ij} K^{ij} \right), \quad (5.40)$$

where $R[g]$ is the Ricci scalar of the $D+1$ dimensional metric g_{ij} and K_{ij} is the extrinsic curvature of the constant- ξ hypersurfaces,

$$K_{ij} = \frac{1}{2} \partial_\xi g_{ij}, \quad K = g^{ij} K_{ij}. \quad (5.41)$$

A detailed derivation of (5.40) is provided in App. D. Thus, each ξ -slice is described by $D+1$ dimensional Einstein–Hilbert dynamics with a positive cosmological constant $\Lambda = \frac{1}{2}D(D-1)$.

We now proceed to extract the mass spectrum of this theory. We decompose the metric as $g_{ij} = \gamma_{ij} + H_{ij}$ and denote its trace by $H = \gamma^{ij}H_{ij}$. Expanding (5.40) to quadratic order in H_{ij} , we obtain

$$S^{(2)} = \frac{1}{8} \int d\xi e^{D\xi} \int_{S^{D+1}} [d^{D+1}x] \left(L_{FP}(H) - (\partial_\xi H_{ij} \partial_\xi H^{ij} - (\partial_\xi H)^2) \right), \quad (5.42)$$

where all covariant derivatives are taken with respect to γ_{ij} , indices are raised using γ^{ij} , and

$$\begin{aligned} L_{FP}(H) = & -\nabla_k H_{ij} \nabla^k H^{ij} + 2\nabla_i H_{jk} \nabla^j H^{ik} - 2\nabla_i H^{ij} \nabla_j H \\ & + \nabla_k H \nabla^k H + 2D \left(H_{ij} H^{ij} - \frac{1}{2} H^2 \right) \end{aligned} \quad (5.43)$$

is the Fierz–Pauli Lagrangian on S^{D+1} (see App. E). The last term does not represent a genuine mass term; instead, it reflects the fact that the background spacetime is a sphere [111–113].

To absorb the overall factor $e^{D\xi}$, we rescale the fields as

$$\mathcal{H}_{ij} = e^{\frac{D}{2}\xi} H_{ij}, \quad (5.44)$$

which yields

$$S^{(2)} = \frac{1}{8} \int d\xi \int_{S^{D+1}} [d^{D+1}x] \left(L_{FP}(\mathcal{H}) - (\partial_\xi \mathcal{H}_{ij} \partial_\xi \mathcal{H}^{ij} - (\partial_\xi \mathcal{H})^2) - \frac{D^2}{4} (\mathcal{H}_{ij} \mathcal{H}^{ij} - \mathcal{H}^2) \right), \quad (5.45)$$

where $\mathcal{H} = \gamma^{ij} \mathcal{H}_{ij}$.

Next, we perform a Fourier transform along the ξ -direction,

$$\begin{aligned} h_{ij}(\lambda, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi e^{-i\lambda\xi} \mathcal{H}_{ij}(\xi, x), \\ \mathcal{H}_{ij}(\xi, x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\lambda e^{+i\lambda\xi} h_{ij}(\lambda, x), \end{aligned} \quad (5.46)$$

with the reality condition $h_{ij}(-\lambda, x) = h_{ij}(\lambda, x)^\dagger$. Substituting this into the action, we find

$$S^{(2)} = \frac{1}{8} \int_{-\infty}^{\infty} d\lambda \int_{S^{D+1}} [d^{D+1}x] \left(L_{FP}^C(h(\lambda)) - m_\lambda^2 (h_{ij}(\lambda) h^{\dagger ij}(\lambda) - h(\lambda) h^\dagger(\lambda)) \right), \quad (5.47)$$

where

$$m_\lambda = \sqrt{\frac{D^2}{4} + \lambda^2}, \quad (5.48)$$

and

$$\begin{aligned} L_{FP}^C(h) = & -\nabla_k h_{ij}^\dagger \nabla^k h^{ij} + 2\nabla_i h_{jk}^\dagger \nabla^j h^{ik} - 2\nabla_i h^{\dagger ij} \nabla_j h \\ & + \nabla_k h^\dagger \nabla^k h + 2D \left(h_{ij}^\dagger h^{ij} - \frac{1}{2} h^\dagger h \right), \end{aligned} \quad (5.49)$$

with $h = \gamma^{ij} h_{ij}$. The corresponding conformal dimensions are therefore given by

$$\Delta_\pm = \frac{D}{2} \pm i\lambda. \quad (5.50)$$

where we have used (5.30).

It is often convenient to replace the coordinate ξ by the radial variable $r = e^\xi$ and reinterpret the Fourier transform as a Mellin transform in r . In particular, by going to a basis labelled by the conformal weight, we can express the correspondence as

$$h_{ij}(\lambda, x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dr}{r} r^{\Delta-2} \mathcal{H}_{ij}(r, x), \quad (5.51)$$

where $\mathcal{H}_{ij}$ denotes the radial-gauge perturbation of the original metric G_{IJ} . Using the Weyl rescaling (5.39), $\mathcal{H}_{ij}$ is related to H_{ij} via

$$H_{ij}(\xi, x) = e^{-2\xi} \mathcal{H}_{ij}(\xi, x). \quad (5.52)$$

This is the gravitational analogue of the formula (5.33) in Maxwell theory. Similarly, the counterpart of (5.32) can be expressed in the embedding formalism as

$$h_{IJ}(\lambda, x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dr}{r} r^{\Delta-} \mathcal{H}_{IJ}(r, x). \quad (5.53)$$

In analogy with (5.32), we define

$$\frac{\partial x^I}{\partial x^i} \frac{\partial x^J}{\partial x^j} h_{IJ} := h_{ij}, \quad (5.54)$$

and impose the transverse condition $x^I h_{IJ} = 0$.

Let's considering a pure-gauge perturbation $2\partial_{(I}\zeta_{J)}$ in flat $D+2$ dimensional space, subject to the radial gauge condition $X^I \partial_{(I}\zeta_{J)} = 0$. Equivalently, ζ_I obeys

$$(X \cdot \partial)\zeta_I - \zeta_I + \partial_I(X \cdot \zeta) = 0. \quad (5.55)$$

Solving this equation, one finds that the most general residual diffeomorphism preserving the radial gauge is

$$\zeta = f(x)\partial_r + \left(\frac{1}{r} \nabla^i f(x) + W^i(x) \right) \partial_i, \quad (5.56)$$

where f is a scalar function on S^{D+1} and W^i is a vector field on S^{D+1} .

For later convenience, we decompose the residual symmetry into its scalar and vector parts.

Scalar part f : supertranslations Setting $W^i = 0$ in (5.56) yields

$$\zeta = f\partial_r + \frac{1}{r}\nabla^i f\partial_i. \quad (5.57)$$

We then analytically continue this expression to Minkowski space using the parametrization (see detail in the next subsection). In these variables, (r, t) provide Rindler-like coordinates on Minkowski spacetime. To compare (5.57) with the standard asymptotic symmetry near future null infinity, we introduce retarded Bondi coordinates (U, R) via

$$U = -re^{-t}, \quad R = r \cosh t. \quad (5.58)$$

Under this change of coordinates, the basis vectors transform as

$$\partial_r = -e^{-t}\partial_U + \cosh t\partial_R, \quad \partial_t = re^{-t}\partial_U + r \sinh t\partial_R. \quad (5.59)$$

Using (5.59) and treating $f = f(t, \Omega)$, one finds that, up to subleading terms in $1/R$, (5.57) can be written as

$$\zeta = e^{-t}(-1 - \partial_t)f\partial_U + (f \cosh t - (\partial_t f) \sinh t)\partial_R + \frac{1}{r \cosh^2 t}\nabla^\alpha f\partial_\alpha + \mathcal{O}(R^{-2}). \quad (5.60)$$

Preservation of the asymptotic structure of asymptotically flat spacetimes requires that the leading behavior of f be of the form

$$f(t, x^\alpha) = -\frac{1}{2}e^t F(\Omega) + (\text{subleading}), \quad (5.61)$$

and, under this assumption, one recovers the standard supertranslation generator near future null infinity (large R),

$$\zeta = F(x^\alpha)\partial_U + G(x^\alpha)\partial_R - \frac{U + 2R}{2R^2}\nabla^\alpha F(x^\alpha)\partial_\alpha + \mathcal{O}(R^{-2}). \quad (5.62)$$

Here the coefficient $G(x^\alpha)$ of ∂_R is partly fixed by the leading contribution of f , which gives $-F(x^\alpha)$, but it can be further shifted by subleading contributions of order e^{-t} . We therefore denote the full coefficient, including these subleading pieces, by $G(x^\alpha)$. Indeed, the coefficient of ∂_R in the supertranslation generator is known to be non-universal (i.e., gauge dependent) [17, 39–41], and in particular, any choice of $G(x)$ will not affect our subsequent analysis.

At leading order, this expression agrees with the standard result in [17]. Transforming (5.62) back to the original (r, t, Ω) coordinates using (5.58), we obtain the compact radial-gauge form

$$\begin{aligned} \zeta &= e^t \left(-\frac{F}{2}\partial_r + \frac{F}{2r}\partial_t + \frac{1}{r \cosh^2 t}\nabla^\alpha F\partial_\alpha \right) \\ &\sim \frac{e^t}{2}F \left(-\partial_r + \frac{1}{r}\partial_t \right) \quad (t \rightarrow \infty). \end{aligned} \quad (5.63)$$

Vector part W^i : superrotations Likewise, setting $f = 0$ in (5.56), we obtain

$$\zeta = W^t \partial_t + \frac{1}{\cosh^2 t} W^\alpha \partial_\alpha. \quad (5.64)$$

In close analogy with the scalar analysis, we now use Bondi coordinates to relate this structure to the asymptotic symmetry algebra. By prescribing the large- t behavior of the vector field W^i , we can reproduce the superrotation transformations, just as the supertranslations emerged from the scalar sector above. The required asymptotics for W^i are

$$\begin{aligned} W^t &= -\frac{1}{D} \nabla_\alpha Y^\alpha + (\text{subleading}), \\ W^\alpha &= \frac{1}{4} e^{2t} Y^\alpha + (\text{subleading}), \end{aligned} \quad (5.65)$$

where Y^α is an arbitrary vector field on S^D , i.e. $Y \in \text{Diff}(S^D)$. The corresponding family of asymptotic symmetries is referred to as the generalized BMS group [61].¹⁵

Using (5.59), and keeping only the leading terms, one then finds that (5.64) takes the form

$$\zeta \sim Y^\alpha \partial_\alpha - \frac{1}{D} (U + R) \nabla_\alpha Y^\alpha \partial_R + \frac{U}{D} \nabla_\alpha Y^\alpha \partial_U. \quad (5.67)$$

For even values of D , this result agrees with the asymptotic symmetry generated by $\text{Diff}(S^D)$ in D dimensions, as derived in [114].

Gauge transformation in de Sitter action The diffeomorphism generated by ζ induces the following variation of the metric perturbation H_{ij} :

$$\delta \mathcal{H}_{ij} = 2e^{\pm(\frac{D}{2}-1)\xi} (\nabla_i \nabla_j + \gamma_{ij}) f + e^{\pm\frac{D}{2}\xi} (\nabla_i W_j + \nabla_j W_i), \quad (5.68)$$

where ∇_i denotes the covariant derivative with respect to γ_{ij} . The $\pm$ signs appearing here arise, as in the Maxwell case, from the symmetry under $\xi \rightarrow -\xi$. Performing a Fourier transform along ξ with spectral parameter λ , we obtain

$$\delta h_{ij} = 2(\nabla_i \nabla_j + \gamma_{ij}) f \delta^G \left(\lambda \pm i \left(\frac{D}{2} - 1 \right) \right) + (\nabla_i W_j + \nabla_j W_i) \delta^G \left(\lambda \pm i \frac{D}{2} \right). \quad (5.69)$$

The first term corresponds, on dS_{D+1} slices, to the partially massless gauge variation $\delta h_{ij} \sim (\nabla_i \nabla_j + \gamma_{ij}) f$.¹⁶ The second term is the massless diffeomorphism generated by a transverse vector on S^{D+1} . Both structures arise as residual $D+2$ dimensional gauge transformations preserving the radial gauge and, after mapping to the cylinder, are identified with the (partially) massless modes.

¹⁵In the special case $D = 2$, one may restrict Y to be a local conformal Killing vector on S^2 , obeying

$$\nabla_{(\alpha} Y_{\beta)} = \frac{1}{D} \gamma_{\alpha\beta} \nabla \cdot Y. \quad (5.66)$$

Under this restriction, the asymptotic symmetry algebra reduces to the extended BMS symmetry [55, 58].

¹⁶We review the gauge transformation of the partially massless graviton in App.E

5.3 QFT in dS_{D+1} from QFT in M_{D+2}

From the correspondence about the scalar field derived in the previous section, the n -point functions of the modes ψ_λ on the sphere S^{D+1} can be expressed in terms of correlators of the original scalar field Φ on Euclidean space $\mathbb{R}^{D+2}$ as

$$\langle \psi_{\lambda_1}(x_1) \cdots \psi_{\lambda_n}(x_n) \rangle_{S^{D+1}} = \int \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \frac{dr_i}{r_i} r_i^{\Delta_i} e^{-\epsilon r_i} \right] \langle \Phi(r_1, x_1) \cdots \Phi(r_n, x_n) \rangle_{\mathbb{R}^{D+2}}. \quad (5.70)$$

To prepare for analytic continuation, it is convenient to introduce polar coordinates on the unit sphere via

$$x^{D+2} = -\cos \theta, \quad x = (x^1, \dots, x^{D+1}) = \sin \theta \hat{n}, \quad (5.71)$$

where $\hat{n}$ is a unit vector on $\mathbb{R}^{D+1}$.

We now analytically continue the positions of the external insertions by complexifying the polar angle of each point on S^{D+1} according to

$$\theta_i = \frac{\pi}{2} + it_i. \quad (5.72)$$

Under this continuation, the correlators on $\mathbb{R}^{D+2}$ appearing on the right-hand side of (5.70) can be reinterpreted as correlation functions in the Poincaré vacuum of Minkowski spacetime M_{D+2} , with the external points confined to the spherical Rindler region $X^2 > 0$. At the same time, the correlators on S^{D+1} on the left-hand side can be viewed as correlators in de Sitter spacetime dS_{D+1} evaluated in the Bunch–Davies (BD) vacuum. The relation then takes the Minkowski form

$$\langle \psi_{\lambda_1}(x_1) \cdots \psi_{\lambda_n}(x_n) \rangle_{\text{dS}^{D+1}} = \int \left[\prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \frac{dr_i}{r_i} r_i^{D-\Delta_i} e^{-\epsilon r_i} \right] \langle \Phi(r_1, x_1) \cdots \Phi(r_n, x_n) \rangle_{\text{M}_{D+2}}. \quad (5.73)$$

At first sight, the appearance of the BD vacuum in de Sitter space on the left-hand side and the Poincaré vacuum of Minkowski space on the right-hand side may seem surprising. In particular, if one foliates Minkowski spacetime by Milne slicing as indicated in Fig. 3, one usually encounters a description in terms of a union of regions of the form $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ [32, 80].

Despite this apparent tension, the construction above is in fact consistent with the $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ picture. The key point is that the path integral on the sphere can be deformed into a contour in the complex θ -plane that passes through all three regions. Concretely, we analytically continue the locations of the external insertions on S^{D+1} and simultaneously deform the θ -integration contour in the path integral from the original segment $[0, \pi]$ to the contour shown in Fig. 7, which we

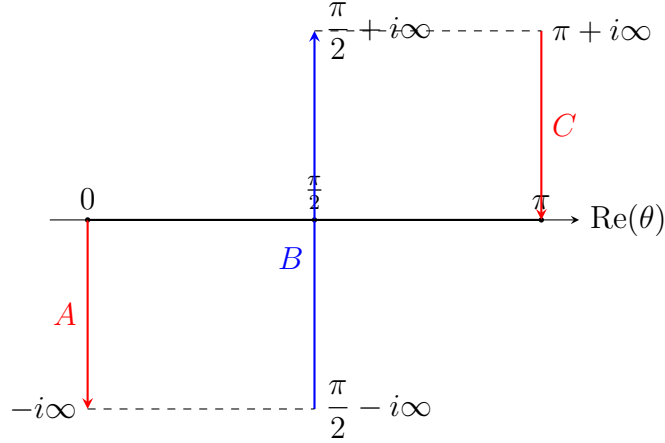


Figure 7: Deformation of the integration contour for the polar angle θ in the path integral. The original contour, running from 0 to π along the real axis, is deformed into the contours A, B, C .

parametrize as follows:¹⁷

$$\begin{aligned}
A : \quad & \theta = -i\chi, \quad \chi : 0 \rightarrow \infty, \\
B : \quad & \theta = \frac{\pi}{2} + it, \quad t : -\infty \rightarrow \infty, \\
C : \quad & \theta = \pi - i\chi, \quad \chi : -\infty \rightarrow 0.
\end{aligned} \tag{5.74}$$

Assuming that the integrand decays sufficiently fast at large $|\theta|$, this deformation is legitimate and the path integral on the sphere is equal to the path integral taken over the contour $A + B + C$.

One can then identify the three pieces of the contour with the different geometric regions: the segment B passes through the de Sitter region, while A and C pass through EAdS^- and EAdS^+ , respectively¹⁸ (See detail in App. B). The Euclidean AdS segments play the role of preparing the Bunch–Davies vacuum at early and late “times” in de Sitter space¹⁹. In this way, our prescription is fully compatible with the earlier $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ constructions.

While this construction is logically consistent, interpreting it as a single quantum field theory defined on the disjoint union $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ requires a somewhat unconventional perspective and exhibits several distinctive features. First, regarding the physical interpretation of the dS segment, if one retains only the de Sitter portion of the contour and imposes the standard Feynman $i\epsilon$ prescription, the resulting correlation functions do not reproduce those of the Bunch–Davies vacuum.²⁰ The BD

¹⁷In deforming the contour into the pieces $A + B + C$, we require the real part of θ to increase monotonically along the path. Otherwise, the contour may cross branch cuts of the integrand, which arise from the propagators. For this reason, the vertical segments in Fig. 7 must be slightly tilted in the complex plane.

¹⁸A closely related discussion in the Poincaré patch can be found in [115].

¹⁹In [10, 116], the path integral in Euclidean AdS is shown to generate the wave function of the Bunch–Davies vacuum for the initial state.

²⁰An explicit example of such a two-point function, which differs from the in-in correlator in the BD vacuum, is provided in [117].

correlators arise only after summing over the entire contour $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$. Second, as for interactions between the EAdS and dS regions, in an interacting theory the fields defined on the EAdS and dS segments necessarily couple to one another: propagators connecting points in EAdS and in dS contribute to correlation functions. From the standpoint of standard QFT, where one usually quantizes on a single connected manifold, such a coupled system living on disconnected pieces lies somewhat outside the conventional framework and requires an extension of the usual intuition. Third, concerning propagators in EAdS, even when both endpoints of a propagator lie within EAdS, the relevant Green's functions are not the usual EAdS propagators with Dirichlet boundary conditions; instead, they are obtained by analytic continuation from the sphere and therefore generally differ from the standard textbook expressions. For these reasons, it is often conceptually clearer, especially in holographic applications, to interpret our prescription primarily as a theory on de Sitter space, with the EAdS segments serving to prepare the BD state, while the $\text{EAdS}^+ + \text{EAdS}^- + \text{dS}$ description is retained mainly as a convenient computational device.

5.4 Cosmological-Celestial Operator Correspondence

The celestial dictionary identifies the bulk scalar field Φ on Minkowski spacetime M_{D+2} with celestial operators $\mathcal{O}_\Delta^\pm$ living on the celestial sphere S^D . In the previous section, we have also related the same bulk field Φ in M_{D+2} to a family of fields ψ_λ propagating in de Sitter spacetime dS_{D+1} . Combining these two correspondences, we can directly relate the celestial operators $\mathcal{O}_\Delta$ on S^D to the asymptotic data of the de Sitter fields ψ_λ . We consider the late/early time behavior of the de Sitter bulk field ψ_λ , which we parametrize as

$$\psi_\lambda \sim e^{\mp(D-\Delta)t} V_{D-\Delta} + e^{\mp\Delta t} W_\Delta \quad (t \rightarrow \pm\infty), \quad (5.75)$$

where the plus (minus) sign corresponds to the late-time (early-time) limit, or equivalently to outgoing (incoming) behavior, and

$$\Delta = \frac{D}{2} + i\lambda. \quad (5.76)$$

The coefficients V_Δ and W_Δ appearing in (5.75) may be interpreted as operators on the sphere S^D . In this section, we establish the precise relation between these operators and the celestial operators $\mathcal{O}_\Delta$.

Recall that the mass of ψ_λ , given in (5.19), satisfies $m_\lambda^2 > D^2/4$, so that ψ_λ belongs to the principal series representation of the de Sitter isometry group. A GKPW-type dictionary for principal series fields in dS/CFT has been discussed, for example, in [118, 119], although their analysis is formulated in the Poincaré patch. In such GKPW-like constructions, V_Δ and W_Δ (or suitable linear combinations thereof) are interpreted as sources that couple to CFT primaries in dS/CFT. By contrast, in a BDHM-like dictionary²¹ in de Sitter space [8] (see also [121, 122]), the combinations of V_Δ and W_Δ are themselves regarded as CFT primaries. Since the

²¹The BDHM prescription [120] in AdS/CFT identifies the boundary limit of bulk operators directly with CFT operators, and thus differs slightly from the GKPW prescription [6, 101], in which

structure of dS/CFT is not as firmly established as in AdS/CFT, we shall refrain from committing to a specific dictionary here. Instead, we treat V_Δ and W_Δ as extrapolated operators associated with the bulk fields in de Sitter space, and we will return to their dS/CFT interpretation later.

We parametrize Minkowski spacetime M_{D+2} using coordinates (t, r, p) as

$$X(t, r, p) = (r \sinh t, r \cosh tp), \quad (5.77)$$

where p is a unit vector on S^D . For later convenience, it is useful to decompose the vector X into null vectors

$$p_- = (1, -p), \quad p = (1, +p), \quad (5.78)$$

so that

$$X = \frac{1}{2} r e^t p - \frac{1}{2} r e^{-t} p_-. \quad (5.79)$$

The embedding of de Sitter space can be written in terms of (t, p) as

$$x(t, p) = (\sinh t, \cosh tp), \quad (5.80)$$

which we will use in what follows²².

With these coordinates, we refine the asymptotic expansion (5.75) as

$$\psi_\lambda(x) \sim e^{-(D-\Delta)t} V_{D-\Delta}(p) + e^{-\Delta t} W_\Delta(p) \quad \text{as } t \rightarrow +\infty, \quad (5.81)$$

$$\psi_\lambda(x) \sim e^{-(D-\Delta)t} V_{D-\Delta}(-p_-) + e^{-\Delta t} W_\Delta(-p_-) \quad \text{as } t \rightarrow -\infty, \quad (5.82)$$

where the arguments p and $-p_-$ indicate the corresponding points on S^D at future and past infinity, respectively. Note that, since $\Delta = D/2 + i\lambda$, both terms on the right-hand side of (5.109) are of the same order in $|t|$.

For definiteness, we focus first on the late-time limit $t \rightarrow +\infty$; the early-time case can be treated in a completely analogous fashion. In the asymptotic region, we may express ψ_λ using the free-field expansion of Φ together with the radial transform:

$$\begin{aligned} \psi_\lambda(x) &= \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \Phi(X) \\ &= \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{D-\Delta} e^{-\epsilon r} \int \frac{\omega^{D-1} d\omega d^D k}{2(2\pi)^{D+1}} (a(\omega k) e^{i\omega k \cdot X - \epsilon \omega}), \end{aligned} \quad (5.83)$$

where $a(\omega k)$ is the annihilation operator in the flat-space mode expansion²³. An ultraviolet regulator $e^{-\epsilon \omega}$ has been introduced by adding a small imaginary part to the time component of X , thereby projecting onto the vacuum²⁴.

the non-normalizable modes of bulk fields play the role of sources J for CFT operators. Correlators are then obtained by functional differentiation with respect to J . While the two dictionaries are equivalent in AdS/CFT, they need not coincide in dS/CFT, as emphasized in [116].

²²Following [8, 9], we use coordinates on the putative dual space S^D such that a point on future infinity and its antipodal point on past infinity share the same angular coordinates.

²³Contributions involving $a_{\text{out}}^\dagger$ would appear in, for example, out-out correlators. Since we will not consider such correlators in this work, we omit these terms. They can be reinstated straightforwardly if needed.

²⁴For notational simplicity we use the same symbol ϵ for the infrared regulator in r and the ultraviolet regulator in ω , although they represent distinct positive infinitesimals.

We now decompose the integral (5.83) into two parts,

$$\psi_\lambda(x) := \psi_\lambda^{D-\Delta}(x) + \psi_\lambda^\Delta(x), \quad (5.84)$$

where $\psi_\lambda^{D-\Delta}$ denotes the contribution from the region $k \neq p$ (i.e. $\int_{k \neq p} d^D k \dots$), while ψ_λ^Δ captures the contribution from a neighbourhood of $k = p$ (i.e. $\int_{k \approx p} d^D k \dots$). We will show that both contributions are of the same order in the limit $t \rightarrow +\infty$. As we shall see, the former reproduces the first term and the latter the second term in the asymptotic expansion (5.109).

A similar extrapolation procedure was considered in [110], where the quantity

$$e^{(D-\Delta)t} \psi_\lambda(x) \sim V_{D-\Delta}(p) + e^{-2i\lambda t} W_\Delta(p) \quad (t \rightarrow +\infty) \quad (5.85)$$

was analyzed. In that setup, the second term is highly oscillatory at large t , so only the first term contributes in a suitable sense. In our prescription, by contrast, we keep both contributions in (5.83); as we will see, the second term is associated with a collinear region of integration and corresponds to the coefficient W_Δ .

Contribution from $k \neq p$

We begin with the contribution from the region $k \neq p$ in (5.83). Using the parametrization of X in terms of p and p_- , we have

$$\omega k \cdot X = \frac{1}{2} r \omega \left(e^t p \cdot k - e^{-t} p_- \cdot k \right). \quad (5.86)$$

Since $k \neq p$ implies $p \cdot k \neq 0$, we can change variables as

$$r = \frac{r'}{\omega e^t (-2p \cdot k)}. \quad (5.87)$$

Expanding the resulting expression at large $|t|$ and retaining the leading contribution, which scales as $e^{-(D+\Delta)t}$, we find

$$\begin{aligned} \psi_\lambda^{D-\Delta} &\sim e^{-(D+\Delta)t} \frac{1}{2(2\pi)^{D+3/2}} \int \frac{dr' d^D k d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} (-2p \cdot k)^{\Delta-D} (a(\omega k) e^{-\frac{i}{4} r'}) \\ &= e^{-(D+\Delta)t} \frac{2^{2D-\Delta} \Gamma(D-\Delta)}{2(2\pi)^{D+3/2}} \int d^D k d\omega \omega^{\Delta-1} (-2p \cdot k)^{\Delta-D} (i^{-(D-\Delta)} a(\omega k)). \end{aligned} \quad (5.88)$$

In the late-time limit $t \rightarrow +\infty$, we may then use the celestial dictionary with $p = (1, p)$ to rewrite this as

$$\begin{aligned} \psi_\lambda^{D-\Delta}(x) &\sim e^{-(D-\Delta)t} S_\Delta \frac{2^{-2i\lambda} \Gamma(D-\Delta)}{2^{5/2} \pi^{D+3/2}} i^{-(D-\Delta)} \tilde{\mathcal{O}}_{D-\Delta}(p) \\ &\sim e^{-(D-\Delta)t} N_{-\lambda} i^{-(D-\Delta)} \tilde{\mathcal{O}}_{D-\Delta}(p), \end{aligned} \quad (5.89)$$

where we defined

$$N_\lambda = \frac{\Gamma(-i\lambda)}{2^{5/2} \pi^{D/2+3/2}}. \quad (5.90)$$

Comparing with the first term in (5.109), we are led to the identification

$$V_\Delta(p) = N_\lambda i^{-\Delta} \tilde{\mathcal{O}}_\Delta(p). \quad (5.91)$$

Here V_Δ depends implicitly on λ through $\Delta = D/2 + i\lambda$; the notation emphasizes the conformal weight rather than the momentum label.

Contribution from the neighbourhood of $k = p$

We now turn to the collinear contribution from a neighbourhood of $k = p$ in (5.83). Assuming that k is close to p (so that in particular $p_- \cdot k \neq 0$), we perform the change of variables

$$r = \frac{r'}{\omega e^{-t}(-p_- \cdot k/2)}. \quad (5.92)$$

Substituting this into (5.83), the contribution $\psi_\lambda^\Delta(x)$ from the collinear region takes the form

$$\begin{aligned} \psi_\lambda^\Delta(x) &\sim e^{-(D-\Delta)t} \frac{1}{2(2\pi)^{D+3/2}} \int \frac{dr' d^D k d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} \left(-\frac{1}{2} p_- \cdot k\right)^{-(D-\Delta)} \\ &\quad \times \left[a(\omega k) \exp\left(-ir' e^{2t} \frac{p \cdot k}{p_- \cdot k} + ir'\right) \right]. \end{aligned} \quad (5.93)$$

To isolate the region where k is nearly collinear with p , we introduce the variable

$$z^2 = -e^{2t} \frac{p \cdot k}{p_- \cdot k}. \quad (5.94)$$

This can be expressed as

$$z = e^t \tan \frac{\phi}{2}, \quad (5.95)$$

where ϕ is the angle between k and p . In terms of z , the angular measure becomes

$$d^D k = \sin^{D-1} \phi d\phi d\Omega_{D-1} = 2^D e^{-Dt} z^{D-1} (1 + e^{-2t} z^2)^{-D} dz d\Omega_{D-1}, \quad (5.96)$$

where $d\Omega_{D-1}$ is the volume element on the $(D-1)$ -sphere. Substituting this into (5.93), we obtain

$$\begin{aligned} \psi_\lambda^\Delta(x) &\sim e^{-\Delta t} \frac{2^D}{2(2\pi)^{D+3/2}} \int \frac{dr' dz d\omega d\Omega_{D-1}}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} z^{D-1} (1 + e^{-2t} z^2)^{-\Delta} \\ &\quad \times \left[a(\omega k) \exp(-ir' z^2 + ir') \right], \end{aligned} \quad (5.97)$$

where the momentum direction k is now encoded in z and the remaining angular coordinates.

In the limit $t \rightarrow +\infty$, we can expand in powers of $e^{-2t} z^2 \ll 1$ and keep the leading contribution, which localizes k near $\phi = 0$. This yields

$$\begin{aligned} \psi_\lambda^\Delta(t, p) &\sim e^{-\Delta t} \frac{2^D \Omega_{D-1}}{2(2\pi)^{D+3/2}} \int \frac{dr' dz d\omega}{r'} r'^{D-\Delta} e^{-\epsilon r'} \omega^{\Delta-1} z^{D-1} \\ &\quad \times \left[a(\omega p) \exp(-ir' z^2 + ir') \right], \end{aligned} \quad (5.98)$$

where Ω_{D-1} is the volume of the unit $D-1$ sphere (with the convention $\Omega_0 = 2$ for $D = 1$).

Performing the integrals over r' and z , and then using the celestial dictionary, we finally obtain

$$\psi_\lambda^\Delta(t, p) \sim e^{-\Delta t} N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p), \quad (5.99)$$

for $t \rightarrow \infty$. Comparing with the second term in (5.109), we arrive at the identification

$$W_\Delta(p) = N_\lambda i^{-\Delta} \mathcal{O}_\Delta(p). \quad (5.100)$$

The analysis for the early-time limit $t \rightarrow -\infty$ proceeds in the same way. An explicit computation shows that the resulting expressions coincide with those obtained at late times, reflecting the fact that our definition of cosmological operators is compatible with the flat-space crossing symmetry underlying the celestial dictionary.

Spinning particle case We now aim to generalize the correspondence between cosmological operators and celestial CFT operators, which was established for scalar fields above, to the case of fields with spin. For simplicity, we first analyze a spin-1 gauge field.

As a preliminary step, we express the cosmological spin-1 operators using the embedding-space formalism, so that the subsequent discussion parallels the scalar analysis. By definition, cosmological operators in de Sitter space are extracted as the leading coefficients of the bulk field in the late/early time limits $t \rightarrow \pm\infty$, where the coordinates are parametrized as in (5.5). The angular components $a_{\alpha,\lambda}(t, \mathbf{p}(x^\alpha))$ of the gauge field on de Sitter exhibit the asymptotic form (see, for example, [123])

$$a_{\alpha,\lambda}(t, \mathbf{p}(x^\alpha)) \sim e^{\mp(\Delta-1)t} V_{\alpha,\Delta-}(x^\alpha) + e^{\mp(\Delta+1)t} W_{\alpha,\Delta+}(x^\alpha) \quad (t \rightarrow \pm\infty), \quad (5.101)$$

where x^α denote coordinates on the celestial sphere.

We next lift this asymptotic behavior to the embedding space by employing the de Sitter embedding coordinates

$$x^I(t, \mathbf{p}) = (\sinh t, \cosh t \mathbf{p}), \quad p^I = (1, \mathbf{p}). \quad (5.102)$$

In analogy with [68], we introduce embedding-space operators V_I and W_I via²⁵

$$\frac{\partial p^I}{\partial x^\alpha} V_{I,\Delta}(p) := V_{\alpha,\Delta}(p), \quad \frac{\partial p^I}{\partial x^\alpha} W_{I,\Delta}(p) := W_{\alpha,\Delta}(p). \quad (5.105)$$

²⁵The embedding-space operators $V_{I,\Delta}(p)$ and $W_{I,\Delta}(p)$ are not uniquely defined: they can be shifted according to

$$V_{I,\Delta}(p) \rightarrow V_{I,\Delta}(p) + \beta(p) p_I, \quad (5.103)$$

$$W_{I,\Delta}(p) \rightarrow W_{I,\Delta}(p) + \beta(p) p_I, \quad (5.104)$$

for an arbitrary scalar function $\beta(p)$. This is the usual redundancy on the null cone appearing in the embedding formalism for spinning primary operators [68].

Using (5.32), we then find

$$\frac{\partial x^I}{\partial x^\alpha} a_{I,\lambda}(x) \sim e^{\mp(\Delta_- - 1)t} \frac{\partial p^I}{\partial x^\alpha} V_{I,\Delta_-}(p) + e^{\mp(\Delta_+ - 1)t} \frac{\partial p^I}{\partial x^\alpha} W_{I,\Delta_+}(p) \quad (t \rightarrow \pm\infty). \quad (5.106)$$

Equivalently, this relation can be rewritten as

$$e^{\pm t} \frac{\partial p^I}{\partial x^\alpha} a_{I,\lambda}(x) \sim e^{\mp(\Delta_- - 1)t} \frac{\partial p^I}{\partial x^\alpha} V_{I,\Delta_-}(p) + e^{\mp(\Delta_+ - 1)t} \frac{\partial p^I}{\partial x^\alpha} W_{I,\Delta_+}(p) \quad (t \rightarrow \pm\infty), \quad (5.107)$$

where we have used the asymptotic relation

$$\frac{\partial x^I}{\partial x^\alpha} \rightarrow e^{\pm t} \frac{\partial p^I}{\partial x^\alpha} \quad (t \rightarrow \pm\infty). \quad (5.108)$$

Removing the Jacobian factor and absorbing any overall normalization into the operator definitions, we obtain the embedding-space asymptotics

$$a_{I,\lambda}(t, \mathbf{p}) \sim e^{\mp\Delta_- t} V_{I,\Delta_-}(p) + e^{\mp\Delta_+ t} W_{I,\Delta_+}(p) \quad (t \rightarrow \pm\infty). \quad (5.109)$$

To relate this to the celestial dictionary, we now focus on the late-time regime $t \rightarrow +\infty$. In this asymptotic region of Minkowski spacetime, the gauge field admits the plane-wave expansion

$$A_I(X) = \sum_\alpha \int \frac{\omega^{D-1} d\omega d^D k}{2(2\pi)^{D+1}} (a^\alpha(\omega k) f_I^\alpha(X) + a^{\alpha\dagger}(\omega k) f_I^{\alpha*}(X)), \quad (5.110)$$

where α labels the polarization states. Substituting this expansion into (5.32) yields

$$a_{I,\lambda}(x) = \frac{1}{\sqrt{2\pi}} \sum_\alpha \int \frac{dr}{r} r^\Delta \int \frac{\omega^{D-1} d\omega d^D k}{2(2\pi)^{D+1}} (a^\alpha(\omega k) f_I^\alpha(X) + a^{\alpha\dagger}(\omega k) f_I^{\alpha*}(X)). \quad (5.111)$$

The mode functions $f_I^\alpha(X)$ take the form

$$f_I^\alpha(X) = \left(\varepsilon_I^\alpha(k) - \frac{\varepsilon^\alpha(k) \cdot x}{k \cdot x} k_I \right) e^{i\omega k \cdot X} = \varepsilon_I^{\prime\alpha}(k) e^{i\omega k \cdot X}, \quad (5.112)$$

where $\varepsilon_I^\alpha(k) := \partial^\alpha p_I(x^\alpha)$ denote the polarization vectors, and the modified polarization $\varepsilon_I^{\prime\alpha}(k)$ satisfies the radial gauge condition $\varepsilon'(k) \cdot x = 0$. In the asymptotic regions $t \rightarrow \pm\infty$, it is convenient to rewrite the polarization as

$$f_I^\alpha(X) = \left(\varepsilon_I^\alpha(k) - \frac{\varepsilon^\alpha(k) \cdot p_\pm}{k \cdot p_\pm} k_I \right) e^{i\omega k \cdot X} = \varepsilon_I^{\prime\alpha}(k, p_\pm) e^{i\omega k \cdot X}, \quad (5.113)$$

where $p_\pm = (1, \pm \mathbf{p})$ are null vectors specifying the asymptotic direction.²⁶

²⁶As $t \rightarrow \infty$ and k approaches $p_\pm$, the particular choice of polarization ε' above becomes singular. In this regime, one must switch to a different gauge. For instance, one may simply take

$$\varepsilon'_I(k) := \varepsilon_I^\alpha(k) = \partial^\alpha p_I(x^\alpha). \quad (5.114)$$

We now introduce the polarization-contracted annihilation operator

$$a_I(\omega \mathbf{k}) := \sum_{\alpha} \varepsilon_I^{\prime \alpha}(k, p_{\pm}) a^{\alpha}(\omega k), \quad (5.115)$$

and define the corresponding creation operator in an analogous way. Plugging this into (5.111), we obtain

$$a_{I,\lambda}(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dr}{r} r^{\Delta} \int \frac{\omega^{D-1} d\omega d^D k}{2(2\pi)^{D+1}} \left(a_I(\omega k) e^{i\omega k \cdot X} + a_I^{\dagger}(\omega k) e^{-i\omega k \cdot X} \right). \quad (5.116)$$

Proceeding as in [30], we analyze the late-time limit, apply the celestial dictionary (5.9), and perform the Mellin transform over ω . In this way, we can match the leading coefficients $V_{I,\Delta_-}(p)$ and $W_{I,\Delta_+}(p)$ appearing in the de Sitter asymptotics (5.109) with spin-1 celestial primary operators and their shadows. More explicitly, up to an overall normalization and possible phase factors, the relations between cosmological and celestial operators can be written as²⁷

$$V_{I,\Delta_-}(p) \propto i^{-\Delta_-} \widetilde{\mathcal{O}_{I,\Delta_+}}(p), \quad (5.119)$$

and

$$W_{I,\Delta_+}(p) \propto i^{-\Delta_+} \mathcal{O}_{I,\Delta_+}(p). \quad (5.120)$$

Here $\mathcal{O}_{I,\Delta_{\pm}}(p)$ denote spin-1 celestial primary operators of conformal dimensions $\Delta_{\pm}$, and $\widetilde{\mathcal{O}}$ stands for their shadow transforms. (5.119) and (5.120) thus encode the operator dictionary between de Sitter cosmological operators for spin-1 fields and spin-1 operators in the celestial CFT.

An analogous structure arises for spin-2 fields, for which we find

$$V_{IJ,\Delta_-}(p) \propto i^{-\Delta_-} \widetilde{\mathcal{O}_{IJ,\Delta_+}}(p), \quad (5.121)$$

and

$$W_{IJ,\Delta_+}(p) \propto i^{-\Delta_+} \mathcal{O}_{IJ,\Delta_+}(p). \quad (5.122)$$

5.5 Ward-Takahashi Identity

In this section, we employ the operator dictionary constructed in Sec. 5.4 to compare Ward-Takahashi identities in de Sitter spacetime with their counterparts in celestial

²⁷A more complete expression reads

$$V_{I,\Delta_-}(p) \propto i^{-\Delta_-} \widetilde{\mathcal{O}_{I,\Delta_+}}(p) + i^{\Delta_-} (\widetilde{\mathcal{O}_{I,\Delta_-}}(p))^{\dagger}, \quad (5.117)$$

and

$$W_{I,\Delta_+}(p) \propto i^{-\Delta_+} \mathcal{O}_{I,\Delta_+}(p) + i^{\Delta_+} (\mathcal{O}_{I,\Delta_-}(p))^{\dagger}. \quad (5.118)$$

In the present work, we omit the latter terms since they become relevant only in the analysis of two-point functions.

holography. In Sec. 5.2, we implemented a dimensional reduction procedure that relates fields in Minkowski spacetime to fields on de Sitter space. There, we showed that gauge transformations in Minkowski space descend to gauge symmetries acting on the massless and partially massless sectors in de Sitter. This allows us to define, directly in de Sitter space, the conserved currents associated with these gauge fields and to study the Ward–Takahashi identities they generate. Throughout this section we specialize to $D = 2$, so that the celestial sphere is two-dimensional and the dual theory is a two-dimensional celestial CFT.

5.5.1 Maxwell : Leading

As a warm-up, let us consider a $U(1)$ gauge theory in which a gauge field a_I is coupled to a single massive complex scalar field ϕ . We start from the Ward–Takahashi identity for the $U(1)$ current in de Sitter space,

$$\langle D_I J^I(p) \prod_i O_{\Delta_i}(p_i) \rangle = \sum_i \delta(p - p_i) \langle \delta O_{\Delta_i}(p_i) \prod_{j \neq i} O_{\Delta_j}(p_j) \rangle, \quad (5.123)$$

where J^I is the conserved current associated with the $U(1)$ symmetry and O_{Δ_i} are scalar boundary operators.

From (5.36), we have seen that the pure-gauge sector of the bulk $U(1)$ field corresponds to modes with

$$\lambda = \pm i \frac{D-2}{2}. \quad (5.124)$$

The associated cosmological gauge-field modes admit the late-time expansions

$$\begin{aligned} a_{I,\lambda=i\frac{D-2}{2}} &\sim e^{-(D-1)t} V_{\Delta=D-1,I}(p) + e^{-t} W_{\Delta=1,I}(p), \\ a_{I,\lambda=-i\frac{D-2}{2}} &\sim e^{-t} V_{\Delta=1,I}(p) + e^{-(D-1)t} W_{\Delta=D-1,I}(p). \end{aligned} \quad (5.125)$$

For $D = 2$, the two branches become degenerate and one finds $V_{\Delta=1} = W_{\Delta=1}$ at the level of conformal weights. To construct a complete basis of modes, one should therefore take suitable linear combinations. This degeneracy is a familiar feature in de Sitter analyses, and an analogous recombination is required in celestial holography. Since our present aim is merely to identify the current that appears in the Ward identity, this subtlety will not play a role, and we keep working with the operators V and W defined above.

The scalar field admits the standard late-time expansion

$$\phi_\lambda \sim e^{-\Delta_- t} V_{\Delta_-}(p) + e^{-\Delta_+ t} W_{\Delta_+}(p), \quad (5.126)$$

where $\Delta_\pm$ are the two characteristic exponents of the massive scalar in de Sitter. The action of a $U(1)$ gauge transformation then induces the variation

$$\delta O_\Delta(p) \sim iq O_\Delta(p), \quad (5.127)$$

for a field of charge q . The cosmological operators associated with the scalar field are related to celestial CFT operators via the dictionary

$$\begin{aligned} W_{I,\Delta_+}(p) &\propto i^{-\Delta_+} \mathcal{O}_{I,\Delta_+}(p), \\ V_{I,\Delta_-}(p) &\propto i^{-\Delta_-} \widetilde{\mathcal{O}_{I,\Delta_+}}(p), \end{aligned} \quad (5.128)$$

where $\mathcal{O}_{I,\Delta+}$ is a spin-1 celestial operator and $\widetilde{\mathcal{O}_{I,\Delta+}}$ denotes its shadow.

In particular, we choose to identify²⁸ the $U(1)$ current with the mode of conformal dimension $\Delta = D - 1$:²⁹

$$J^I = V_{\Delta=D-1}^I \sim i^{-(D-1)} \widetilde{\mathcal{O}_{\Delta=1}^I}. \quad (5.129)$$

With this identification, the Ward–Takahashi identity (5.123) can be recast directly in terms of celestial operators as

$$\left\langle D_I \widetilde{\mathcal{O}_{\Delta=1}^I}(p) \prod_i \mathcal{O}_{\Delta_i} \right\rangle \propto \sum_i q \delta(p - p_i) \left\langle \prod_i \mathcal{O}_{\Delta_i} \right\rangle. \quad (5.130)$$

This relation is precisely the conformally soft photon theorem [124, 125], thereby demonstrating that the Ward–Takahashi identity for cosmological correlators with an inserted $U(1)$ current is equivalent to the conformally soft photon theorem.

In the special case $D = 2$, the shadow and non-shadow operators of scaling dimension $\Delta = 1$ become degenerate. Working on the Poincaré section of the celestial sphere, identified with the complex plane $\mathbb{C}$ equipped with coordinates $(z, \bar{z})$ and flat metric $\gamma_{z\bar{z}} = \gamma_{\bar{z}z} = 1$,³⁰ we find

$$\left\langle \mathcal{O}_{z,\Delta=1}(z, \bar{z}) \prod_i \mathcal{O}_{\Delta_i}(z_i, \bar{z}_i) \right\rangle \propto \sum_i \frac{q}{z - z_i} \left\langle \prod_i \mathcal{O}_{\Delta_i}(z_i, \bar{z}_i) \right\rangle, \quad (5.132)$$

where we have made use of the standard distributional identity

$$\partial_{\bar{z}} \left(\frac{1}{z - w} \right) = 2\pi \delta^{(2)}(z - w). \quad (5.133)$$

This reproduces the celestial Ward identity in $D = 2$ derived in [126].

5.5.2 Gravity : Leading

We now turn to the gravitational analogue and derive the celestial Ward–Takahashi identity associated with the leading soft graviton, starting from the partially massless

²⁸Within this framework, one could in principle also regard $W_{\Delta=D-1}$ as a $U(1)$ current solely based on its conformal weight. The subtlety is likely tied to the fact that we have not specified the edge-mode degrees of freedom at the asymptotic boundary of Minkowski spacetime. A careful treatment of these boundary modes is beyond the scope of the present work. Here we simply adopt the convention that it is V_{D-1} that plays the role of the current.

²⁹To be more precise, in the standard treatment of celestial holography for $D = 2$, the current is defined by multiplying the operator by a prefactor $\Delta - 1$ and subsequently taking the limit $\Delta \rightarrow 1$. This prescription cancels the divergence appearing in the normalization of the celestial dictionary and produces a finite answer. In the discussion below, we will not keep explicit track of this overall factor.

³⁰The Poincaré section is obtained by parametrizing a null vector p^I in terms of $(z, \bar{z})$ as

$$p^I(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}). \quad (5.131)$$

gauge symmetry in de Sitter space. We postulate the de Sitter Ward identity for the partially massless current J^{IJ} at the future boundary,

$$\left\langle D_I D_J J^{IJ}(p) \prod_i O_{\Delta_i}(p_i) \right\rangle = \sum_i \delta(p - p_i) \left\langle \delta O_{\Delta_i}(p_i) \prod_{j \neq i} O_{\Delta_j}(p_j) \right\rangle, \quad (5.134)$$

where δO_{Δ_i} is the variation of the scalar boundary operator induced by the partially massless gauge transformation.

The partially massless graviton modes in de Sitter admit the late-time expansion

$$h_{IJ, \lambda=i \frac{D-2}{2}} \sim e^{-(D-1)t} V_{\Delta=D-1, IJ}(p) + e^{-t} W_{\Delta=1, IJ}(p), \quad (5.135)$$

where V and W are the two independent branches of boundary data.

To extract the action of the partially massless symmetry on boundary operators, we first consider its effect on a bulk scalar field ϕ . Let $F(p)$ denote the scalar gauge parameter on the boundary sphere. In the asymptotic region near future infinity, the partially massless transformation induces

$$\delta \phi \sim \frac{1}{2} F e^t \left(-\partial_r + \frac{1}{r} \partial_t \right) \phi + \frac{1}{r e^t} D^\alpha F \partial_\alpha \phi + O(e^{-t}), \quad (5.136)$$

where (r, t) are radial and time coordinates and D_α is the covariant derivative on the boundary sphere.

We next pass to the cosmological modes $\psi_\lambda(x)$ obtained from ϕ by the radial transform

$$\psi_\lambda(x) \sim \frac{1}{\sqrt{2\pi}} \int_0^\infty dr r^{\Delta_- - 1} \phi(r, t, p), \quad (5.137)$$

with $\Delta_\pm$ the two characteristic exponents. Using (5.136), we find at leading order in the late-time expansion

$$\begin{aligned} \delta \psi_\lambda(x) &\sim \frac{1}{\sqrt{2\pi}} \int_0^\infty dr r^{\Delta_- - 1} \delta \phi \\ &\sim \frac{1}{2\sqrt{2\pi}} F e^t \int_0^\infty dr r^{\Delta_- - 1} \left(-\partial_r + \frac{1}{r} \partial_t \right) \phi \\ &\sim \frac{1}{2} F e^t (\Delta_- - 1 + \partial_t) \psi_{\lambda-i} + \mathcal{O}(e^{-t}), \end{aligned} \quad (5.138)$$

where the mode $\psi_{\lambda-i}$ appears because the partially massless shift effectively decreases the radial exponent by one unit.

The late-time behaviour of $\psi_{\lambda-i}$ can be organized as

$$\psi_{\lambda-i}(x) \sim e^{-(\Delta_- - 1)t} V_{\Delta_- - 1}(p) + e^{-(\Delta_+ + 1)t} W_{\Delta_+ + 1}(p) + \dots, \quad (5.139)$$

where ellipses indicate further subleading terms in powers of e^{-2t} . Substituting (5.139) into (5.138) and matching powers of e^{-t} , we see that the leading contribution proportional to $e^{-(\Delta_- - 1)t}$ cancels:

$$(\Delta_- - 1 + \partial_t) e^{-(\Delta_- - 1)t} = 0. \quad (5.140)$$

Thus the variation δO_{Δ_-} vanishes at leading order. The first non-vanishing contribution is controlled by the subleading term in (5.136), together with the subleading pieces in the asymptotic expansion of $\psi_{\lambda-i}$,

$$\psi_{\lambda-i}(x) \sim e^{-(\Delta_- - 1)t} \left(V_{\Delta_- - 1} + e^{-2t} V_{\Delta_- - 1}^{(2)} + \dots \right) + e^{-(\Delta_+ + 1)t} \left(W_{\Delta_+ + 1} + \dots \right). \quad (5.141)$$

The subleading term in (5.136) carries two derivatives after integration by parts, and, in analogy with the Fefferman–Graham expansion, the coefficient $V_{\Delta_- - 1}^{(2)}$ can be interpreted as a level-two descendant of $V_{\Delta_- - 1}$ [127–129]. It follows that δV_{Δ_-} has conformal weight $\Delta_- + 1$, which matches the conformal dimension of the operator obtained by first acting with a supertranslation (weight shift) on $\mathcal{O}_{\Delta_+}$ and subsequently applying the shadow transform.

By contrast, the variation of the “plus” branch is already non-vanishing at leading order:

$$\begin{aligned} \delta W_{\Delta_+} &\sim \frac{1}{2} (\Delta_- - \Delta_+ - 2) W_{\Delta_+ + 1} \\ &\sim -(1 + i\lambda) W_{\Delta_+ + 1}, \end{aligned} \quad (5.142)$$

so the partially massless symmetry acts non-trivially on the W -branch by shifting its conformal weight by one unit.

We are now in a position to employ the cosmological–celestial dictionary in order to rewrite (5.134) as the conformally soft graviton theorem. In direct analogy with the $U(1)$ case, the partially massless boundary current is associated with the graviton mode of conformal weight $\Delta = D - 1$ and is mapped, on the celestial side, to the shadow of a spin-2 conformal primary operator. Accordingly, we identify the partially massless current with the $\Delta = D - 1$ mode as

$$J^{IJ} = V_{\Delta=D-1}^{IJ} \sim -i \widetilde{\mathcal{O}_{\Delta=1}^{IJ}}. \quad (5.143)$$

Denoting the corresponding celestial operator by $\widetilde{\mathcal{O}_{\Delta=1}^{IJ}}(p)$ and using the normalization specified in (5.142), the Ward–Takahashi identity (5.134) can be rewritten as

$$\left\langle D_I D_J \widetilde{\mathcal{O}_{\Delta=1}^{IJ}}(p) \prod_i \mathcal{O}_{\Delta_i}(p_i) \right\rangle \propto \sum_i \delta(p - p_i) \left\langle \mathcal{O}_{\Delta_i+1}(p_i) \prod_{j \neq i} \mathcal{O}_{\Delta_j}(p_j) \right\rangle. \quad (5.144)$$

This identity coincides with the celestial Ward–Takahashi relation derived in [124, 126].

Next, invoking the Mellin representation that connects momentum-space scattering amplitudes to celestial correlators, we can express the insertion of $\mathcal{O}_{\Delta_i+1}(p_i)$

as the action of an energy operator $\hat{\omega}_i$ on the celestial correlator:³¹

$$\left\langle \mathcal{O}_{\Delta_i+1}(p_i) \prod_{j \neq i} \mathcal{O}_{\Delta_j}(p_j) \right\rangle = \hat{\omega}_i \left\langle \prod_j \mathcal{O}_{\Delta_j}(p_j) \right\rangle. \quad (5.146)$$

Specializing now to $D = 2$, and using the fact that shadow and non-shadow operators at $\Delta = 1$ are degenerate [79], we can perform a partial integration of (5.144) to obtain

$$\left\langle \mathcal{O}_{zz,\Delta=1}(z, \bar{z}) \prod_i \mathcal{O}_{\Delta_i}(z_i, \bar{z}_i) \right\rangle \propto \sum_i \frac{\bar{z} - \bar{z}_i}{z - z_i} \hat{\omega}_i \left\langle \prod_j \mathcal{O}_{\Delta_j}(z_j, \bar{z}_j) \right\rangle. \quad (5.147)$$

This expression agrees with the celestial Ward–Takahashi identity associated with supertranslations, which is equivalent to the leading soft graviton theorem in $D = 2$ [43, 53].

5.5.3 Gravity : Subleading

Finally, we derive the celestial Ward–Takahashi identity associated with the sub-leading soft graviton, starting from diffeomorphism invariance in de Sitter space. We begin with the Ward identity for the massless graviton current,

$$\int [d^2p] Y_J(p) \left\langle D_I T^{IJ}(p) \prod_i O_{\Delta_i} \right\rangle = \int [d^2p] \sum_i \delta(p - p_i) \left\langle \delta_Y O_{\Delta_i} \prod_{j \neq i} O_{\Delta_j} \right\rangle, \quad (5.148)$$

where T^{IJ} is the boundary current constructed from the massless graviton, Y^I is a conformal Killing vector (CKV) on the boundary sphere (the superrotation parameter), and O_{Δ_i} are scalar operators.

The massless graviton modes in de Sitter exhibit the late-time behaviour

$$h_{IJ,\lambda=-i} \sim V_{\Delta=2,IJ}(p) + e^{-2t} W_{\Delta=0,IJ}(p), \quad (5.149)$$

so that, for $D = 2$, the $\Delta = 2$ branch corresponds to the usual massless graviton data, while the $\Delta = 0$ branch can be viewed as its shadow partner.

To determine the action of diffeomorphisms generated by Y^I on scalar fields, we write the bulk vector field as W^a and decompose it into time and angular components. On de Sitter space, one finds

$$\delta_W \phi = W^a \partial_a \phi = \left(-W^t \partial_t + \frac{1}{\cosh^2 t} W^\alpha \partial_\alpha \right) \phi. \quad (5.150)$$

Using the late-time asymptotics $W^t \sim -\frac{1}{2} D_\alpha Y^\alpha$ and $W^\alpha \sim \frac{1}{4} e^{2t} Y^\alpha$, this becomes

$$\delta_W \phi \sim \left(\frac{1}{2} D_\alpha Y^\alpha \partial_t + Y^\alpha \partial_\alpha \right) \phi \quad (t \rightarrow +\infty). \quad (5.151)$$

³¹In our conventions, celestial operators are introduced via the Mellin transform

$$\mathcal{O}_\Delta(p) = \int_0^\infty d\omega \omega^{\Delta-1} a(p), \quad (5.145)$$

so that multiplying a momentum-space amplitude by the energy ω corresponds to shifting the conformal weight by one unit, $\Delta \rightarrow \Delta + 1$. We denote this weight-shifting operation by $\hat{\omega}$.

Passing to the cosmological modes ψ_λ obtained by the radial transform preserves this asymptotic structure, so that

$$\delta\psi_\lambda(x) \sim \left(\frac{1}{2}D_\alpha Y^\alpha \partial_t + Y^\alpha \partial_\alpha\right)\psi_\lambda(x). \quad (5.152)$$

Expanding the cosmological mode near future infinity as

$$\psi_\lambda \sim e^{-\Delta-t}V_{\Delta-}(p) + e^{-\Delta+t}W_{\Delta+}(p), \quad (5.153)$$

and matching coefficients of $e^{-\Delta t}$, we deduce that the associated boundary operators $O_\Delta(p)$ transform as

$$\delta_Y O_\Delta(p) \sim \left(-\frac{\Delta}{2}D_\alpha Y^\alpha(p) + Y^\alpha(p)\partial_\alpha\right)O_\Delta(p). \quad (5.154)$$

The current appearing in (5.148) is associated with the $\Delta = 2$ mode in (5.149). In analogy with the $U(1)$ case, we identify

$$T^{IJ}(p) \sim V_{\Delta=2}^{IJ}(p) \propto -\widetilde{\mathcal{O}_{\Delta=0}^{IJ}}(p) \equiv -\mathcal{T}_{\Delta=2}^{IJ}(p), \quad (5.155)$$

where $\mathcal{T}_{\Delta=2}^{IJ}$ denotes the spin-2 celestial operator (the celestial stress tensor).

We now work on the Poincaré section of the celestial sphere. Inserting the transformation law (5.154) into the right-hand side of (5.148), and expressing the left-hand side in terms of $\mathcal{T}_{\Delta=2}^{IJ}$ via the dictionary introduced above, we arrive at the celestial Ward–Takahashi identity

$$\int_{\mathbb{C}} d^2z Y_\beta(z, \bar{z}) \left\langle \partial_\alpha \mathcal{T}_{\Delta=2}^{\alpha\beta}(z, \bar{z}) \prod_{i=1}^n \mathcal{O}_{\Delta_i}(z_i, \bar{z}_i) \right\rangle = \sum_{i=1}^n \left[\frac{\Delta_i}{2} \partial_\alpha Y^\alpha(z_i, \bar{z}_i) - Y^\alpha(z_i, \bar{z}_i) \partial_{z_i^\alpha} \right] \left\langle \prod_{j=1}^n \mathcal{O}_{\Delta_j} \right\rangle. \quad (5.156)$$

Performing an integration by parts on the left-hand side of (5.156), we can rewrite it as

$$\int_{\mathbb{C}} d^2z Y_\beta \left\langle \partial_\alpha \mathcal{T}_{\Delta=2}^{\alpha\beta} \prod_{i=1}^n \mathcal{O}_{\Delta_i} \right\rangle = - \int_{\mathbb{C}} d^2z (\partial_\alpha Y_\beta) \left\langle \mathcal{T}_{\Delta=2}^{\alpha\beta} \prod_{i=1}^n \mathcal{O}_{\Delta_i} \right\rangle. \quad (5.157)$$

On the complex plane, the only non-zero components of the spin-2 celestial operator are the holomorphic and anti-holomorphic pieces,

$$T_{\text{cel}}(z) := 2\pi \mathcal{T}_{zz, \Delta=2}(z, \bar{z}), \quad \bar{T}_{\text{cel}}(\bar{z}) := 2\pi \mathcal{T}_{\bar{z}\bar{z}, \Delta=2}(z, \bar{z}), \quad (5.158)$$

while the mixed components vanish, $\mathcal{T}_{\Delta=2}^{z\bar{z}} = \mathcal{T}_{\Delta=2}^{\bar{z}z} = 0$. Using the relations $Y_z = Y^{\bar{z}}$ and $Y_{\bar{z}} = Y^z$, the contraction takes the form

$$2\pi(\partial_\alpha Y_\beta) \mathcal{T}_{\Delta=2}^{\alpha\beta} = (\partial_z Y^{\bar{z}}) \bar{T}_{\text{cel}}(\bar{z}) + (\partial_{\bar{z}} Y^z) T_{\text{cel}}(z). \quad (5.159)$$

To isolate the local celestial Ward–Takahashi identity associated with the holomorphic stress tensor, we choose a conformal Killing vector that probes only the holomorphic sector,

$$Y^{\bar{z}}(z, \bar{z}) = 0, \quad Y^z(z, \bar{z}) = \frac{1}{z - w}, \quad (5.160)$$

where $w \in \mathbb{C}$ is the insertion point and Y^z is holomorphic. For this choice, $\partial_\alpha Y^\alpha = \partial_z Y^z$, and the Ward identity becomes

$$-\int_{\mathbb{C}} d^2z \frac{1}{2\pi} (\partial_{\bar{z}} Y^z) \langle T_{\text{cel}}(z) \prod_{i=1}^n \mathcal{O}_{\Delta_i} \rangle = \sum_{i=1}^n \left[\frac{\Delta_i}{2} \partial_z Y^z(z_i) - Y^z(z_i) \partial_{z_i} \right] \left\langle \prod_{j=1}^n \mathcal{O}_{\Delta_j} \right\rangle. \quad (5.161)$$

Exploiting the distributional identity (5.133) and writing the scalar celestial primaries as $\mathcal{O}_{\Delta_i}(z_i, \bar{z}_i)$ with conformal weights $(h_i, \bar{h}_i) = (\Delta_i/2, \Delta_i/2)$, we find

$$\langle T_{\text{cel}}(w) \prod_{i=1}^n \mathcal{O}_{\Delta_i}(z_i, \bar{z}_i) \rangle = \sum_{i=1}^n \left[\frac{\Delta_i}{2} \frac{1}{(w - z_i)^2} + \frac{1}{w - z_i} \partial_{z_i} \right] \left\langle \prod_{j=1}^n \mathcal{O}_{\Delta_j}(z_j, \bar{z}_j) \right\rangle, \quad (5.162)$$

along with the anti-holomorphic counterpart, obtained by exchanging $z \leftrightarrow \bar{z}$ and $T_{\text{cel}} \leftrightarrow \bar{T}_{\text{cel}}$. This expression is the holomorphic celestial Ward–Takahashi identity, and through the celestial dictionary it is equivalent to the subleading soft graviton theorem in four-dimensional Minkowski space, in agreement with [78].

We now turn to the second massless mode with $\lambda = -i$, whose late-time behavior is

$$h_{IJ, \lambda=-i} \sim V_{IJ, \Delta=0}(p) + e^{-2t} W_{IJ, \Delta=2}(p). \quad (5.163)$$

From this mode we can construct an additional operator of conformal dimension $\Delta = 2$,

$$T'^{IJ} \sim W_{\Delta=2}^{IJ}(p) \propto \mathcal{O}_{\Delta=2}^{IJ}(p) \equiv \mathcal{T}_{\Delta=2}^{\prime IJ}(p). \quad (5.164)$$

The operator $\mathcal{T}_{\Delta=2}^{\prime IJ}$ is identified with the “dual stress tensor” introduced in [82]. In the framework of celestial holography it is known that this dual stress tensor does not, in general, define a conserved current [130]. From the point of view of our de Sitter construction, however, the origin of this non-conservation is not completely obvious, and clarifying this issue requires a more detailed investigation, which we leave for future work.

6 Summary and Conclusion

In this thesis, we study a consistent formulation of the dictionary of celestial holography and relationship between celestial holography and holography in other spacetimes. Asymptotically flat spacetimes, which provide the natural arena for celestial holography, have been extensively investigated from the viewpoints of geometry and quantum field theory. However, their holographic properties have remained much less understood. Celestial holography proposes to identify scattering amplitudes with correlation functions of a CFT living on the celestial sphere, and a dictionary between bulk fields and celestial operators has been widely employed.

Nevertheless, this original dictionary suffers from a serious inconsistency: the celestial two-point functions, for massless case obtained via the standard Mellin transform of scattering amplitudes, exhibit distributional (delta-function-like) behavior rather than the familiar power-law form expected in a generic CFT two-point function. This indicates that the naive celestial dictionary requires modification. In order to resolve this issue, we propose a rectified dictionary for celestial CFT operators. Concretely, for massless case, we adopt the convention that outgoing particles are mapped to celestial non-shadow operators, while incoming particles are mapped to their shadow counterparts. With this prescription, the celestial two-point functions acquire the standard CFT power-law structure and become consistent with higher-point correlators. Moreover, when massive external states are included, the resulting expressions admit a smooth massless limit, providing further support for the consistency of our proposal. These results suggest that, upon implementing our rectified dictionary, celestial holography can be regarded as a well-defined CFT framework.

Our broader objective is to elucidate the still poorly understood structure of celestial holography by re-examining it from the vantage point of more established holographic frameworks. As a first approach, we consider $d+1$ dimensional Minkowski spacetime and employ a Milne slicing to decompose it into three regions (wedges). We then apply the idea of wedge holography, originally proposed in the context of the AdS/BCFT correspondence, to associate these three wedges with a $d-1$ dimensional CFT. Within this setup, we are able to determine both the central charge of the CFT and the spectrum of a scalar field, and we find that these results are fully consistent with previous proposals in celestial holography.

As a second approach, we construct a correspondence between theories on Minkowski and de Sitter spacetimes by starting from the action of a Euclidean flat-space theory, performing a Weyl transformation and a Fourier transform, and then analytically continuing the resulting theory. Through this procedure, we find that a massless field theory on $D+2$ dimensional Minkowski spacetime is mapped to a field theory with a continuous mass spectrum on $D+1$ dimensional de Sitter spacetime. We explicitly carry out this analysis for scalar fields, Maxwell theory, and linearized gravity. This construction enables us to relate operators in the celestial CFT, defined via Mellin transformations of Minkowski scattering states, to cosmological operators in de Sitter space, defined from the early/late time limits of bulk fields. We refer to this correspondence as the cosmological–celestial dictionary. In the case of gauge theories, we further show that pure-gauge modes in Minkowski spacetime

correspond to (partially) massless modes in de Sitter space. This observation allows us to define dual currents from the early/late time behavior of (partially) massless fields on de Sitter space and to derive the associated Ward–Takahashi identities. By applying the cosmological–celestial dictionary, we demonstrate that these de Sitter Ward–Takahashi identities are equivalent to the celestial soft theorems. In particular, for linearized gravity, an analogous analysis of the massless mode of the Fierz–Pauli field on de Sitter space shows that the celestial stress tensor can be understood as the boundary limit of this massless de Sitter graviton mode.

Acknowledgement

First, I would like to express my sincere gratitude to Tadashi Takayanagi for his invaluable advice, continuous support, and patience during my Ph.D. study. I am grateful to the staff in Kyoto University, Sotaro Sugishita, for their mentorship. Furthermore, I would also like to thank all members of the particle physics group in the Yukawa Institute for Theoretical Physics (YITP) for having fruitful discussions and suggestions. I am especially grateful to Takahiro Waki, Hideo Furugori, Takashi Tsuda, Shunta Takahashi, Peng-Xiang Hao, Kazuki Doi, Kotaro Shinmyo, Francesco Di Filippo, Yu-ki Suzuki and Shinji Mukohyama, Rei Nishiura for great discussions. I am deeply grateful to my parents, Masazumi and Atsuko, for their great support and for allowing me to study in Kyoto University. Lastly, I would like to say thank you to Mirei Ota for making my life happy.

A Stereographic Coordinate

In the analysis of asymptotically flat spacetimes, it is convenient to use null coordinates on Minkowski space together with complex stereographic coordinates on the celestial two-sphere. This appendix collects the basic definitions and identities in a self-contained form. Our conventions follow the standard literature.

On $\mathbb{R}^{1,3}$, we write

$$ds^2 = -dt^2 + d\vec{x}^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2, \quad (\text{A.1})$$

where $r = |\vec{x}|$ and the unit two-sphere metric is

$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (\text{A.2})$$

Introduce complex stereographic coordinates $(z, \bar{z})$ by projecting the round sphere to the Riemann sphere:

$$z = e^{i\varphi} \cot \frac{\theta}{2}, \quad \bar{z} = e^{-i\varphi} \cot \frac{\theta}{2}. \quad (\text{A.3})$$

The inverse relations are

$$\cos \theta = \frac{1 - z\bar{z}}{1 + z\bar{z}}, \quad \sin \theta e^{i\varphi} = \frac{2z}{1 + z\bar{z}}, \quad \sin \theta e^{-i\varphi} = \frac{2\bar{z}}{1 + z\bar{z}}. \quad (\text{A.4})$$

In these coordinates the round metric becomes conformally flat,

$$d\Omega_2^2 = 2\gamma_{z\bar{z}} dz d\bar{z}, \quad \gamma_{z\bar{z}} \equiv \frac{2}{(1 + z\bar{z})^2}, \quad \gamma^{z\bar{z}} = \frac{(1 + z\bar{z})^2}{2}. \quad (\text{A.5})$$

For a scalar f the covariant Laplacian is

$$D^2 f \equiv D_a D^a f = 2\gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} f. \quad (\text{A.6})$$

Let $\hat{x}(\theta, \varphi)$ be the unit vector on S^2 . In stereographic variables,

$$\hat{x}(z, \bar{z}) = \frac{1}{1 + z\bar{z}} \left(z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z} \right), \quad (\text{A.7})$$

which obeys

$$\partial_z \hat{x} \cdot \partial_{\bar{z}} \hat{x} = \gamma_{z\bar{z}}, \quad \hat{x} \cdot \partial_z \hat{x} = \hat{x} \cdot \partial_{\bar{z}} \hat{x} = 0. \quad (\text{A.8})$$

A future-directed null vector pointing in direction $(z, \bar{z})$ is conveniently packaged as

$$q^\mu(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}), \quad q^\mu q_\mu = 0, \quad (\text{A.9})$$

so that a massless momentum is $p^\mu = \omega q^\mu$ with energy $\omega > 0$. These building blocks are standard in the celestial representation of scattering [17].

Retarded and advanced null coordinates. Define retarded and advanced times by

$$u = t - r, \quad v = t + r. \quad (\text{A.10})$$

With $(u, r, z, \bar{z})$, the Minkowski metric becomes

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}, \quad (\text{A.11})$$

while in $(v, r, z, \bar{z})$ one finds

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \quad (\text{A.12})$$

The level sets $u = \text{const.}$ ($v = \text{const.}$) are null hypersurfaces generated by the outgoing (ingoing) null congruence. The celestial sphere at large r with fixed $(u, z, \bar{z})$ (or fixed $(v, z, \bar{z})$) furnishes angular coordinates on future (past) null infinity $\mathcal{I}^+$ ($\mathcal{I}^-$).

B Contour Deformation

In this appendix we consider a $D+2$ dimensional Euclidean space with coordinates $(X^1, \dots, X^{D+1}, X^{D+2}) = rx$, where x denotes the embedding coordinates on the unit sphere S^{D+1} . We begin with the integral

$$I = \int_{S^{D+1}} [d^{D+1}x] f(x), \quad (\text{B.1})$$

where f is an arbitrary function that decays sufficiently fast at infinity. Our aim is to deform the contour of the polar angle on S^{D+1} as indicated in Fig. 7 and (5.74).

We parametrize the sphere using the polar angle θ via

$$x^{D+2} = -\cos\theta, \quad \mathbf{x} = (x^1, \dots, x^{D+1}) = \sin\theta \hat{n}, \quad \hat{n} \in S^D. \quad (\text{B.2})$$

In these coordinates, the Euclidean volume element becomes $[d^{D+1}x] = (\sin\theta)^D d\theta d^D\hat{n}$.

In section 5.3, we perform the analytic continuation $\theta = \pi/2 + it$. Under this transformation, the coordinates are mapped to

$$x^{D+2} \rightarrow i \sinh t =: ix^0, \quad \mathbf{x} \rightarrow \cosh t \hat{n}. \quad (\text{B.3})$$

The pair $(x^0, \mathbf{x})$ obeying $-(x^0)^2 + |\mathbf{x}|^2 = 1$ then describes the standard embedding of global coordinates on dS_{D+1} .

For the analytic continuation $\theta = -i\chi$, the embedding coordinates become

$$x^{D+2} \rightarrow -\cosh \chi =: -iy^{D+2} = y^0, \quad \mathbf{x} \rightarrow -i \sinh \chi \hat{n} =: -i\mathbf{y}. \quad (\text{B.4})$$

The resulting coordinates $(\mathbf{y}, y^0)$, which satisfy $-(y^0)^2 + |\mathbf{y}|^2 = -1$, realize the embedding of global coordinates on EAdS_{D+1}^- , namely the $(D+1)$ -dimensional unit hyperbolic space H^- .

Similarly, for the continuation $\theta = \pi - i\chi$, we obtain

$$x^{D+2} \rightarrow \cosh \chi =: -iy^{D+2} = y^0, \quad \mathbf{x} \rightarrow i \sinh \chi \hat{n} =: -i\mathbf{y}. \quad (\text{B.5})$$

Again, the coordinates $(\mathbf{y}, y^0)$ satisfy $-(y^0)^2 + |\mathbf{y}|^2 = -1$ and now parametrize the embedding of global coordinates on EAdS_{D+1}^+ (the hyperbolic space H^+).

For a test function $f(x^1, \dots, x^{D+2})$ that decays sufficiently rapidly, we can rewrite the original integral as

$$\begin{aligned} I &= \int_{S^{D+1}} [d^{D+1}x] f(\sin \theta \hat{n}, -\cos \theta) \\ &= \int_{S^D} [d^D \hat{n}] \int_0^\pi d\theta \sin^D \theta f(\sin \theta \hat{n}, -\cos \theta). \end{aligned} \quad (\text{B.6})$$

The real interval $\theta \in [0, \pi]$ can then be deformed as in Eq. (5.74) and Fig. 7. The additional horizontal segments at $\text{Im } \theta = \pm\infty$ do not contribute, provided f decays sufficiently fast in those directions. We also assume that f is analytic in the complex θ -plane so that the contour deformation is legitimate. Under these assumptions, we may decompose the integral as $I = I_A + I_B + I_C$, with

$$I_A = i^{-D-1} \int_{S^D} [d^D \hat{n}] \int_0^\infty d\chi \sinh^D \chi f(-i \sinh \chi \hat{n}, -\cosh \chi), \quad (\text{B.7a})$$

$$I_B = i \int_{S^D} [d^D \hat{n}] \int_{-\infty}^\infty dt \cosh^D t f(\cosh t \hat{n}, i \sinh t), \quad (\text{B.7b})$$

$$I_C = i^{-D-1} \int_{S^D} [d^D \hat{n}] \int_0^\infty d\chi \sinh^D \chi f(-i \sinh \chi \hat{n}, \cosh \chi). \quad (\text{B.7c})$$

Collecting these contributions, we arrive at

$$I = I_A + I_B + I_C = i \left(i^{-D-2} \int_{\text{EAdS}^-} [d^{D+1}y] + \int_{\text{dS}} [d^{D+1}y] + i^{-D-2} \int_{\text{EAdS}^+} [d^{D+1}y] \right) f(x),$$

where

$$x = \begin{cases} -iy & (y \in \text{EAdS}), \\ y & (y \in \text{dS}). \end{cases} \quad (\text{B.8})$$

C Generalized Delta Function

Following [67], we introduce a generalized delta function as a distributional extension of the ordinary Dirac delta to complex arguments. Let $f(x)$ be a function defined on $\mathbb{R}$, and let its analytic continuation to a complex variable $z \in \mathbb{C}$ be denoted by the same symbol $f(z)$. We then define $f(z)$ via

$$f(z) = \int_{-\infty}^\infty dx f(x) \delta^G(x - z), \quad (\text{C.1})$$

where the generalized delta function $\delta^G(z)$ is given, as a distribution, by

$$\delta^G(z) = \frac{1}{2\pi} \int_{-\infty}^\infty dt e^{itz}. \quad (\text{C.2})$$

To see that this prescription reduces to the usual Dirac delta for real arguments, write $z = x' + iy'$ with $x', y' \in \mathbb{R}$. Then

$$\begin{aligned}
f(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt f(x) e^{it(x-(x'+iy'))} \\
&= \frac{1}{2\pi} \int_{-\infty+iy'}^{\infty+iy'} dx \int_{-\infty}^{\infty} dt f(x) e^{it(x-(x'+iy'))} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt f(x + iy') e^{it(x-x')} \\
&= \int_{-\infty}^{\infty} dx f(x + iy') \delta(x - x').
\end{aligned} \tag{C.3}$$

In the second line, we have deformed the x -integration contour from the real axis to the horizontal line $\text{Im } x = y'$. This is justified if f admits an analytic continuation to a strip containing both contours and decays sufficiently rapidly at complex infinity so that the contribution from the closing arcs vanishes. The last line then reproduces the expected sampling property: the generalized delta function localizes the integral at $x = x'$, while the imaginary part y' is encoded in the analytic continuation $f(x' + iy')$.

D Dimensional Reduction of Gravity Action

Let us consider the $(D+2)$ -dimensional metric of the form

$$G_{IJ} dX^I dX^J = e^{2\xi} (d\xi^2 + g_{ij}(\xi, x) dx^i dx^j). \tag{D.1}$$

We now apply the Weyl rescaling

$$G_{IJ} = e^{2\xi} \tilde{G}_{IJ}, \tag{D.2}$$

under which the $(D+2)$ -dimensional Ricci scalar transforms according to

$$\begin{aligned}
R'[G] &= e^{-2\xi} \left[R'[\tilde{G}] - 2(D+1)\tilde{\square}\xi - D(D+1)(\tilde{\nabla}\xi)^2 \right] \\
&= e^{-2\xi} \left[R'[\tilde{G}] - 2(D+1)K - D(D+1) \right],
\end{aligned} \tag{D.3}$$

where $\tilde{\square}$ and $\tilde{\nabla}$ are constructed from $\tilde{G}_{IJ}$, and K denotes the trace of the extrinsic curvature of the constant- ξ hypersurfaces, with $K_{ij} = \frac{1}{2}\partial_\xi g_{ij}$ and $K = g^{ij}K_{ij}$.

The Gauss–Codazzi relation for the induced metric g_{ij} on the constant- ξ slices can be written as

$$\begin{aligned}
R'[\tilde{G}] &= R[g] - K_{ij}K^{ij} + K^2 + 2\tilde{\nabla}_I(n^J\tilde{\nabla}_J n^I - n^I\tilde{\nabla}_J n^J) \\
&= R[g] - K_{ij}K^{ij} + K^2 - 2\tilde{\nabla}_I(n^I K) \\
&= R[g] - K_{ij}K^{ij} + K^2 - 2(K^2 + \partial_\xi K),
\end{aligned} \tag{D.4}$$

where n^I is the unit normal vector to these slices with respect to $\tilde{G}_{IJ}$.

Combining the above relations, we arrive at

$$\begin{aligned}
\sqrt{G}R'[G] &= e^{(D+2)\xi}\sqrt{\tilde{G}}e^{-2\xi}\left[R'[\tilde{G}] - 2(D+1)K - D(D+1)\right] \\
&= e^{D\xi}\sqrt{g}\left[R[g] - K_{ij}K^{ij} + K^2 - 2(K^2 + \partial_\xi K) - 2(D+1)K - D(D+1)\right] \\
&\sim e^{D\xi}\sqrt{g}\left[R[g] - K_{ij}K^{ij} + K^2 - D(D-1)\right].
\end{aligned} \tag{D.5}$$

Here the symbol $\sim$ indicates equality modulo total derivative terms, namely terms that reduce to boundary contributions after integration by parts.³²

We work with the $(D+1)$ -dimensional metric $g_{ij} = \gamma_{ij} + H_{ij}$, where H_{ij} denotes a small fluctuation and its trace $H \equiv \gamma^{ij}H_{ij}$.

The background metric γ_{ij} is maximally symmetric and satisfies

$$R_{ij}[\gamma] = D\gamma_{ij}, \quad R[\gamma] = D(D+1), \quad R_{ikjl}[\gamma] = \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}, \tag{D.6}$$

so that the combination $R[\gamma] - D(D-1) = 2D$ is constant.

We consider the Lagrangian density

$$\mathcal{L}[g] = \sqrt{g}(R[g] - D(D-1)), \tag{D.7}$$

and expand it to second order in H_{ij} :

$$\mathcal{L}[g] = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{O}(H^3). \tag{D.8}$$

Metric, inverse metric, and determinant

We first expand the inverse metric and the determinant. From the defining relation $g^{ik}g_{kj} = \delta^i_j$ one obtains

$$g^{ij} = \gamma^{ij} - H^{ij} + H^i_k H^{kj} + \mathcal{O}(H^3), \tag{D.9}$$

where $H^{ij} := \gamma^{ik}\gamma^{jl}H_{kl}$. The determinant can be expanded using $\det g = \det \gamma \det(\mathbf{1} + \gamma^{-1}H)$ and $\ln \det(\mathbf{1} + M) = \text{tr}M - \frac{1}{2}\text{tr}M^2 + \frac{1}{2}(\text{tr}M)^2 + \dots$, which yields

$$\sqrt{g} = \sqrt{\gamma} \left[1 + \frac{1}{2}H + \frac{1}{8}(H^2 - 2H_{ij}H^{ij}) + \mathcal{O}(H^3) \right], \tag{D.10}$$

with $H^2 := H^2$ and $H_{ij}H^{ij} := \gamma^{ik}\gamma^{jl}H_{ij}H_{kl}$.

Linearized curvature

The variation of the Levi-Civita connection with respect to the metric is

$$\delta\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\nabla_i\delta g_{jl} + \nabla_j\delta g_{il} - \nabla_l\delta g_{ij}). \tag{D.11}$$

³²In particular, we use

$$\partial_\xi(e^{D\xi}\sqrt{g}) = e^{D\xi}\sqrt{g}(D+K), \quad \partial_\xi(e^{D\xi}\sqrt{g}K) = e^{D\xi}\sqrt{g}(DK + K^2 + \partial_\xi K).$$

To first order in H_{ij} , we may replace g^{kl} by γ^{kl} and δg_{ij} by H_{ij} , so that

$$\Gamma^k_{ij}[g] = \Gamma^k_{ij}[\gamma] + C^k_{ij} + \mathcal{O}(H^2), \quad (\text{D.12})$$

$$C^k_{ij} = \frac{1}{2}\gamma^{kl}(\nabla_i H_{jl} + \nabla_j H_{il} - \nabla_l H_{ij}). \quad (\text{D.13})$$

The Ricci tensor satisfies the Palatini identity

$$\delta R_{ij} = \nabla_k \delta \Gamma^k_{ij} - \nabla_i \delta \Gamma^k_{kj}. \quad (\text{D.14})$$

Using the above expression for C^k_{ij} , one finds the standard form of the linearized Ricci tensor on a general background,

$$R^{(1)}_{ij} = \frac{1}{2} \left(-\square H_{ij} - \nabla_i \nabla_j H + \nabla_k \nabla_i H^k_j + \nabla_k \nabla_j H^k_i \right) - R_{(i}{}^k H_{j)k}, \quad (\text{D.15})$$

where $\square := \nabla^k \nabla_k$ and parentheses denote symmetrization. On the maximally symmetric background specified above, $R_{ikjl} = \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}$ and $R_{ij} = D\gamma_{ij}$ imply

$$R_{(i}{}^k H_{j)k} = DH_{ij}, \quad (\text{D.16})$$

so that

$$R^{(1)}_{ij} = \frac{1}{2} \left(-\square H_{ij} - \nabla_i \nabla_j H + \nabla_k \nabla_i H^k_j + \nabla_k \nabla_j H^k_i \right) - DH_{ij}. \quad (\text{D.17})$$

The linearized Ricci scalar is then

$$R^{(1)} = g^{ij} R^{(1)}_{ij} \Big|_{\mathcal{O}(H)} = \gamma^{ij} R^{(1)}_{ij} - H^{ij} R^{(0)}_{ij}, \quad (\text{D.18})$$

with $R^{(0)}_{ij} = D\gamma_{ij}$. A short computation gives

$$\gamma^{ij} R^{(1)}_{ij} = \nabla_i \nabla_j H^{ij} - \square H, \quad (\text{D.19})$$

$$-H^{ij} R^{(0)}_{ij} = -DH, \quad (\text{D.20})$$

and hence

$$R^{(1)} = -DH + \nabla_i (\nabla_j H^{ij} - \nabla^i H). \quad (\text{D.21})$$

Zeroth and first order of the Lagrangian

Using the above ingredients, we expand

$$\begin{aligned} & \sqrt{g}(R[g] - D(D-1)) \\ &= \sqrt{\gamma} \left(1 + \frac{1}{2}H + \frac{1}{8}(H^2 - 2H_{ij}H^{ij}) + \dots \right) \left(R^{(0)} + R^{(1)} + R^{(2)} + \dots - D(D-1) \right). \end{aligned} \quad (\text{D.22})$$

At zeroth order one finds

$$\mathcal{L}^{(0)} = \sqrt{\gamma}(R^{(0)} - D(D-1)) = \sqrt{\gamma}2D. \quad (\text{D.23})$$

The first-order term is

$$\mathcal{L}^{(1)} = \sqrt{\gamma} \left(R^{(1)} + \frac{1}{2} H (R^{(0)} - D(D-1)) \right) = \sqrt{\gamma} (R^{(1)} + DH). \quad (\text{D.24})$$

Inserting the expression for $R^{(1)}$ obtained above, one obtains

$$R^{(1)} + DH = \nabla_i (\nabla_j H^{ij} - \nabla^i H), \quad (\text{D.25})$$

so that

$$\mathcal{L}^{(1)} = \sqrt{\gamma} \nabla_i (\nabla_j H^{ij} - \nabla^i H). \quad (\text{D.26})$$

This is a total derivative and vanishes upon integration on the compact manifold S^{D+1} . Thus γ_{ij} is indeed a stationary point of the action.

Quadratic order

The quadratic contribution $\mathcal{L}^{(2)}$ comes from

$$\mathcal{L}^{(2)} = \sqrt{\gamma} \left[R^{(2)} + \frac{1}{2} H R^{(1)} + \frac{1}{8} (H^2 - 2H_{ij}H^{ij}) (R^{(0)} - D(D-1)) \right]. \quad (\text{D.27})$$

Using $R^{(0)} - D(D-1) = 2D$, the last term simplifies to

$$\frac{1}{8} (H^2 - 2H_{ij}H^{ij}) (R^{(0)} - D(D-1)) = -\frac{D}{2} \left(H_{ij}H^{ij} - \frac{1}{2} H^2 \right). \quad (\text{D.28})$$

The combination $R^{(2)} + \frac{1}{2} H R^{(1)}$ is obtained by expanding the Ricci tensor and scalar to second order using the above expressions for the connection and the Palatini identity. The resulting expression contains terms with second derivatives of H_{ij} . After performing integrations by parts so as to write everything in terms of first derivatives of H_{ij} , and using repeatedly the background identities $R_{ij} = D\gamma_{ij}$ and $R_{ikjl} = \gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}$, one finds that the second-order Einstein–Hilbert part can be written as

$$\begin{aligned} \sqrt{\gamma} \left(R^{(2)} + \frac{1}{2} H R^{(1)} \right) = \sqrt{\gamma} \left[-\frac{1}{2} \nabla_k H_{ij} \nabla^k H^{ij} + \nabla_i H_{jk} \nabla^j H^{ik} - \nabla_i H^{ij} \nabla_j H + \frac{1}{2} \nabla_k H \nabla^k H \right. \\ \left. + \frac{3}{2} D \left(H_{ij} H^{ij} - \frac{1}{2} H^2 \right) \right] + (\text{total derivative}). \end{aligned} \quad (\text{D.29})$$

Adding the contribution from the cosmological term, which supplies

$$-\sqrt{\gamma} \frac{D}{2} \left(H_{ij} H^{ij} - \frac{1}{2} H^2 \right), \quad (\text{D.30})$$

one obtains the full quadratic Lagrangian density (up to total derivatives) as

$$\begin{aligned} \mathcal{L}^{(2)} = \sqrt{\gamma} \left[-\frac{1}{2} \nabla_k H_{ij} \nabla^k H^{ij} + \nabla_i H_{jk} \nabla^j H^{ik} - \nabla_i H^{ij} \nabla_j H + \frac{1}{2} \nabla_k H \nabla^k H \right. \\ \left. + D \left(H_{ij} H^{ij} - \frac{1}{2} H^2 \right) \right]. \end{aligned} \quad (\text{D.31})$$

Up to an overall normalization and the omission of total derivative terms, this is precisely the Fierz–Pauli Lagrangian for a massless spin-2 field propagating on the round sphere S^{D+1} . Furthermore, the extrinsic curvature term become mass term

$$\sqrt{g} [-K_{ij} K^{ij} + K^2] \implies \sqrt{\gamma} \frac{1}{4} [-\partial_\xi H_{ij} \partial^\xi H^{ij} + (\partial_\xi H)^2]. \quad (\text{D.32})$$

E Partially Massless Graviton

We begin by considering linearized gravity on a $(D + 1)$ -dimensional maximally symmetric background spacetime [111–113]. The metric is decomposed as $g_{ij} + h_{ij}$, where g_{ij} is the background metric and h_{ij} denotes the fluctuation. The quadratic action for h_{ij} is

$$S = \int [d^{D+1}x] \left[-\frac{1}{2} \nabla_k h_{ij} \nabla^k h^{ij} + \nabla_i h_{jk} \nabla^j h^{ik} - \nabla_i h^{ij} \nabla_j h + \frac{1}{2} \nabla_k h \nabla^k h + \frac{R}{D+1} \left(h_{ij} h^{ij} - \frac{1}{2} h^2 \right) \right], \quad (\text{E.1})$$

where ∇ is the Levi-Civita connection associated with g_{ij} , and $h \equiv g^{ij} h_{ij}$ is the trace of the fluctuation. This action is invariant under the linearized diffeomorphism symmetry

$$\delta h_{ij} = \nabla_i \xi_j + \nabla_j \xi_i. \quad (\text{E.2})$$

To study massive deformations, we add the Fierz–Pauli mass term

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} m^2 (h^{ij} h_{ij} - h^2), \quad (\text{E.3})$$

with a nonzero mass parameter $m^2 \neq 0$, which explicitly breaks the above gauge invariance. In the resulting massive theory, the equations of motion imply the (linearized) Bianchi identity

$$\nabla^i h_{ij} = \nabla_j h, \quad (\text{E.4})$$

together with the trace constraint³³

$$[m^2 - (D - 1)] h = 0. \quad (\text{E.5})$$

For generic mass values

$$m^2 \neq D - 1, \quad (\text{E.6})$$

this constraint enforces $h = 0$, and the propagating content is that of a massive spin-2 field in $(D + 1)$ dimensions, with the standard Fierz–Pauli number of degrees of freedom

$$N_{\text{FP}} = \frac{(D - 1)(D + 2)}{2}. \quad (\text{E.7})$$

A qualitatively different situation arises at the special mass value corresponding to a partially massless spin-2 field [131–134],

$$m^2 = D - 1. \quad (\text{E.8})$$

At this point, the theory acquires an additional scalar gauge symmetry

$$\delta_\alpha h_{ij} = (\nabla_i \nabla_j + g_{ij}) \alpha, \quad (\text{E.9})$$

³³We set the de Sitter radius $H = 1$.

which eliminates one more propagating degree of freedom.³⁴ Consequently, the number of physical degrees of freedom carried by the partially massless graviton is

$$N = \frac{(D-1)(D+2)}{2} - 1. \quad (\text{E.11})$$

The corresponding conformal weights of the partially massless mode and its shadow partner are

$$\Delta_+ = D - 1, \quad \Delta_- = 1. \quad (\text{E.12})$$

In global coordinates (t, x^α) , this leads to the following late-time asymptotics for the boundary components of the metric perturbation:

$$h_{\alpha\beta}(t, x) \sim e^{-t} V_{\alpha\beta}(x) + e^{-(D-1)t} W_{\alpha\beta}(x). \quad (\text{E.13})$$

Under the standard choice of quantization, we treat $V_{\alpha\beta}$ as the fixed source; this choice and its implications are discussed in the main text.

For this quantization, one finds the current-type identity [136]

$$D_\alpha D_\beta W^{\alpha\beta} = 0, \quad (\text{E.14})$$

where D_α denotes the covariant derivative with respect to the boundary metric. In correlation functions, including insertions of other local operators, this current generates a transformation of the form

$$\int d^D x \, \alpha(x) \left\langle D_\alpha D_\beta W^{\alpha\beta}(x) \prod_n O_n(x_n) \right\rangle = \left\langle \delta_\alpha O_m(x_m) \prod_{n \neq m} O_n(x_n) \right\rangle. \quad (\text{E.15})$$

³⁴In de Sitter space, this Fierz-Pauli mass saturates the lower endpoint of the Higuchi bound [135],

$$m^2 \geq D - 1. \quad (\text{E.10})$$

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