

Probabilistic Construction of Solutions To Some Integral Equations

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§1. Notations and assumptions

For simplicity, let $D_0 := \mathbb{R}^3 \setminus \{0\}$, and we put $\mathbb{R}_+ := [0, \infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, we use the symbol $\alpha \cdot \beta$ for the inner product, and we define $e_x := x/|x|$ for every $x \in D_0$. In this article we consider the following deterministic nonlinear integral equation:

$$\begin{aligned} e^{\lambda t|x|^2} u(t, x) &= u_0(x) + \frac{\lambda}{2} \int_0^t ds e^{\lambda s|x|^2} \int p(s, x, y; u) n(x, y) dy \\ &\quad + \frac{\lambda}{2} \int_0^t e^{\lambda s|x|^2} f(s, x) ds, \quad \text{for } \forall (t, x) \in \mathbb{R}_+ \times D_0. \end{aligned} \quad (1)$$

Here $u \equiv u(t, x)$ is an unknown function : $\mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \rightarrow \mathbb{C}^3$ is the initial data such that $u(t, x)|_{t=0} = u_0(x)$. Moreover, $f(t, x) : \mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$ is a given function satisfying $f(t, x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$. The integrand p in (1) is given by

$$p(t, x, y; u) = u(x, y) \cdot e_x \{u(t, x - y) - e_x(u(t, x - y) \cdot e_x)\}. \quad (2)$$

On the other hand, we consider a Markov kernel $K : D_0 \rightarrow D_0 \times D_0$. Actually, for every $z \in D_0$, $K_z(dx, dy)$ lies in the space $\mathcal{P}(D_0 \times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel k is given by $k(x, y) = i|x|^{-2}n(x, y)$, then we define K_z as a Markov kernel satisfying that for any positive measurable function $h = h(x, y)$ on $D_0 \times D_0$,

$$\iint h(x, y) K_z(dx, dy) = \int h(x, z - x) k(x, z) dx. \quad (3)$$

Moreover, we assume that for every measurable functions $f, g > 0$ on $\mathbb{R}^+$,

$$\int h(|z|) \nu(dz) \int g(|x|) K_z(dx, dy) = \int g(|z|) \nu(dz) \int h(|y|) K_z(dx, dy) \quad (4)$$

holds, where the measure ν is given by $\nu(dz) = |z|^{-3} dz$.

§2. Main result

In this section we shall state our main result, which asserts the existence and uniqueness of solutions to the nonlinear integral equation (1). As a matter of fact, the solution $u(t, x)$ can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of the below-mentioned branching particle systems and branching models. Let

$$M_{\star}^{\langle u_0, f \rangle}(\omega) = \prod_{\star} \star_{[x_m]} \Xi_{m_2, m_3}^{m_1} [u_0, f](\omega), \quad (5)$$

be a probabilistic representation in terms of tree-based star-product functional with weight (u_0, f) . For the details of the definition, see the succeeding sections. On the other hand, $M_{\star}^{\langle U, F \rangle}(\omega)$ denotes the corresponding $\star$ -product functional with weight (U, F) . In fact, as to be seen in what follows, in a similar manner as the case of a star-product functional we can construct a (U, F) -weighted tree-based $\star$ -product functional $M_{\star}^{\langle U, F \rangle}(\omega)$, which is indexed by the nodes (x_m) of a binary tree. Here we suppose that $U = U(x)$ (resp. $F = F(t, x)$) is a non-negative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each x . Indeed, in construction of the $\star$ -product functional, the product in question is taken as ordinary multiplication $*$ instead of the star-product $\star$ in the definition of star-product functional.

Theorem 1. *Suppose that $|u_0(x)| \leq U(x)$ for $\forall x$ and $|\tilde{f}(t, x)| \leq F(t, x)$ for $\forall t, x$, and also that for some $T > 0$ ($T \gg 1$ sufficiently large),*

$$E_{T,x}[M_{\star}^{\langle U, F \rangle}] < \infty, \quad \text{a.e.} - x \quad (6)$$

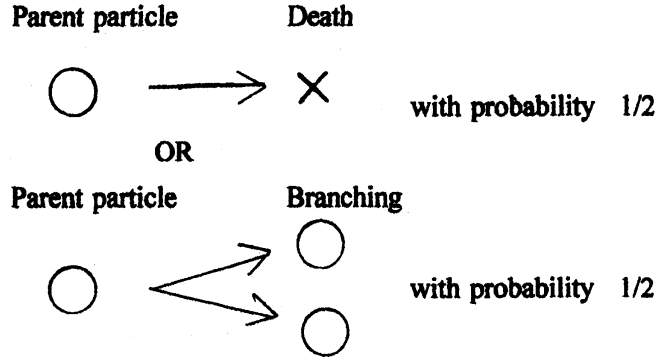
Then there exists a (u_0, f) -weighted tree-based star $\star$ -product functional $M_{\star}^{\langle u_0, f \rangle}(\omega)$, indexed by a set of node labels accordingly to the tree structure which a binary critical branching process $Z^{K_x}(t)$ determines. Furthermore, the function

$$u(t, x) = E_{t,x}[M_{\star}^{\langle u_0, f \rangle}] \quad (7)$$

gives a unique solution to the integral equation (1). Here $E_{t,x}$ denotes the expectation with respect to a probability measure $P_{t,x}$ as the time-reversed law of $Z^{K_x}(t)$.

§3. Branching model

In this section we consider a continuous time binary critical branching process $Z^{K_x}(t)$ on D_0 , whose branching rate is given by a parameter $\lambda|x|^2$, whose branching mechanism is binary with equi-probability, and whose descendant branching particle behavior (or distribution) is determined by the kernel K_x . Next, taking



⊠ 1: Binary Branching

notice of the tree structure which the process $Z^{K_x}(t)$ determines, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \quad (8)$$

by Ω . Furthermore, we write the time-reversed law of $Z^{K_x}(t)$ being a probability measure on Ω as $P_{t,x} \in \mathcal{P}(\Omega)$. Here t denotes the birth time of common ancestor, and the particle x_m dies when $\eta_m = 0$, while it generates two descendants x_{m1}, x_{m2} when $\eta_m = 1$. On the other hand,

$$\mathcal{V} = \bigcup_{\ell \geq 0} \{1, 2\}^\ell$$

is a set of all labels, namely, finite sequences of symbols with length ℓ , which describe the whole tree structure given. For $\omega \in \Omega$ we denote by $\mathcal{N}(\omega)$ the totality of nodes being branching points of tree, and let $N_+(\omega)$ be the set of all nodes m being a member of $\mathcal{V} \setminus \mathcal{N}(\omega)$, whose direct predecessor lies in $\mathcal{N}(\omega)$ and which satisfies the condition $t_m(\omega) > 0$, and let $N_-(\omega)$ be the same set as described above, but satisfying $t_m(\omega) \leq 0$. Finally we put

$$N(\omega) = N_+(\omega) \cup N_-(\omega). \quad (9)$$

§4. Star-product functional and *-product functional

In what follows we shall introduce a tree-based star-product functional in order to construct a probabilistic solution to the class of integral equations (1). First of all, we denote by the symbol $\text{Proj}^z(\cdot)$ a projection of the objective element

onto its orthogonal part of the z component in $\mathbb{C}^3$, and we define a $\star$ -product of β, γ for $z \in D_0$ as

$$\beta \star_{[z]} \gamma = -i(\beta \cdot e_z) \text{Proj}^z(\gamma). \quad (10)$$

We shall define $\Theta^m(\omega)$ for each $\omega \in \Omega$ realized as follows. When $m \in N_+(\omega)$, then $\Theta^m(\omega) = \tilde{f}(t_m(\omega), x_m(\omega))$, while $\Theta^m(\omega) = u_0(x_m(\omega))$ if $m \in N_-(\omega)$. Then we define

$$\Xi_{m_2.m_3}^{m_1}(\omega) \equiv \Xi_{m_2.m_3}^{m_1}[u_0, f](\omega) := \Theta^{m_2}(\omega) \star_{[x_{m_1}]} \Theta^{m_3}(\omega), \quad (11)$$

where as for the product order in the star-product $\star$, when we write $m \prec m'$ lexicographically with respect to the natural order $\prec$, the term Θ^m labelled by m necessarily occupies the left-hand side and the other $\Theta^{m'}$ labelled by m' occupies the right-hand side by all means. And besides, as abuse of notation we write

$$\Xi_{m,\emptyset}^\emptyset(\omega) \equiv \Xi_{m,\emptyset}^\emptyset[u_0, f](\omega) := \Theta^m(\omega), \quad (12)$$

especially when $m \in \mathcal{V}$ is a label of single terminal point in the restricted tree structure in question.

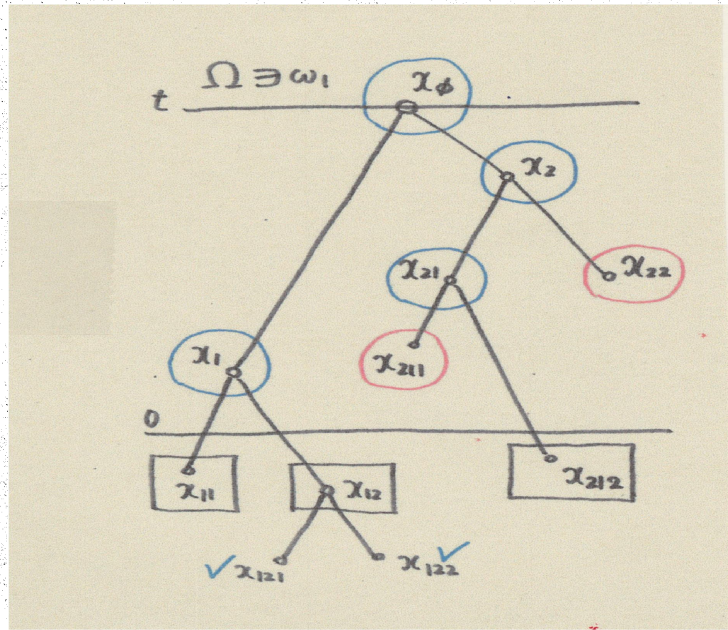


图 2: Example: A realized Tree

Under these circumstances, we consider a random quantity which obtained by executing the star-product $\star$ inductively at each node in $\mathcal{N}(\omega)$, and we call it a tree-based $\star$ -product functional, and we express it symbolically as

$$M_{\star}^{(u_0, f)}(\omega) = \prod \star_{[x_{\tilde{m}}]} \Xi_{m_2.m_3}^{m_1}[u_0, f](\omega), \quad (13)$$

where $m_1 \in \mathcal{N}(\omega)$ and $m_2, m_3 \in N(\omega)$, and by the symbol $\prod \star$ (as a product relative to the star-product) we mean that the star-products $\star$'s should be successively executed in a lexicographical manner with respect to $x_{\tilde{m}}$ such that $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$ when $|m_1| = \ell$.

Example 1. Now let us suppose that a tree structure $\omega_1 (\in \Omega)$ has been realized here (see Figure 2).

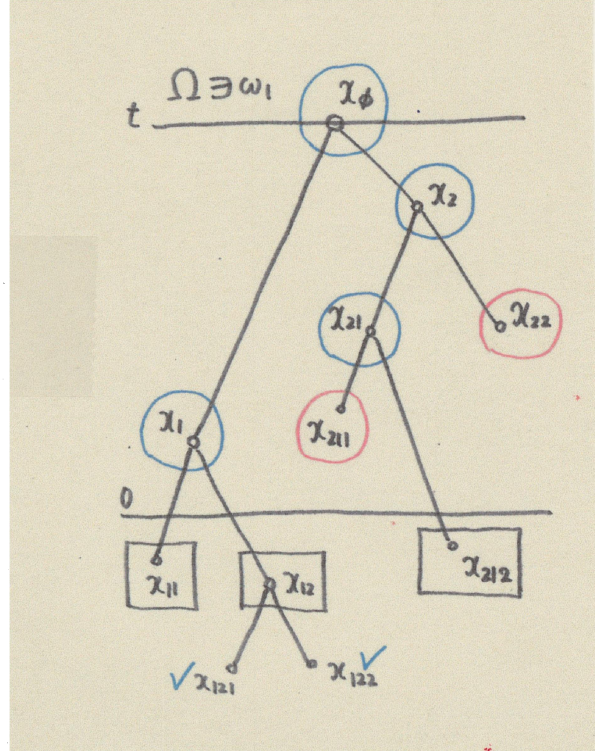


Fig 3: Classification of Nodes

Next we shall classify those nodes in the realized tree ω_1 . As a matter of fact, as to those two particles located at x_{11} and x_{12} with nodes of the level $|m| = \ell = 2$ accompanied by the pivoting node x_1 , we can construct

$$\Xi_{11,12}^1(\omega_1) = \Theta^{11}(\omega_1) \star_{[x_1]} \Theta^{12}(\omega_1)$$

by a star-product $u_0(x_{11}(\omega_1)) \star_{[x_1]} u_0(x_{12}(\omega_1))$ in accordance with the rule, because both $m_1 = 11$ and $m_2 = 12$ lie in $N_-(\omega)$. As to the node x_{21} , how to construct $\Xi(\omega_1)$ is the almost same thing as described above. In fact, it goes similarly because x_{211} lies in $N_+(\omega_1)$ and x_{212} lies in $N_-(\omega_1)$. According to the rule, it follows that

$$\Theta^{211}(\omega_1) = \tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \quad \text{and} \quad \Theta^{212}(\omega_1) = u_0(x_{212}(\omega_1)),$$

hence $\Xi_{211,212}^{21}(\omega_1)$ is given by $\tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \star_{[x_{21}]} u_0(x_{212}(\omega_1))$, see Figure 3. Consequently, we obtain finally

$$M_{\star}^{(u_0, f)}(\omega_1) = (u_0(x_{11}) \star_{[x_1]} u_0(x_{12})) \star_{[x_\phi]} \left\{ \left(\tilde{f}(t_{211}, x_{211}) \star_{[x_{21}]} u_0(x_{212}) \right) \star_{[x_2]} \tilde{f}(t_{22}, x_{22}) \right\}. \quad (14)$$

□

§5. Outline of proof: tree-based star-product functional as a solution

In this section we are first going to construct a (U, F) -weighted tree-based $\star$ -product functional $M_{\star}^{(U, F)}(\omega)$, which is indexed by the nodes (x_m) of a binary tree. Here recall that $U = U(x)$ (resp. $F = F(t, x)$) is a non-negative measurable function on D_0 (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each x . Moreover, in construction of the functional, the product is taken as ordinary multiplication $*$ instead of the star-product $\star$.

In what follows we shall give an outline of the proof of Theorem 1. We need the following technical lemma, which plays an essential role in the proof.

Lemma 2. *For $0 \leq t \leq T$ and $x \in D_0$, the function $V(t, x) = E_{t,x}[M_{\star}^{(U, F)}(\omega)]$ satisfies*

$$e^{\lambda t |x|^2} V(t, x) = U(x) + \int_0^t ds \frac{\lambda |x|^2}{2} e^{\lambda s |x|^2} \left\{ F(s, x) + \int V(s, y) V(s, z) K_x(dy, dz) \right\}. \quad (15)$$

As a matter of fact, the mapping : $[0, T] \ni t \mapsto e^{\lambda |x|^2 t} V(t, x) \in \bar{\mathbb{R}}_+$ is non-decreasing, so that, it proves to be that

$$E_{t,x}[M_{\star}^{(U, F)}(\omega)] < \infty \quad (16)$$

holds for $\forall t \in [0, T]$ and $x \in E_c$, where E_c is a measurable set on which the validity of $E_{t,x}[M_{\star}^{(U, F)}] < \infty$ may be kept. Another important aspect for the proof consists in establishment of the following $M_{\star}$ -control inequality. That is to say, we have

$$|M_{\star}^{(u_0, f)}(\omega)| \leq |M_{\star}^{(U, F)}(\omega)| \quad (17)$$

because of the validity of a simple inequality

$$|w \star_{[x]} v| \leq |w| \cdot |v| \quad \text{for } w, v \in \mathbb{C}^3 \quad \text{and } x \in D_0.$$

On the other hand, it is derived that the space of solutions to (1) is formed by the condition

$$\int_0^T ds \int |u(s, y)| \cdot |u(s, z)| K_x(dy, dz) < \infty \quad \text{for } x \in E_c.$$

A similar discussion as above leads to

$$u(t, x) = E_{t,x}[M_{\star}^{(u_0, f)}(\omega)] = e^{-\lambda t|x|^2} u_0(x) + \int_0^t ds \lambda |x|^2 e^{-\lambda(t-s)|x|^2} \times \\ \times \frac{1}{2} \left\{ \tilde{f}(s, x) + \iint E_{s,x_1}[M_{\star}] \star_{[x]} E_{s,x_2}[M_{\star}] K_x(dx_1, dx_2) \right\}. \quad (18)$$

Finally we can deduce that $u(t, x) = E_{t,x}[M_{\star}^{(u_0, f)}(\omega)]$ satisfies the integral equation (1), and this $u(t, x)$ is a solution lying in the space $\mathcal{D}$. Actually, $\mathcal{D}$ is a space of all functions $\varphi : \mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$, being continuous in t and measurable such that

$$\int_0^\infty ds \int |p(s, x, y; \varphi)| K_x(dy, dz) < \infty, \quad \text{a.e. } -x.$$

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