

$PSL(2, \mathbf{Z})$ の有限型不変量について

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0. PRELIMINARIES

$PSL(2, \mathbf{Z})$ is the group of 2×2 matrices over $\mathbf{Z}$ with determinant 1 modulo $\pm E$. This group has the following generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfying the relations

$$S^2 = (TS)^3 = E.$$

Any element of $PSL(2, \mathbf{Z})$ can be presented as follows by using S and T ,

$$PSL(2, \mathbf{Z}) \ni T^{b_1} S T^{b_2} S \cdots T^{b_l} S.$$

From now on, we use the following sequence of integers to indicate the element.

$$[b_1, b_2, \dots, b_l]$$

Then we get the following relations by using this symbol.

$$[b_1, b_2, \dots, b_i, 0, b_{i+2}, \dots, b_l] = [b_1, b_2, \dots, b_i + b_{i+2}, \dots, b_l]$$

$$[b_1, b_2, \dots, b_i, 1, 1, 1, b_{i+4}, \dots, b_l] = [b_1, b_2, \dots, b_i, b_{i+4}, \dots, b_l]$$

It is known that two symbols present the same element in $PSL(2, \mathbf{Z})$ if and only if they can be transformed to each other by finite sequence of the above relations.

1. THE DEFINITION OF THE FINITE TYPE INVARIANT OF $PSL(2, \mathbf{Z})$

Let $\bar{\Gamma}$ denote the free abelian group generated by all the elements in $PSL(2, \mathbf{Z})$ and $\bar{\Gamma}_n$ denote the group spanned by the following set

$$\left\{ \sum_{c_{i_j}=\pm 1} (-1)^{\text{the number of } (-1)\text{'s in } \{c_{i_j}\}} \times [b_1, b_2, \dots, b_l]_{c_{i_1}, c_{i_2}, \dots, c_{i_n}} \right\},$$

where

$$\begin{aligned} & [b_1, b_2, \dots, b_{i_1}, \dots, b_{i_2}, \dots, b_{i_n}, \dots, b_l]_{c_{i_1}, c_{i_2}, \dots, c_{i_n}} \\ &= [b_1, b_2, \dots, b_{i_1} - c_{i_1} + 1, \dots, b_{i_2} - c_{i_2} + 1, \dots, b_{i_n} - c_{i_n} + 1, \dots, b_l]. \end{aligned}$$

Note that if c_{i_j} is 1, then b_{i_j} does not change and that if c_{i_j} is -1 , then b_{i_j} is changed to $b_{i_j} + 2$.

Now we define the finite type invariant of $PSL(2, \mathbf{Z})$ as following.

Definition. An additive map from $\bar{\Gamma}/\bar{\Gamma}_{n+1}$ to $\mathbf{Q}$ is called an invariant of type n .

Let $\sim_n$ (we call this n -equivalence) denote the equivalence relation defined by $\bar{\Gamma}_{n+1}$ in $\bar{\Gamma}$.

2. ON TYPE 0, 1 AND 2 INVARIANTS

Theorem 1.

$$\bar{\Gamma}/\bar{\Gamma}_1 = \mathbf{Z}\{[], [0], [1], [0, 1], [1, 0], [1, 1]\}.$$

Moreover, 0-equivalence class of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is determined by its congruence class modulo 2.

From now on, we restrict ourselves to the matrices 0-equivalent to the identity E and consider finite type invariants. Let $\bar{\Gamma}(2)$ be the span over $\mathbf{Z}$ of matrices 0-equivalent to the identity. We know that any element of $\bar{\Gamma}(2)$ can be presented as a sequence of even integers with even length, subject to the following relation

$$\begin{aligned} & [2a_1, 2a_2, \dots, 2a_i, 0, 2a_{i+2}, \dots, 2a_{2m}] \\ &= [2a_1, 2a_2, \dots, 2(a_i + a_{i+2}), \dots, 2a_{2m}]. \end{aligned}$$

By similar calculation, we have the following

Theorem 2.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_2 = \mathbf{Z}\{[], [0, 2], [2, 0]\}.$$

In fact, any element of $\bar{\Gamma}(2)$ is 1-equivalent to

$$(1 - A)[] + A_0[0, 2] + A_1[2, 0],$$

where

$$A = \sum_{i=1}^{2m} a_i, \quad A_0 = \sum_{i=1}^m a_{2i}, \quad A_1 = \sum_{i=1}^m a_{2i-1}.$$

Moreover, $1 - A$, A_0 , A_1 are well-defined.

If $[2a_1, 2a_2, \dots, 2a_{2m}] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$A_0 = \sum_{i=1}^{\gamma/2} (-1)^{[(2i-1)\frac{\alpha}{\gamma}]}, \quad A_1 = \sum_{i=1}^{\gamma/2} (-1)^{[(2i-1)\frac{\delta}{\gamma}]}.$$

Where $[]$ denotes the greatest integer function.

To prove the formulas, we use Tuler's result of the linking number of a 2-bridge link ([2]).

Corollary 2.1. *Any type 1 invariant is of the form*

$$c_1(1 - A) + c_2A_0 + c_3A_1,$$

where c_i 's are constants.

Theorem 3.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_3 = \mathbf{Z}\{[], [0, 2], [2, 0], [2, 2], [0, 2, 2, 0], [0, 4], [4, 0]\}.$$

In fact, any element of $\bar{\Gamma}(2)$ is 2-equivalent to

$$\begin{aligned} & \frac{(A-1)(A-2)}{2} [] - A_0(A-2)[0, 2] + A_1(A-2)[2, 0] \\ & + \sum_{i=1}^m \sum_{j=i}^m a_{2i-1}a_{2j} [2, 2] + \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i}a_{2j+1} [0, 2, 2, 0] \end{aligned}$$

$$+ \frac{A_0(A_0 - 1)}{2} [0, 4] + \frac{A_1(A_1 - 1)}{2} [4, 0].$$

If $[2a_1, 2a_2, \dots, 2a_{2m}] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then

$$\sum_{i=1}^m \sum_{j=i}^m a_{2i-1} a_{2j} = \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\gamma}{\alpha}] + [2j\frac{\gamma}{\alpha}]},$$

$$\sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i} a_{2j+1} = \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\gamma}{\alpha}] + [2j\frac{\gamma}{\alpha}]}.$$

To prove the formulas, we use the result of the Casson knot invariant of a 2-bridge knot ([1]).

Corollary 3.1. *Any type 2 invariant is of the form*

$$d_1 \frac{(A-1)(A-2)}{2} + d_2 A_0(A-2) + d_3 A_1(A-2)$$

$$+ d_4 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\gamma}{\alpha}] + [2j\frac{\gamma}{\alpha}]} + d_5 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\gamma}{\alpha}] + [2j\frac{\gamma}{\alpha}]}$$

$$+ d_6 \frac{A_0(A_0 - 1)}{2} + d_7 \frac{A_1(A_1 - 1)}{2},$$

where d_i 's are constants.

Detail will appear elsewhere.

REFERENCES

- [1] Y. Mizuma: *A formula for the Casson knot invariant of a 2-bridge knot*, to appear in J. Knot Theory Ramifications.
- [2] R. Tuler: *On the linking number of a 2-bridge link*, Bull. London Math. Soc. **13** (1981), 540–544.