

Remarks on perturbation of defect operators on Hilbert function spaces

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1 Introduction

Let $(\mathcal{H}, k_\lambda, \Omega)$ be a reproducing kernel Hilbert space consisting of analytic functions on a domain Ω in $\mathbb{C}^n$ with the variable $z = (z_1, \dots, z_n)$ and the reproducing kernel $k_\lambda = k(\lambda, \cdot)$, where λ is a point in Ω . Without loss of generality, we may assume that Ω contains the origin. Moreover, we assume that $\mathcal{H}$ is invariant under pointwise multiplication of any polynomial in $\mathbb{C}[z_1, \dots, z_n]$. Then a family of operators encoding structure of $(\mathcal{H}, k_\lambda, \Omega)$ is obtained under appropriate conditions. In this note, these operators will be denoted by Δ_λ . We should mention that $\Delta = \Delta_0$ has been studied already by many researchers on some Hilbert function spaces.

This note has been organized as follows. In Section 2 and Section 3, we will give a partial announcement of results obtained in [8], where we dealt with Δ_λ 's of submodules in Hardy space over the bidisk. In Section 4, we revisit the Hardy space over the unit disk from our point of view. In Section 5, we studies Δ_λ 's of submodules in the Bergman space over the unit disk.

2 Rudin's module

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $H^2(\mathbb{D})$ be the Hardy space over $\mathbb{D}$. The Hardy space over the bidisk $\mathbb{D}^2$ will be denoted by $H^2(\mathbb{D}^2)$, or H^2 for short. Then $z = (z_1, z_2)$ will denote the variable of functions in H^2 . We note that H^2 can be defined as the tensor product Hilbert space $H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$. A will denote the bidisk algebra. Then,

under pointwise multiplication, H^2 becomes a Hilbert module over A . A closed subspace $\mathcal{M}$ of H^2 is called a submodule if $\mathcal{M}$ is invariant under the module action, that is, a submodule is an invariant subspace of H^2 under multiplication of each function in A . $[S]$ denotes the submodule generated by a set S . The rank of a submodule $\mathcal{M}$ is the least cardinality of a generating set of $\mathcal{M}$ as a Hilbert module, and which will be denoted by $\text{rank } \mathcal{M}$, and the following inequality is well known:

$$\dim \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] \leq \text{rank } \mathcal{M} \quad ((\lambda_1, \lambda_2) \in \mathbb{D}). \quad (2.1)$$

Set $\alpha_n = 1 - n^{-3}$ ($n \in \mathbb{N}$), and let b_{α_n} be the Blaschke factor whose zero is α_n . Then

$$\mathcal{M} = \sum_{j=0}^{\infty} q_j H^2(\mathbb{D}) \otimes z_2^j \quad \left(\text{where } q_j = \prod_{n=j}^{\infty} b_{\alpha_n}^{n-j} \right)$$

has been called Rudin's module (cf. Rudin [7]). The striking fact on Rudin's module is that the module rank is infinity. Indeed, for any $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{D}^2$, we have

$$\dim \mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] = \begin{cases} n + 1 & (\lambda = (\alpha_n, 0)) \\ 1 & (\text{otherwise}). \end{cases}$$

As n tends to infinity, we have $\text{rank } \mathcal{M} = \infty$ by (2.1).

Therefore we are interested in the following family of quotient vector spaces.

$$\mathcal{M}/[(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}] \quad ((\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

3 $H^2(\mathbb{D}^2)$ case

Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then R_f denotes the compression of a Toeplitz operator T_f into $\mathcal{M}$, that is, we set $R_f = P_{\mathcal{M}} T_f|_{\mathcal{M}}$ where $P_{\mathcal{M}}$ is the

orthogonal projection of H^2 onto a submodule $\mathcal{M}$. The following operator is called the defect operator of a submodule $\mathcal{M}$.

$$\Delta = I_{\mathcal{M}} - R_{z_1} R_{z_1}^* - R_{z_2} R_{z_2}^* + R_{z_1} R_{z_2} R_{z_1}^* R_{z_2}^*,$$

which has been introduced by Yang in [9, 10] (see, also Guo [3] and Guo-Yang [5]). Moreover, we introduce the following operator valued function:

$$\Delta_{\lambda} = I_{\mathcal{M}} - R_{b_{\lambda_1}(z_1)} R_{b_{\lambda_1}(z_1)}^* - R_{b_{\lambda_2}(z_2)} R_{b_{\lambda_2}(z_2)}^* + R_{b_{\lambda_1}(z_1)} R_{b_{\lambda_2}(z_2)} R_{b_{\lambda_1}(z_1)}^* R_{b_{\lambda_2}(z_2)}^*,$$

where

$$(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2)) = \left(\frac{z_1 - \lambda_1}{1 - \overline{\lambda_1} z_1}, \frac{z_2 - \lambda_2}{1 - \overline{\lambda_2} z_2} \right) \quad (\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2).$$

Since $(b_{\lambda_1}(z_1), b_{\lambda_2}(z_2))$ defines an automorphism of $\mathbb{D}^2$ (i.e. a biholomorphic map acting on $\mathbb{D}^2$), Δ_{λ} can be seen as a defect operator perturbed by an automorphism. The following theorem is the reason why we are interested in Δ_{λ} , which was shown in Guo-Yang [5] for the case where $\lambda = 0$ (see also Guo-Wang [4]), and their proof can be applied to the general case.

Theorem 3.1 (Guo-Yang [5], Guo-Wang [4]) *Let $\mathcal{M}$ be a submodule of $H^2(\mathbb{D}^2)$. Then for any $\lambda \in \mathbb{D}^2$,*

$$\ker(I_{\mathcal{M}} - \Delta_{\lambda}) = \mathcal{M} / [(z_1 - \lambda_1)\mathcal{M} + (z_2 - \lambda_2)\mathcal{M}].$$

Yang defined a new class of submodules in $H^2(\mathbb{D}^2)$.

Definition 3.1 ([10]) *A submodule $\mathcal{M}$ in H^2 is said to be Hilbert-Schmidt if Δ is Hilbert-Schmidt.*

Yang showed that Hilbert-Schmidt class includes Rudin's module and submodules generated by polynomials.

Theorem 3.2 (S [8]) *Let $\mathcal{M}$ be a submodule of H^2 .*

- (i) *If Δ_{μ} is Hilbert-Schmidt for some μ in $\mathbb{D}^2$, then Δ_{λ} is Hilbert-Schmidt for any λ in $\mathbb{D}^2$.*

(ii) If $\mathcal{M}$ is Hilbert-Schmidt then $\|\Delta_\lambda - \Delta_\mu\|_2 \rightarrow 0$ ($\lambda \rightarrow \mu$).

Theorem 3.3 (S [8]) *Let $\mathcal{M}$ be a Hilbert-Schmidt submodule such that $\dim \ker(I - \Delta_\mu) = n > 1$ for some μ in $\mathbb{D}^2$. Then, for any neighborhood U_1 of 1 such that $\sigma(\Delta_\mu) \cap \overline{U_1} = \{1\}$, there exists a neighborhood U_μ of μ such that $\sigma(\Delta_\lambda) \cap U_1 = \{1, \sigma_1(\lambda), \dots, \sigma_{n-1}(\lambda)\}$ for any λ in U_μ , counting multiplicity.*

Example 3.1 (Yang [9], S [8]) Let $q_1 = q_1(z_1)$ and $q_2 = q_2(z_2)$ be one variable inner functions, and let $\mathcal{M}$ be the submodule generated by q_1 and q_2 in $H^2(\mathbb{D}^2)$. Then we have

$$\dim \ker(I_{\mathcal{M}} - \Delta_\lambda) = \begin{cases} 2 & (\text{if } q_1(\lambda_1) = q_2(\lambda_2) = 0) \\ 1 & (\text{otherwise}). \end{cases}$$

and

$$\sigma(\Delta_\lambda) = \{0, 1, \pm\sigma(\lambda)\},$$

where we set

$$\sigma(\lambda) = \sqrt{(1 - |q_1(\lambda_1)|^2)(1 - |q_2(\lambda_2)|^2)}.$$

This calculation has been done already in the case where $(\lambda_1, \lambda_2) = (0, 0)$ by Yang in [9]. If $\sigma(\lambda) \neq 1$ then the eigenfunction corresponding to $\sigma(\lambda)$ is

$$\begin{aligned} e(\lambda) = & \left(\sqrt{1 - |q_2(\lambda_2)|^2} - \sqrt{1 - |q_1(\lambda_1)|^2} \right) \frac{q_1(z_1)q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)} \\ & - \frac{q_2(\lambda_2)}{\sqrt{1 - |q_2(\lambda_2)|^2}} \frac{q_1(z_1)(1 - \overline{q_2(\lambda_2)}q_2(z_2))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)} \\ & + \frac{q_1(\lambda_1)}{\sqrt{1 - |q_1(\lambda_1)|^2}} \frac{q_2(z_2)(1 - \overline{q_1(\lambda_1)}q_1(z_1))}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)} \end{aligned}$$

If $\sigma(\lambda) = 1$ then the eigenfunctions corresponding to $\sigma(\lambda)$ are

$$\frac{q_1(z_1)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}, \quad \frac{q_2(z_2)}{(1 - \overline{\lambda_1}z_1)(1 - \overline{\lambda_2}z_2)}.$$

Note that $e(\lambda)$ converges to 0 as $\sigma(\lambda)$ tends to 1.

4 $H^2(\mathbb{D})$ case

The defect operator of a submodule $\mathcal{M}$ in $H^2(\mathbb{D})$ is as follows:

$$\Delta = I_{\mathcal{M}} - R_z R_z^* = \text{Proj}(\mathcal{M}/z\mathcal{M}) = q \otimes q,$$

where q is the inner function corresponding to a submodule $\mathcal{M}$ by Beurling's theorem. The definition of Δ_λ is similar to that given in Section 3, and we have

$$\Delta_\lambda = I_{\mathcal{M}} - R_{b_\lambda} R_{b_\lambda}^* = \text{Proj}(\mathcal{M}/(z - \lambda)\mathcal{M}) = qK_\lambda \otimes qK_\lambda,$$

where we set $b_\lambda = (z - \lambda)/(1 - \bar{\lambda}z)$ and K_λ denotes the normalized Szegő kernel. These facts are well known.

5 $L_a^2(\mathbb{D})$ case

In this section, we deal with the defect operator of a submodule in Bergman space over $\mathbb{D}$. The Bergman space over $\mathbb{D}$ is defined as follows:

$$L_a^2(\mathbb{D}) = \left\{ f \in \text{Hol}(\mathbb{D}) : \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^2 dx dy < \infty \ (z = x + iy) \right\}.$$

The reproducing kernel is

$$k_\lambda(z) = \frac{1}{(1 - \bar{\lambda}z)^2} \quad (\text{the Bergman kernel}),$$

and the operator $S_z : f \mapsto zf$ acting on $L_a^2(\mathbb{D})$ is called the Bergman shift. The definition of submodules in $L_a^2(\mathbb{D})$ is the same as that of $H^2(\mathbb{D}^2)$. We summarize well known facts on submodules of $L_a^2(\mathbb{D})$.

Theorem 5.1 Let $\mathcal{M}$ be a submodule of $L_a^2(\mathbb{D})$.

- (i) $\dim \mathcal{M}/(z - \lambda)\mathcal{M}$ is independent of choice of λ in $\mathbb{D}$ (Richter [6]).

(ii) For every n in $\{1, 2, \dots, \infty\}$, there exists a submodule $\mathcal{M}$ such that $\dim \mathcal{M}/z\mathcal{M} = n$ (Apostol-Bercovici-Foiaş-Pearcy [1]).

(iii) $\mathcal{M}/z\mathcal{M}$ is a generating set of $\mathcal{M}$ (Aleman-Richter-Sundberg [2]).

The defect operator of a submodule of $L_a^2(\mathbb{D})$ is as follows:

$$\Delta = I_{\mathcal{M}} - 2R_z R_z^* + R_z^2 R_z^{*2},$$

which was introduced by Yang-Zhu [11] (they called this the root operator of $\mathcal{M}$). The definition of Δ_λ is similar to that given in Section 3,

$$\Delta_\lambda = I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*2},$$

where we set $b_\lambda = (z - \lambda)/(1 - \bar{\lambda}z)$. The following theorem was shown in Yang-Zhu [11] in the case where $\lambda = 0$, and their proof can be applied to the general case.

Theorem 5.2 (Yang-Zhu [11])

$$\ker(I_{\mathcal{M}} - \Delta_\lambda) = \mathcal{M}/(z - \lambda)\mathcal{M}.$$

The Hilbert-Schmidt class of submodules in $L_a^2(\mathbb{D})$ is defined as same as that given in Section 3.

Theorem 5.3 (S) Let $\mathcal{M}$ be a Hilbert-Schmidt submodule of $L_a^2(\mathbb{D})$. Then

(i) Δ_λ is Hilbert-Schmidt for any λ in $\mathbb{D}$,

(ii) $\|\Delta_\lambda - \Delta_\mu\|_2 \rightarrow 0$ ($\lambda \rightarrow \mu$).

Proof First, we shall show (i). Setting $k_z^{\mathcal{M}} = P_{\mathcal{M}}k_z$, we have

$$\begin{aligned} (\Delta_\lambda f)(z) &= \langle \Delta_\lambda f, k_z^{\mathcal{M}} \rangle \\ &= \langle (I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*2})f, k_z^{\mathcal{M}} \rangle \\ &= \langle f, (I_{\mathcal{M}} - 2R_{b_\lambda} R_{b_\lambda}^* + R_{b_\lambda}^2 R_{b_\lambda}^{*2})k_z^{\mathcal{M}} \rangle \\ &= \langle f, (1 - 2\overline{b_\lambda(z)}R_{b_\lambda} + \overline{b_\lambda(z)}^2 R_{b_\lambda}^2)k_z^{\mathcal{M}} \rangle \\ &= \langle f, (1 - 2\overline{b_\lambda(z)}b_\lambda + \overline{b_\lambda(z)}^2 b_\lambda^2)k_z^{\mathcal{M}} \rangle \\ &= \int_{\mathbb{D}} f(w) \overline{(1 - \overline{b_\lambda(z)}b_\lambda(w))^2} k_z^{\mathcal{M}}(w) dA(w), \end{aligned}$$

where $dA(w) = \pi^{-1}dxdy$ ($w = x + iy$). Hence Δ_λ is Hilbert-Schmidt if and only if

$$(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w)$$

is square integrable with respect to the Lebesgue measure on $\mathbb{D}^2$. We note that

$$\frac{(1 - \overline{b_\lambda(z)}b_\lambda(w))^2}{(1 - \bar{z}w)^2} = \left(\frac{1 - |\lambda|^2}{(1 - \lambda\bar{z})(1 - \bar{\lambda}w)} \right)^2. \quad (5.1)$$

Hence we have

$$\begin{aligned} (1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w) &= \frac{(1 - \overline{b_\lambda(z)}b_\lambda(w))^2}{(1 - \bar{z}w)^2} (1 - \bar{z}w)^2 k_z^{\mathcal{M}} \\ &= \left(\frac{1 - |\lambda|^2}{(1 - \lambda\bar{z})(1 - \bar{\lambda}w)} \right)^2 (1 - \bar{z}w)^2 k_z^{\mathcal{M}}. \end{aligned} \quad (5.2)$$

Since trivially (5.1) is bounded on $\mathbb{D}^2$, (5.2) is square integrable on $\mathbb{D}^2$. This concludes (i).

Next, we shall show (ii). Since the integral kernel of Δ_λ is

$$(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w),$$

and using (5.2), we have

$$\begin{aligned} &\|\Delta_\lambda - \Delta_\mu\|_2^2 \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} |(1 - \overline{b_\lambda(z)}b_\lambda(w))^2 k_z^{\mathcal{M}}(w) - (1 - \overline{b_\mu(z)}b_\mu(w))^2 k_z^{\mathcal{M}}(w)|^2 dA(z)dA(w) \\ &= \int_{\mathbb{D}} \int_{\mathbb{D}} \left| \left(\frac{1 - |\lambda|^2}{(1 - \lambda\bar{z})(1 - \bar{\lambda}w)} \right)^2 - \left(\frac{1 - |\mu|^2}{(1 - \mu\bar{z})(1 - \bar{\mu}w)} \right)^2 \right|^2 \\ &\quad \times |(1 - \bar{z}w)^2 k_z^{\mathcal{M}}|^2 dA(z)dA(w) \\ &\rightarrow 0 \quad (\lambda \rightarrow \mu) \end{aligned}$$

by the Lebesgue dominated convergence theorem. This concludes (ii).

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