

A BOUNDARY ELEMENT GALERKIN METHOD FOR
THE DIRICHLET PROBLEM OF THE HEAT EQUATION

大西 和 榮

K. Onishi

Applied Mathematics Department

Fukuoka University

Jonan-ku, Fukuoka 814-01 (Japan)

1. INTRODUCTION

Recently, numerical solutions of initial-boundary value problems of the heat equation are often obtained by boundary element methods based on boundary integral equations, because the approach enables us to treat heat conduction problems with domains extending to infinity, with polygonal domains and non-smooth data with much ease. For the Dirichlet problem, direct methods lead to the approximation of a Volterra integral equation of the first kind. The kernel function involved in the boundary integral equation corresponds to the single-layer heat potential, which is weakly singular.

The approximation of boundary integral equations in transient heat conduction problems has been considered by several authors; see Brebbia et al. (1984) for example in engineering applications. They used the collocation method with boundary finite elements as trial functions on the boundary. As regard to the mathematical analysis, Costabel et al. (1987) and Onishi (1987) discussed the Neumann problem and they showed the existence of the solution of a corresponding Volterra integral equation of the second kind on a non-smooth boundary. They showed the convergence and the

stability of the projection method in the space of continuous functions. Recently, Yang (1986) and Arnold and Noon (1987) presented some attempts at boundary element methods using the single-layer heat potential to the solution of Dirichlet problem on a smooth surface. More recently, Okamoto (1988) showed an application of Fourier transform to the Dirichlet problem and proved unconditional stability as well as conditional convergence of the boundary element approximation for the heat operator in L^2 -sense.

In this paper, we will show the convergence property and the stability of Galerkin's method applied to the solution of the boundary integral equation of the Dirichlet problem of heat conduction in a solid with piecewise Liapunov surface with corners and edges in a more general class. The discussion is done for three-dimensional problems, but the validity of the results remains also for problems in two dimensions.

2. DIRICHLET PROBLEM IN A NON-SMOOTH DOMAIN

We shall confine the geometry of the domain in question. Let Ω be a simply connected bounded open domain in the three-dimensional Euclidean space R^3 and assume that the closed bounded surface $\Gamma = \partial\Omega$ consists of a finite number of open smooth subsurfaces Γ_i ($i=1, \dots, N$) so that $\Gamma = \bigcup_{1 \leq i \leq N} \bar{\Gamma}_i$, where $\bar{\Gamma}_i = \Gamma_i \cup \partial\Gamma_i$. Then the surface has a tangent plane at every point $x \in \bar{\Gamma}_i$ if the tangent plane at the edge point of Γ_i is understood to be the corresponding half plane. Moreover, the angle v between the exterior normal vector $n(x)$ to Γ_i at $x \in \Gamma_i$ and the vector $(x-y)$ for an arbitrary point $y \in \Gamma_i$ ($x \neq y$) satisfies the Liapunov condition, see Michlin (1987, p.285) for example:

$$|\cos v| \leq L(\Gamma) |y-x|^\kappa \quad (0 < \kappa < 1) \quad , \quad (2.1)$$

where L is a global constant depending only on Γ . The set of points on Γ where the surface is not smooth forms corners and edges. This is denoted by $\delta\Gamma = \bigcup_{1 \leq i \leq N} \partial\Gamma_i$, which has zero Lebesgue volume measure.

Let $d\Theta_x(y)$ denote an infinitesimal solid angle at $x \in R^3$ subtending the infinitesimal surface area $d\Gamma(y)$ at $y \in \Gamma - \delta\Gamma$; see Michlin (1978, p.287). Then

$$\begin{aligned} d\Theta_x(y) &= - \frac{\partial}{\partial n(y)} \left(\frac{1}{|y-x|} \right) d\Gamma(y) \\ &= \frac{(y-x) \cdot n(y)}{|y-x|^3} d\Gamma(y) \end{aligned} \quad (2.2)$$

Remarks. Let $I(x)$ be the index set attributed to the point x , for which $x \in \bar{\Gamma}_i$ with $i \in I(x)$. If $x \notin \Gamma$, then $I(x)$ is the null set. Put $I^C(x) = \{1, 2, \dots, N\} - I(x)$. For $i \in I(x)$ it follows from (2.1) that $|(y-x) \cdot n(y)| / |y-x|^3 = |\cos \nu| / |y-x|^2 \leq L / |y-x|^{2-\kappa}$ for any $y \in \bar{\Gamma}_i$. Therefore, the integral $\int_{\bar{\Gamma}_i} d\Theta_x(y)$ is absolutely convergent. For $j \in I^C(x)$, $\int_{\bar{\Gamma}_j} d\Theta_x(y)$ is also convergent since $|y-x| \geq C(x) > 0$ for $y \in \bar{\Gamma}_j$. Hence, $\Theta(x) = \int_{\Gamma} d\Theta_x(y)$ is well defined for every $x \in R^3$.

For $x \in \Gamma$, $\Theta(x)$ is equal to the interior solid angle at the vertex x of the cone, whose side surface is constructed by all the half ray tangential lines to the surface Γ radiating from x . For a piecewise Liapunov surface Γ it follows that

$$\sup_{x \in R^3} \int_{\Gamma} |d\Theta_x(y)| = \sup_{x \in R^3} \int_{\Gamma} \frac{|(y-x) \cdot n(y)|}{|y-x|^3} d\Gamma(y) = A < +\infty \quad (2.3)$$

with some constant A . In addition we require Γ to satisfy

$$\lim_{\delta \rightarrow 0} \sup_{x \in \Gamma} W_{\delta}(x) = \omega < 1, \quad (2.4)$$

where $W_{\delta}(x)$ is defined by the expression:

$$W_{\delta}(x) := \frac{1}{2\pi} \left\{ \int_{0 < |y-x| \leq \delta} |d\Theta_x(y)| + |2\pi - \Theta(x)| \right\}. \quad (2.5)$$

Remarks. The piecewise Liapunov surface satisfying (2.4) is called Wendland surface. The condition (2.4) (Wendland 1965) allows the splitting of the integral operator of the double-layer potential into the sum of a contraction operator and a completely continuous one in $C(\Gamma)$, which is basic for the validity of the Fredholm-Radon method in potential theory.

We consider the heat equation for unknown temperature $u(x, t)$:

$$\frac{\partial u}{\partial t} = \Delta u, \quad (x, t) \in (\Omega \cup \Omega^e) \times (0, T] \quad (2.6)$$

for some finite value T , in which Δ is the Laplacian in R^3 with respect to the variable x and Ω^e denotes the exterior of the domain Ω .

On the boundary we consider the Dirichlet condition:

$$u(x, t) = \hat{u}(x, t), \quad (x, t) \in \Gamma \times [0, T]. \quad (2.7)$$

In addition, we consider the initial condition:

$$u(x, 0) = u_0(x), \quad x \in \Omega \cup \Omega^e \quad (2.8)$$

for the bounded Cauchy datum u_0 in $C(\Omega \cup \Omega^e)$. In Ω^e , the corresponding Cauchy datum u_0 may be assumed to grow at most exponentially:

$$|u_0(x)| \leq \alpha_1 \exp[\beta_1 |x|^\sigma] \quad (2.9)$$

with some constants $\alpha_1 > 0$, $\beta_1 > 0$ and $0 < \sigma < 2$; see Krzyzanski (1971, p.455) for example. We can assume without loss of generality by considering the Weierstrass integral that $u_0 = 0$ in $\Omega \cup \Omega^e$.

3. BOUNDARY INTEGRAL EQUATION OF THE FIRST KIND

We shall derive a boundary integral equation corresponding to the Dirichlet problem (2.6)-(2.8) and investigate properties of the integral operator. We start the discussion with definitions of single-layer heat potential :

$$Gq(x,t) := \int_0^t \int_{\Gamma} q(y,\tau) v(y,\tau;x,t) d\Gamma(y) d\tau, \quad (3.1)$$

with the density q and double-layer heat potential:

$$Hu(x,t) := \int_0^t \int_{\Gamma} u(y,\tau) \frac{\partial v(y,\tau;x,t)}{\partial n(y)} d\Gamma(y) d\tau, \quad (3.2)$$

with the density u , where $n(y)$ is the external normal at y to the boundary Γ . Here, v is the fundamental solution to the heat operator $\partial/\partial t - \Delta$:

$$v(y,\tau;x,t) = \begin{cases} \left(\frac{1}{2\sqrt{\pi(t-\tau)}} \right)^3 \exp\left[-\frac{r^2}{4(t-\tau)} \right] & (t > \tau) \\ 0 & (t < \tau) \end{cases} \quad (3.3)$$

with $r = |y-x|$.

Put

$$g(x,t) := \frac{1}{2} \hat{u}(x,t) + H\hat{u}(x,t), \quad x \in \Gamma \quad (3.4)$$

with the expression:

$$\hat{H}u(x,t) = (1 - \frac{\Theta(x)}{2\pi})\hat{u}(x,t) + \int_0^t \int_{\Gamma} \hat{u}(y,\tau) \frac{r^3}{(t-\tau)} v(y,\tau;x,t) d\Theta_x(y) d\tau. \quad (3.5)$$

According to Costabel et al. (1987), unknown boundary flux $q = \partial u / \partial n$ in the normal direction is given as a solution of the linear Volterra-Fredholm boundary integral equation of the first kind:

$$Gq(x,t) = g(x,t), \quad (x,t) \in \Sigma = \Gamma \times [0,T] \quad (3.6)$$

Next lemma shows that G can be understood as a linear bounded operator from $C(L^p(\Gamma):[0,T])$ into $C(\Sigma)$.

Lemma 3.1. The operator $G: C(L^p(\Gamma):[0,T]) \rightarrow C(\Sigma)$ defined by (3.1) is bounded for $p > 2$.

Proof. Using the idea in Pogorzelski (1966, p.353), we have for any μ ($0 < \mu < \frac{3}{2}$) the inequality:

$$\begin{aligned} v(y,\tau;x,t) &= \frac{1}{2^{2\mu}\pi^{3/2}} \frac{1}{(t-\tau)^\mu} \frac{1}{r^{3-2\mu}} \left[\frac{r^2}{4(t-\tau)} \right]^{(3/2)-\mu} \exp\left[-\frac{r^2}{4(t-\tau)}\right] \\ &\leq \frac{1}{(t-\tau)^\mu} \frac{G_1}{r^{3-2\mu}} \end{aligned} \quad (3.7)$$

with $G_1 = s^s e^{-s} / (2^{2\mu}\pi^{3/2})$ and $s = \frac{3}{2} - \mu$. We apply the Hölder's inequality twice to $Gq(x,t)$ and obtain from (3.1) that

$$\begin{aligned} |Gq(x,t)| &\leq \int_0^t \left\{ \int_{\Gamma} |q(y,\tau)|^p d\Gamma \right\}^{1/p} \left\{ \int_{\Gamma} |v|^{p'} d\Gamma \right\}^{1/p'} d\tau \\ &\leq \int_0^t \|q(\cdot,\tau)\|_p \left\{ \left(\int_{\Gamma} |v|^{p' \frac{1}{p}} d\Gamma \right)^{p'} \left(\int_{\Gamma} 1^{\frac{1}{p}} d\Gamma \right)^p \right\}^{1/p'} d\tau \end{aligned}$$

$$= \text{mes}(\Gamma)^{p/p'} \int_0^t \|q(\cdot, \tau)\|_p \int_{\Gamma} |v| d\Gamma d\tau$$

with $\frac{1}{p} + \frac{1}{p'} = 1$, where $\text{mes}(\Gamma) = \int_{\Gamma} d\Gamma(y)$. In view of the inequality (3.7) we can see that inequalities above are valid only for such values of μ satisfying $(3-2\mu)p' < 2$, i.e., $\frac{1}{2}(3 - \frac{2}{p'}) < \mu$. Then we have

$$\|Gq(x, t)\| \leq \{\text{mes}(\Gamma)\}^{\frac{p}{p'}} \int_0^t \frac{d\tau}{(t-\tau)^{\mu}} \int_{\Gamma} \frac{d\Gamma(y)}{r^{3-2\mu}} \|q\|_{C(L^p(\Gamma):[0,T])}.$$

If we take μ with $\frac{1}{2} < \mu < 1$, the integrals are convergent. Consequently, there exists a constant C depending only on Γ and T such that $\|Gq\| \leq C \|q\|_{C(L^p(\Gamma):[0,T])}$. The assumption $p > 2$ is equivalent to $\frac{1}{2}(3 - \frac{2}{p'}) < 1$.

Remarks. In order to apply the Hilbert space approach in the approximation method in the next section, we shall regard G as an operator:

$$G: H^{-1/2, -1/4}(\Sigma) \rightarrow H^{1/2, 1/4}(\Sigma).$$

Lemma 3.2. Under the assumption (2.3) and for $u \in C(\Sigma)$, the continuous function $g(x, t)$ of (3.4) satisfies the inequality:

$$\|g\| \leq \left(\frac{3}{2} + \frac{A}{2\pi} \right) \|u\|.$$

Proof. The continuity of $g(x, t)$ is shown in Costabel et al. (1987). We shall prove the inequality of the lemma: By the variable transformation; $\tau \mapsto$

$\sigma = r/2\sqrt{t-\tau}$, $\hat{u}(x, t)$ in (3.5) can be expressed as

$$\hat{u}(x, t) = \left(1 - \frac{\Theta(x)}{2\pi}\right) \hat{u}(x, t)$$

$$+ \frac{1}{2\pi} \int_{\Gamma} \left\{ \frac{4}{\sqrt{\pi}} \int_{r/2\sqrt{t}}^{\infty} \sigma^2 e^{-\sigma^2} \hat{u}(y, t - \frac{r^2}{4\sigma^2}) d\sigma \right\} d\Theta_x(y) .$$

Consequently, we have

$$\begin{aligned} |\hat{Hu}(x, t)| &\leq \left| 1 - \frac{\Theta(x)}{2\pi} \right| + \frac{1}{2\pi} \int_{\Gamma} \left(\frac{4}{\sqrt{\pi}} \int_0^{\infty} \sigma^2 e^{-\sigma^2} d\sigma \right) |d\Theta_x(y)| |\hat{u}| \\ &\leq \left(1 + \frac{A}{2\pi} \right) \|\hat{u}\| . \end{aligned}$$

The last inequality follows from (2.3) and from $0 < \Theta(x) < 4\pi$, $\int_0^{\infty} \sigma^2 e^{-\sigma^2} d\sigma = \sqrt{\pi}/4$.

Properties of the integral operator G are now discussed in the space $H^{1/2, 1/4}(\Sigma)$ and its dual space $H^{-1/2, -1/4}(\Sigma)$, introduced by Lions and Magenes (1968, p.10 and p.44): Let $H^{1/2, 1/4}(\Sigma)$ be a Sobolev space defined by

$$H^{1/2, 1/4}(\Sigma) = L^2(H^{1/2}(\Gamma); [0, T]) \cap H^{1/4}(L^2(\Gamma); [0, T])$$

equipped with the norm:

$$\begin{aligned} \|w\|_{H^{1/2, 1/4}(\Sigma)}^2 &= \int_0^T \|w(\cdot, t)\|_{H^{1/2}(\Gamma)}^2 dt \\ &\quad + \int_0^T \int_0^T \frac{\|w(\cdot, t) - w(\cdot, s)\|_{L^2(\Gamma)}^2}{|t-s|^{3/2}} ds dt . \end{aligned}$$

We denote by $((\cdot, \cdot))_0$ the scalar product:

$$((w_1, w_2))_0 := \int_0^T (w_1(\cdot, t), w_2(\cdot, t))_{L^2(\Gamma)} dt .$$

Next two important lemmas are much due to Costabel (1987).

Lemma 3.3. There exists a constant $\alpha > 0$ depending only on Σ such that

$$\alpha^{-1} |||q|||_{H^{-1/2, -1/4}(\Sigma)} \leq |||Gq|||_{H^{1/2, 1/4}(\Sigma)} \leq \alpha |||q|||_{H^{-1/2, -1/4}(\Sigma)}.$$

The next lemma shows strong coerciveness of the operator G .

Lemma 3.4. There exists a constant $\beta > 0$ depending only on Σ such that

$$((Gq, q))_0 \geq \beta |||q|||_{H^{-1/2, -1/4}(\Sigma)}^2$$

for all q in $H^{-1/2, -1/4}(\Sigma)$.

4. APPROXIMATION ON THE BOUNDARY

In this section, we shall consider the semi-discretization of the solution by Galerkin method using boundary finite elements. We shall show convergence and accuracy of the semi-discretized approximate solution. The way of arguments is much due to Nedelec and Planchard (1973) as well as Hsiao and Wendland (1977).

Let V_h be finite-dimensional subspaces of the Hilbert space $H^{-1/2}(\Gamma)$, approximating the solution $q(x, t)$ of the Volterra-Fredholm integral equation (3.1) and (3.6), such that $\bigcup_{h>0} V_h$ is dense in $L^2(\Gamma)$ and $V_h \subset V_{h'}$ for $h > h'$. Put $\dim(V_h) = n$ by assuming that $n = 1/h$ for $n = 1, 2, \dots$. Let $\{\phi_j(x)\}_{j=1,2,\dots,n}$ denote the basis of V_h . We consider the approximation of $q(x, t)$ in the form:

$$q_h(x, t) = \sum_{j=1}^n \hat{q}_j(t) \phi_j(x) \quad (4.1)$$

with coefficient functions $\hat{q}_j(t)$ ($0 \leq t \leq T$) to be determined.

We shall consider the semi-discrete Galerkin approximation: Find unknown q_h in $H^{-1/2, -1/4}(\Sigma)$ satisfying that

$$((Gq_h, q'_h))_0 = ((g_h, q'_h))_0 \quad \text{for all } q'_h \in V_h, \quad (4.2)$$

where g_h is an L^2 -orthogonal projection of $g \in H^{1/2, 1/4}(\Sigma)$ into $L^2(V_h; (0, T))$: That is, with the projector

$$P_h : g \in L^2(\Sigma) \rightarrow g_h \in L^2(V_h; (0, T)) \quad .$$

We assume that $\|P_h g\|_{H^{1/2, 1/4}(\Sigma)} \leq \|g\|_{H^{1/2, 1/4}(\Sigma)}$. This is equivalent to the proposition:

$$((Gq_h, \varphi_i))_0 = ((g, \varphi_i))_0 \quad \text{for all } \varphi_i \in V_h, \quad i = 1, 2, \dots, n. \quad (4.3)$$

Theorem 4.1. Let q be the solution of (3.6) in $H^{-1/2, -1/4}(\Sigma)$ and q_h be a solution of (4.2). Then, there exists a constant $\rho(\Sigma) > 0$ such that

$$\begin{aligned} & \| \|q - q_h\| \|_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq \rho \left\{ \inf_{q'_h \in V_h} \| \|q - q'_h\| \|_{H^{-1/2, -1/4}(\Sigma)} + \| \|g - g_h\| \|_{H^{1/2, 1/4}(\Sigma)} \right\}. \end{aligned} \quad (4.4)$$

Proof. From (3.6) we have

$$((Gq, q'))_0 = ((g, q'))_0 \quad \text{for all } q' \in H^{-1/2, -1/4}(\Sigma). \quad (4.5)$$

For an arbitrary q'_h in V_h , it follows from Lemma 3.4 that

$$((G(q_h - q'_h), q_h - q'_h))_0 \geq \beta |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)}^2.$$

On the other hand, we can see that

$$\begin{aligned} ((G(q_h - q'_h), q_h - q'_h))_0 &= ((G((q - q'_h) - (q - q_h)), q_h - q'_h))_0 \\ &= ((G(q - q'_h), q_h - q'_h))_0 - ((G(q - q_h), q_h - q'_h))_0 \\ &= ((G(q - q'_h), q_h - q'_h))_0 - ((g - g_h, q_h - q'_h))_0 \\ &\leq |||G(q - q'_h)|||_{H^{1/2, 1/4}(\Sigma)} |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ &\quad + |||g - g_h|||_{H^{1/2, 1/4}(\Sigma)} |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ &\leq (\alpha |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + |||g - g_h|||_{H^{1/2, 1/4}(\Sigma)}) \\ &\quad \times |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)}. \end{aligned}$$

The third equality follows from (4.2) and (4.5). The last inequality follows from Lemma 3.3. Combining the above inequalities, we have

$$\begin{aligned} &\beta |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ &\leq \alpha |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + |||g - g_h|||_{H^{1/2, 1/4}(\Sigma)}, \end{aligned}$$

from which it follows that

$$|||q - q_h|||_{H^{-1/2, -1/4}(\Sigma)}$$

$$\begin{aligned}
&\leq |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\
&\leq (1 + \frac{\alpha}{\beta}) |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + \frac{1}{\beta} |||g - g_h|||_{H^{1/2, 1/4}(\Sigma)}.
\end{aligned}$$

This leads to the desired inequality (4.4) with $\rho = \max\{(1+\alpha/\beta), 1/\beta\}$.

We can obtain a stronger result in the next theorem, which shows the optimal rate of convergence of the Galerkin approximation in $H^{-1/2, -1/4}(\Sigma)$.

Theorem 4.2 (Cea's lemma). The semi-discrete Galerkin approximation (4.2) is inverse stable: For the Galerkin solution q_h it holds that

$$\begin{aligned}
&|||q - q_h|||_{H^{-1/2, -1/4}(\Sigma)} \\
&\leq (1 + \frac{\alpha}{\beta}) \inf_{q'_h \in V_h} |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)}.
\end{aligned}$$

Proof. The Galerkin approximation (4.2) is equivalent to the problem of finding the unknown q_h of the form (4.1) in $H^{-1/2, -1/4}(\Sigma)$, satisfying the operator equation:

$$P_h G P_h q_h = P_h g \quad (4.6)$$

By the same way of arguments as in Wendland (1982, p.21), we can see from Lemma 3.4 that

$$\begin{aligned}
\beta |||q_h|||_{H^{-1/2, -1/4}(\Sigma)}^2 &\leq ((Gq_h, q_h))_0 = ((P_h G P_h q_h, q_h))_0 \\
&\leq |||P_h G P_h q_h|||_{H^{1/2, 1/4}(\Sigma)} |||q_h|||_{H^{-1/2, -1/4}(\Sigma)}.
\end{aligned}$$

The first equality followed from the relation:

$$((GP_h q_h, q_h))_0 - ((P_h GP_h q_h, q_h))_0 = (((I - P_h)GP_h q_h, q_h))_0 = 0,$$

since $q_h \in V_h$ and $(I - P_h)GP_h q_h \in V_h^\perp$, the orthogonal complement of V_h , for every $t \in [0, T]$. Then we have

$$\beta |||q_h|||_{H^{-1/2, -1/4}(\Sigma)} \leq |||P_h GP_h q_h|||_{H^{1/2, 1/4}(\Sigma)}.$$

Since this inequality holds for all q_h , we know that $P_h GP_h: H^{-1/2, -1/4}(\Sigma) \rightarrow H^{1/2, 1/4}(\Sigma)$ is invertible. The inverse is bounded as follows:

$$\begin{aligned} & |||(P_h GP_h)^{-1}|||_{H^{1/2, 1/4}(\Sigma), H^{-1/2, -1/4}(\Sigma)} \\ & := \sup_{g_h \neq 0} \frac{|||(P_h GP_h)^{-1}g_h|||_{H^{-1/2, -1/4}(\Sigma)}}{|||g_h|||_{H^{1/2, 1/4}(\Sigma)}} \leq \frac{1}{\beta} \quad (4.7) \end{aligned}$$

see Kantorowitsch and Akilow (1964, Satz 2,2.V) for example. From (3.6) and (4.6) it follows that

$$q_h = (P_h GP_h)^{-1} P_h Gq.$$

This defines the Galerkin projector $G_h = (P_h GP_h)^{-1} P_h G: H^{-1/2, -1/4}(\Sigma) \rightarrow H^{-1/2, -1/4}(\Sigma)$. We shall show that G_h is bounded. For this purpose, we put $g_h = P_h Gq$. From (4.7) and Lemma 3.3 it follows that

$$\begin{aligned} & |||(P_h GP_h)^{-1} P_h Gq|||_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq |||(P_h GP_h)^{-1}|||_{H^{1/2, 1/4}(\Sigma), H^{-1/2, -1/4}(\Sigma)} |||P_h Gq|||_{H^{1/2, 1/4}(\Sigma)} \end{aligned}$$

$$\leq \frac{\alpha}{\beta} |||q|||_{H^{-1/2, -1/4}(\Sigma)}.$$

This implies that G_h is bounded as desired: Namely

$$|||G_h|||_{H^{-1/2, -1/4}(\Sigma), H^{-1/2, -1/4}(\Sigma)} \leq \frac{\alpha}{\beta}. \quad (4.8)$$

Note that $G_h q'_h = q'_h$ for all $q'_h \in V_h$ because

$$P_h G P_h q'_h = P_h G q'_h. \quad (4.9)$$

Consequently we have

$$\begin{aligned} & |||q - q_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + |||q_h - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ & = |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + |||G_h q - G_h q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq (1 + |||G_h|||_{H^{-1/2, -1/4}(\Sigma), H^{-1/2, -1/4}(\Sigma)}) \\ & \quad \times |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} \end{aligned}$$

which leads to the assertion of the theorem from (4.8).

For the concreteness of the discussion, as V_h we shall consider the regular finite element spaces S_h with the following two conditions for some positive integer m ; see Hsiao and Wendland (1981, p.4) for example:

Convergence property: Let $t \leq s$ be such that $-(m+1) \leq t \leq s \leq m+1$, $-m \leq s$ and $t \leq m$ for some non-negative integer m . Then for any $q \in H^s(\Gamma)$ there exists a $q'_h \in S_h$ such that

$$\|q - q'_h\|_{H^t(\Gamma)} \leq C_1 h^{s-t} \|q\|_{H^s(\Gamma)} \quad (4.10)$$

with some constant C_1 which is independent on q_h and h .

Inverse assumption: Let $t \leq s$ be such that $|t|, |s| \leq m$. Then there exists a constant C_2 independent on h such that

$$\|q_h\|_{H^s(\Gamma)} \leq C_2 h^{t-s} \|q_h\|_{H^t(\Gamma)} \quad \text{for all } q_h \in S_h \quad (4.11)$$

Remarks. Nedelec and Planchard (1973, Lemme 3.1 and Lemme 3.2) showed that, if Γ is a polyhedron, linear triangular finite element spaces satisfy (4.10) and (4.11) with $m = 1$, provided that all the angles θ in the triangulation satisfy $\theta \geq \theta_0 > 0$ with a constant θ_0 , which is independent of the maximum diameter h among all triangles. For constant triangular finite element spaces, the convergence property (4.10) is satisfied with $m = 0$; see Nedelec and Planchard (1973, Lemme 3.4). However, (4.11) holds only for $-1 \leq t \leq s \leq 0$, see Nedelec and Planchard (1973, Lemme 3.3).

As an immediate consequence of Theorem 4.2 and (4.10) we have

Theorem 4.3. For the semi-discrete Galerkin solution q_h with constant boundary finite elements on the triangulation of the polyhedron Γ , it holds that

$$\|q - q_h\|_{H^{-1/2, -1/4}(\Sigma)} \leq \left(1 + \frac{\alpha}{\beta}\right) C_1 h^{s+1/2} \|q\|_{H^{s, -1/4}(\Sigma)}$$

with $0 \leq s \leq 1$.

It is often the case that Dirichlet data $\hat{u}(x,t)$ of (2.7) or Cauchy data $u_0(x)$ of (2.8) are given imprecisely due to measurements. The right hand side $g(x,t)$ of (3.6) can not be obtained exactly because of the approximate evaluation of the term $H\hat{u}(x,t)$ in (3.4) and (3.5). Due to the limitation of a finite number of digits available in the numerical computation, round-off errors are not avoidable. These cause the additional impreciseness involved in the right hand side $g(x,t)$. We assume that the polluted g , denoted here by $\tilde{g}$, belong to $H^{0,1/4}(\Sigma)$. Instead of (3.6), we have to solve the equation:

$$G\tilde{q}(x,t) = \tilde{g}(x,t) \quad , \quad (x,t) \in \Sigma \quad (4.12)$$

In this situation we have the ill-posedness of the Galerkin approximation as next theorem shows.

Theorem 4.4. For the semi-discrete Galerkin solution $\tilde{q}_h$ of (4.12) with $\phi_i \in S_h$, it holds that

$$\begin{aligned} & |||q - \tilde{q}_h|||_{H^{-1/2,-1/4}(\Sigma)} \\ & \leq C_3 \{ h^{s+1/2} |||q|||_{H^{s,-1/4}(\Sigma)} + h^{-1/2} |||g - \tilde{g}|||_{H^{0,1/4}(\Sigma)} \} \end{aligned}$$

with some constant $C_3 > 0$ and $0 \leq s \leq 1$.

Proof. The way of the proof is due to Hsiao and Wendland (1981, p.9). From (4.6) it follows that

$$\begin{aligned} P_h G P_h (q - \tilde{q}_h) &= P_h G P_h q - P_h \tilde{g} \\ &= (P_h G P_h - P_h G) q + P_h (g - \tilde{g}) \quad . \end{aligned}$$

Using (4.9) we can see that

$$P_h G P_h (q - \tilde{q}_h) = (P_h G P_h - P_h G) (q - q'_h) + P_h (g - \tilde{g})$$

for all $q'_h \in S_h$. Application of $(P_h G P_h)^{-1}$ to both side of the equality yields that

$$q - \tilde{q}_h = (I - G_h) (q - q'_h) - (P_h G P_h)^{-1} P_h (g - \tilde{g})$$

Consequently, from (4.7) and (4.8) it follows that

$$\begin{aligned} & |||q - \tilde{q}_h|||_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq (1 + \frac{\alpha}{\beta}) |||q - q'_h|||_{H^{-1/2, -1/4}(\Sigma)} + \frac{1}{\beta} |||P_h (g - \tilde{g})|||_{H^{1/2, 1/4}(\Sigma)} \\ & \leq (1 + \frac{\alpha}{\beta}) C_1 h^{s+1/2} |||q|||_{H^s, -1/4(\Sigma)} + \frac{1}{\beta} C_2 h^{-1/2} |||g - \tilde{g}|||_{H^{0, 1/4}(\Sigma)}. \end{aligned} \quad (4.13)$$

The last inequality followed from (4.10) and (4.11).

Remarks. For constant elements we can obtain only the first inequality of (4.13). Hence it is suggested that, when constant elements are used, numerical computations must proceed so that

$|||P_h (g - \tilde{g})|||_{H^{1/2, 1/4}(\Sigma)}$ is evaluated as small as possible. In other

words, the right hand side is required to be smooth and it should be calculated with high accuracy.

Remarks. A rough estimate of the optimal choice of h may be given from Theorem 4.4 by minimization of the expression in $\{\dots\}$ with respect to h : From the relation

$$h^{s+1} = \frac{1}{2} \frac{|||g - \tilde{g}|||_{H^{0, 1/4}(\Sigma)}}{(s + \frac{1}{2}) |||q|||_{H^s, 1/4(\Sigma)}},$$

we have the guideline:

$$h_{\text{opt}} = O \left(\| \|g - \tilde{g}\| \| \right)^{\frac{1}{s+1}}_{H^{0,1/4}(\Sigma)} .$$

5. APPROXIMATION IN TIME

In this section, we shall consider a constructive theory in the full-discretization of the solution by Galerkin method using one-dimensional finite elements in the time variable. We shall estimate the condition number of the coefficient matrix in the linear system of equations for a time-stepping procedure. We shall obtain convergence and accuracy of the approximate solution.

Let us subdivide the interval $[0, T]$ into N small segments of equal length with nodes $t_k = t_{k-1} + \Delta t$; $k = 1, 2, \dots, N$ ($= T/\Delta t$). Let $T_{\Delta t}$ be corresponding finite element subspaces of $C([0, T])$, approximating coefficient functions $\hat{q}_j(t)$ in the expression (4.1). Let $\{\psi_k(t)\}_{k=0,1,\dots,N}$ denote the basis of $T_{\Delta t}$. From (3.6) it follows that

$$\sum_{j=1}^n \int_0^t \hat{q}_j(\tau) v_{ij}(\tau, t) d\tau = G_i(t) \quad , \quad i=1, 2, \dots, n \quad , \quad (5.1)$$

where

$$v_{ij}(\tau, t) = \int_{\Gamma} \int_{\Gamma} \phi_i(x) \phi_j(y) v(y, \tau; x, t) d\Gamma(x) d\Gamma(y) \quad (5.2)$$

$$G_i(t) = \int_{\Gamma} g(x, t) \phi_i(x) d\Gamma(x) \quad . \quad (5.3)$$

This is the linear system of Volterra integral equations of the first kind for unknowns $\hat{q}_j(t)$ with kernels $v_{ij}(\tau, t)$. Let $q_j(t)$ be the orthogonal projection of $\hat{q}_j(t)$ into $T_{\Delta t}$:

$$q_j(t) = \sum_{k=0}^m q_j^k v_k(t) \quad , \quad 0 \leq t \leq t_m \quad (5.4)$$

with coefficients q_j^k , which stand for approximate values of $\hat{q}_j(t_k)$. As an approximation, we consider the Galerkin method: Namely, we will find unknown $q_j(t)$ satisfying that

$$\int_0^{t_m} \psi_m(t) \left\{ \sum_{j=1}^n \int_0^t q_j(\tau) v_{ij}(\tau, t) d\tau \right\} dt = \int_0^{t_m} \psi_m(t) G_i(t) dt \quad (5.5)$$

for $m = 1, 2, \dots, N$. Substitution of (5.4) into these equations yields the linear system of algebraic equations for unknowns q_j^k :

$$\sum_{j=1}^n \sum_{k=0}^m q_j^k a_{ij}^k = b_i^m \quad , \quad (5.6)$$

where

$$a_{ij}^k = \int_0^{t_m} \psi_m(t) \int_0^t \psi_k(\tau) v_{ij}(\tau, t) d\tau dt \quad , \quad (5.7)$$

$$b_i^m = \int_0^{t_m} \psi_m(t) G_i(t) dt \quad . \quad (5.8)$$

Note that a_{ij}^k depends on the number m of the time step, in general.

Inductively suppose that all q_j^k ($k \leq m-1$) are known. Then, the system of equations (5.6) can be written in the form:

$$\sum_{j=1}^n q_j^m a_{ij}^m = b_i^m - \sum_{j=1}^n \sum_{k=0}^{m-1} q_j^k a_{ij}^k \quad . \quad (5.9)$$

We shall express this form using matrices and column vectors as follows:

$$[A^{(m)}] \{q^m\} = \{b^m\} - \sum_{k=0}^{m-1} [A^{(k)}] \{q^k\} \quad (5.10)$$

Note that all square matrices $[A^{(k)}]$ ($k = 0, 1, \dots, m$) are symmetric since $V_{ij} = V_{ji}$ in view of the reciprocity $v(y, \tau; x, t) = v(x, \tau; y, t)$ in (5.2).

Lemma 5.1. The matrix $[A^{(m)}]$ is symmetric, positive definite, and all eigenvalues $\lambda(A^{(m)})$ satisfy

$$\frac{\beta}{C_3} h \Delta t^2 \lambda_{\min}(B) \leq \lambda(A^{(m)}) \leq \alpha C_4 \Delta t \lambda_{\max}(B) \quad (5.11)$$

with some constant $C_3 > 0$ and $C_4 > 0$, where $[B]$ is the Gram matrix of the basis $\{\varphi_i(x)\}_{i=1,2,\dots,n}$ in L^2 -sense: $b_{ij} = (\varphi_i, \varphi_j)_{L^2(\Gamma)}$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ are smallest and largest eigenvalues of $[B]$, respectively.

Proof. The basic idea of the proof is due to Richter (1978). With real numbers ξ_i ($i = 1, 2, \dots, n$), consider the quadratic form:

$$\begin{aligned} Q(A) &= \sum_{i,j=1}^n a_{ij}^m \xi_i \xi_j \\ &= \int_0^t \psi_m(\tau) \int_0^t \psi_m(\tau) \int_{\Gamma} \int_{\Gamma} \eta_h(x) \eta_h(y) v(y, \tau; x, t) d\Gamma(x) d\Gamma(y) d\tau dt \end{aligned}$$

Here we put: $\eta_h(x) = \sum_{i=1}^n \xi_i \varphi_i(x) \in V_h$. Set $q'_h(x, t) = \psi_m(t) \eta_h(x)$. Then we have

$$Q(A) = \int_0^t (Gq'_h(\cdot, t), q'_h(\cdot, t))_{L^2(\Gamma)} dt \quad (5.12)$$

From Lemma 3.4 with $T=t_m$ it follows for any $g \in H^{1/2,1/4}(\Sigma)$ that

$$\begin{aligned} Q(A) &= ((Gq'_h, q'_h))_0 \geq \beta |||q'_h|||_{H^{-1/2,-1/4}(\Sigma)}^2 \\ &\geq \beta |((q'_h, g))_0|^2 / |||g|||_{H^{1/2,1/4}(\Sigma)}^2. \end{aligned}$$

The last inequality followed from the definition:

$$|||q'_h|||_{H^{-1/2,-1/4}(\Sigma)} = \sup_{g \neq 0} |((q'_h, g))_0| / |||g|||_{H^{1/2,1/4}(\Sigma)}.$$

Take $g(x) = \eta_h(x)$. Then it becomes

$$\begin{aligned} Q(A) &\geq \beta \left| \int_0^T (\eta_h(\cdot), \psi_m(t) \eta_h(\cdot))_{L^2(\Gamma)} dt \right|^2 / |||\eta_h|||_{H^{1/2,1/4}(\Sigma)}^2 \\ &= \beta \left| \int_0^T \psi_m(t) dt \right|^2 |||\eta_h|||_{L^2(\Gamma)}^4 / \{ T |||\eta_h|||_{H^{1/2}(\Gamma)}^2 \} \\ &\geq \frac{\beta}{C_2^2 h} \left| \int_0^T \psi_m(t) dt \right|^2 |||\eta_h|||_{L^2(\Gamma)}^2 / T. \end{aligned}$$

The last inequality followed from the inverse assumption (4.11) with $s = 1/2$, $t = 0$. This implies the positive definiteness of $[A^{(m)}]$. For the finite element base $\psi_k(t)$, there exists an integer ρ , which is independent on k , such that $\text{supp}(\psi_k) \subset [t_{k-\rho}, t_{k+\rho}]$. We have

$$\int_0^{t_m} \psi_m(t) dt = o(\Delta t), \quad \int_0^{t_m} |\psi_m(t)|^2 dt = o(\Delta t)$$

independently of m . We have also that

$$||\eta_h||_{L^2(\Gamma)}^2 = \sum_{i,j=1}^n b_{ij} \xi_i \xi_j.$$

Consequently, there exists a constant $C_3(\Sigma)$ such that

$$Q(A) \geq \frac{\beta}{C_3} h \Delta t^2 \lambda_{\min}(B) |\xi|^2. \quad (5.13)$$

On the other hand, from (5.12) it follows that

$$\begin{aligned} Q(A) &\leq |||Gq'_h|||_{H^{1/2,1/4}(\Sigma)} |||q'_h|||_{H^{-1/2,-1/4}(\Sigma)} \\ &\leq \alpha |||q'_h|||_{H^{-1/2,-1/4}(\Sigma)}^2 \leq \alpha \gamma(\Sigma)^2 |||q'_h|||_{L^2(\Sigma)}^2 \\ &= \alpha \gamma^2 \int_0^{t_m} |\psi_m(t)|^2 dt ||\eta_h||_{L^2(\Gamma)}^2 \\ &\leq \alpha \gamma^2 \Delta t C_3 ||\eta_h||_{L^2(\Gamma)}^2 \end{aligned}$$

with some constant $C_3 > 0$. The second inequality followed from Lemma 3.3.

The third inequality followed from the continuous imbedding: $H^{-1/2,-1/4}(\Sigma) \supset L^2(\Sigma)$ with the constant $\gamma > 0$. Consequently there exists a constant $C_4(\Sigma)$ such that

$$Q(A) \leq \alpha C_4 \Delta t \lambda_{\max}(B) |\xi|^2. \quad (5.14)$$

By combining (5.13) and (5.14), we can obtain (5.11).

Corollary 5.1. The condition number $\kappa(A^{(m)})$ of the coefficient matrix in the linear system of equations (5.10) satisfies

$$\kappa(A^{(m)}) \leq \frac{\alpha}{\beta} C_3 C_4 \frac{1}{h \Delta t} \kappa(B). \quad (5.15)$$

Proof. From (5.11) the assertion follows immediately, since

$$\kappa(A^{(m)}) := \frac{\lambda_{\max}(A^{(m)})}{\lambda_{\min}(A^{(m)})} \leq \frac{\alpha C_3 C_4}{\beta h \Delta t} \frac{\lambda_{\max}(B)}{\lambda_{\min}(B)}.$$

In order to obtain error estimates of fully discretized approximate solution $q_j(t)$ of (5.4), let us introduce the interpolates $q_j^I(t)$ defined by

$$q_j^I(t) = \sum_{k=0}^m \hat{q}_j(t_k) \psi_k(t), \quad 0 \leq t \leq t_m \quad (5.16)$$

and we assume the convergence property:

$$|q_j^I(t) - \hat{q}_j(t)| \leq C_\epsilon \Delta t^\sigma, \quad 0 < t < T \quad (5.17)$$

with some constants $C_\epsilon > 0$ and $\sigma > 0$.

Remarks. If linear finite element shape functions (roof functions) are used for $T_{\Delta t}$ and $\hat{q}_j(t)$ has bounded second derivative, then (5.17) holds with $\sigma = 2$; see Strang and Fix (1973, Theorem 3.1) for example.

From (5.1) we have the next semi-exact equations:

$$\int_0^{t_m} \psi_m(t) \left\{ \sum_{j=1}^n \int_0^t \hat{q}_j(\tau) v_{ij}(\tau, t) d\tau \right\} dt = \int_0^{t_m} \psi_m(t) G_i(t) dt.$$

Subtracting (5.5) from this equation we have

$$\int_0^{t_m} \psi_m(t) \left\{ \sum_{j=1}^n \int_0^t [q_j(\tau) - \hat{q}_j(\tau)] v_{ij}(\tau, t) d\tau \right\} dt = 0.$$

Hence

$$\begin{aligned}
& \int_0^{t_m} \psi_m(t) \left\{ \sum_{j=1}^n \int_0^t [q_j^I(\tau) - q_j(\tau)] v_{ij}(\tau, t) d\tau \right\} dt \\
&= \int_0^{t_m} \psi_m(t) \left\{ \sum_{j=1}^n \int_0^t [q_j^I(\tau) - q_j(\tau)] v_{ij}(\tau, t) d\tau \right\} dt .
\end{aligned}$$

Put $e_j^k = \hat{q}_j(t_k) - q_j^k$. This indicates the error committed in the time-discretization at time-space lattice point (P_j, t_k) with $P_j \in \Gamma$. Since

$$q_j^I(t) - q_j(t) = \sum_{k=0}^m e_j^k \psi_k(t) ,$$

we have

$$\sum_{j=1}^n \sum_{k=0}^m e_j^k a_{ij}^k = v_i^m \quad (5.18)$$

where

$$v_i^m = \sum_{j=1}^n \int_0^{t_m} \psi_m(t) \int_0^t \{q_j^I(\tau) - q_j(\tau)\} v_{ij}(\tau, t) d\tau dt .$$

Theorem 5.1. Let $\xi(h, \Delta t) = h / \{\Delta t \lambda \min(B)\}$. Then the maximum norm $||\{e^m\}||_\infty$ of the error column vector $\{e^m\} = (e_1^m, \dots, e_n^m)'$ satisfies

$$||\{e^m\}||_\infty \leq G_2 \Delta t^\sigma \xi \text{Exp}[G_2 T \xi] \quad (5.19)$$

with some constant $G_2 > 0$.

Proof. By using the inequalities (3.7) and (5.17), v_i^m can be estimated as follows:

$$|v_i^m| \leq C_6 \Delta t^\sigma \int_0^{t_m} |\psi_m(t)| \int_0^t \frac{d\tau}{(t-\tau)^\mu} dt \int_\Gamma |\phi_i(x)| \sum_{j=1}^n \int_\Gamma |\phi_j(y)| \frac{G_1 d\Gamma(y)}{r^{3-2\mu}} d\Gamma(x) .$$

Since finite element bases have the properties; $-1 \leq \phi_i(x) \leq 1$ and $-1 \leq \psi_k(t) \leq 1$, all integrals involved in the right hand side are convergent for $1/2 < \mu < 1$. We can see that

$$\int_0^{t_m} |\psi_m(t)| \int_0^t \frac{d\tau}{(t-\tau)^\mu} dt \leq \frac{1}{1-\mu} \int_{t_{m-p}}^{t_m} t^{1-\mu} dt \leq p \Delta t \frac{t_m^{1-\mu}}{1-\mu}$$

since $\text{supp}(\psi_m) \subset [t_{m-p}, t_m]$. Since ϕ_i has a locally compact support, that is,

$$\int_\Gamma |\phi_i(x)| d\Gamma(x) \leq C_7 h^2$$

with some constant $C_7 > 0$, independent on i , we can see that

$$\begin{aligned} & \int_\Gamma |\phi_i(x)| \sum_{j=1}^n \int_\Gamma |\phi_j(y)| \frac{G_1 d\Gamma(y)}{r^{3-2\mu}} d\Gamma(x) \\ & \leq \int_\Gamma |\phi_i(x)| d\Gamma(x) \max_{x \in \Gamma} \sum_{j=1}^n \int_\Gamma |\phi_j(y)| \frac{G_1 d\Gamma(y)}{r^{3-2\mu}} . \end{aligned}$$

From (2.3) and the fact that ϕ_j has a locally compact support, there exists a constant G_3 , depending only on Γ , such that

$$\max_{x \in \Gamma} \sum_{j=1}^n \int_\Gamma |\phi_j(y)| \frac{G_1 d\Gamma(y)}{r^{3-2\mu}} \leq G_3$$

Consequently we have the estimate:

$$||\{v^m\}||_\infty := \max_i |v_i^m| \leq C_6 C_7 G_3 p h^2 \Delta t^{\sigma+1} \frac{t_m^{1-\mu}}{1-\mu} \leq G_4 h^2 \Delta t^{\sigma+1} \quad (5.20)$$

with the constant $G_4 = C_6 C_7 G_3 p T^{1-\mu}/(1-\mu)$. Similarly, we can see from (5.7) that

$$\begin{aligned} \sum_{j=1}^n |a_{ij}^k| &\leq \int_{t_{m-p}}^{t_m} |\psi_m(t)| \int_{t_{k-p}}^{t_{k+p}} |\psi_k(\tau)| \frac{d\tau}{(t-\tau)^\mu} dt \\ &\quad \times \int_{\Gamma} |\varphi_i(x)| \sum_{j=1}^n \int_{\Gamma} |\varphi_j(y)| \frac{G_1 d\Gamma(y)}{r^{3-2\mu}} d\Gamma(x) \\ &\leq 2p^2 \Delta t^2 C_8 C_7 h^2 G_3. \end{aligned}$$

Here we used the inequality:

$$\int_{t_{k-p}}^{t_{k+p}} |\psi_k(\tau)| \frac{d\tau}{(t-\tau)^\mu} \leq 2p \Delta t C_8$$

with some constant $C_8 > 0$. Therefore, there exists a constant G_5 such that

$$||[A^{(k)}]||_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}^{(k)}| \leq G_5 h^2 \Delta t^2 \quad (5.21)$$

Now we shall show (5.19). For this purpose, we express (5.18) in the matrix form:

$$[A^{(m)}] \{e^m\} = \{v^m\} - \sum_{k=0}^{m-1} [A^{(k)}] \{e^k\}.$$

Using (5.20) and (5.21), we can see that

$$\frac{||\{e^m\}||_\infty}{||[A^{(m)}]^{-1}||_\infty} \leq ||[A^{(m)}]\{e^m\}||_\infty \leq ||\{v^m\}||_\infty + \sum_{k=0}^{m-1} ||[A^{(k)}]||_\infty ||\{e^k\}||_\infty$$

$$\leq h^2 \Delta t^2 (G_4 \Delta t^{\sigma-1} + G_5 \sum_{k=0}^{m-1} ||\{e^k\}||_\infty) .$$

From (5.11) we know that

$$||[A^{(m)}]^{-1}||_\infty \leq \frac{C_3}{\beta h \Delta t^2 \lambda_{\min}(B)} .$$

Consequently we have the recursive relation for $||\{e^m\}||_\infty$:

$$||\{e^m\}||_\infty \leq G_6 \frac{h}{\lambda_{\min}(B)} (\Delta t^{\sigma-1} + \sum_{k=0}^{m-1} ||\{e^k\}||_\infty)$$

with some constant $G_6 > 0$. Using the result in Onishi (1982, Eq.4.45), we can see that

$$||\{e^m\}||_\infty \leq (\hat{\alpha} ||\{e^0\}||_\infty + \hat{\beta}) (1 + \alpha)^{m-1}$$

where

$$\hat{\alpha} = G_6 \frac{h}{\lambda_{\min}(B)} , \quad \hat{\beta} = G_6 \frac{h \Delta t^{\sigma-1}}{\lambda_{\min}(B)} .$$

From the inequality $1+x \leq e^x$ ($x \geq 0$), we have

$$(1 + \hat{\alpha})^{m-1} \leq \text{Exp} [G_6 h m / \lambda_{\min}(B)] .$$

If we can assume that $||\{e^0\}||_\infty = 0$, we arrive at (5.19) by noting that $m \leq T/\Delta t$ and $G_2 = G_6$.

So far we have considered several solutions; the exact solution $q(x,t)$ of the equation (3.6) in $H^{-1/2,-1/4}(\Sigma)$, semi-discretized solution $q_h(x,t)$ of the form (4.1), fully discretized solution

$$q_{h,\Delta t}(x,t) = \sum_{j=1}^n q_j(t) \phi_j(x) \quad \text{in } T_{\Delta t} \times S_h \quad (5.22)$$

with $q_j(t)$ defined by (5.4), and interpolated solution

$$q_h^I(x,t) = \sum_{j=1}^n q_j^I(t) \phi_j(x) \quad \text{in } T_{\Delta t} \times S_h \quad (5.23)$$

with $q_j^I(t)$ defined by (5.16). Put $e(x,t) = q(x,t) - q_{h,\Delta t}(x,t)$. This indicates the total error of the boundary finite element solution $q_{h,\Delta t}$. We shall estimate $e(x,t)$ in $H^{-1/2,-1/4}(\Sigma)$.

Theorem 5.2. Under the assumptions (4.10), (4.11) and (5.17), the total error $e(x,t)$ is bounded by

$$|||e|||_{H^{-1/2,-1/4}(\Sigma)} \leq G_7 h^{s+1/2} |||q|||_{H^s, -1/4(\Sigma)} \quad (5.24)$$

$$+ \frac{\Delta t^{\sigma-1}}{h} \{ G_8 \Delta t \sqrt{\lambda_{\max}(B)} + G_9 h \sqrt{\frac{\kappa(B)}{\lambda_{\min}(B)}} \exp[G_2 T \xi] \}$$

for $0 \leq s \leq 1$, with some constants $G_7 > 0$ and $G_8 > 0$.

Proof. From (4.1), (5.22) and (5.23) it follows that

$$e(x,t) = \{q(x,t) - q_h(x,t)\} + \{q_h(x,t) - q_h^I(x,t)\}$$

$$\begin{aligned}
& + \{q_h^I(x, t) - q_{h, \Delta t}(x, t)\} \\
& = \{q(x, t) - q_h(x, t)\} + \sum_{j=1}^n \{\hat{q}_j(t) - q_j^I(t)\} \varphi_j(x) \\
& \quad + \sum_{j=1}^n \{q_j^I(t) - q_j(t)\} \varphi_j(x) \quad .
\end{aligned} \tag{5.25}$$

The second term on the most right hand side can be bounded as follows:

$$\begin{aligned}
& ||| \sum_{j=1}^n (\hat{q}_j - q_j^I) \varphi_j |||_{H^{-1/2, -1/4}(\Sigma)}^2 \\
& \leq \gamma^2 ||| \sum_{j=1}^n \{\hat{q}_j - q_j^I\} \varphi_j |||_{L^2(\Sigma)}^2 \\
& = \gamma^2 \sum_{i,j=1}^n \int_0^T \{\hat{q}_i(t) - q_i^I(t)\} \{\hat{q}_j(t) - q_j^I(t)\} dt (\varphi_i, \varphi_j)_{L^2(\Gamma)} \\
& \leq \gamma^2 TC_\#^2 \Delta t^{2\sigma} n \lambda_{\max}(B) \quad .
\end{aligned}$$

The last inequality followed from (5.17). The third term on the most right hand side can be bounded similarly as follows:

$$\begin{aligned}
& ||| \sum_{j=1}^n (q_j^I - q_j) \varphi_j |||_{H^{-1/2, -1/4}(\Sigma)}^2 \\
& \leq \gamma^2 ||| \sum_{j=1}^n \{q_j^I - q_j\} \varphi_j |||_{L^2(\Sigma)}^2 \\
& = \gamma^2 ||| \sum_{j=1}^n \{ \sum_{k=0}^m e_j^k \psi_k \} \varphi_j |||_{L^2(\Sigma)}^2 \quad .
\end{aligned}$$

Notice that the inequality:

$$| \sum_{k=0}^m e_j^k \psi_k(t) | \leq C_9 (\max_{0 \leq k \leq m} \| \{e^k\} \|_{\infty})$$

holds with some constant $C_9 > 0$. It follows from Theorem 5.1 that

$$\begin{aligned} & \| \sum_{j=1}^n (q_j^I - q_j) \varphi_j \|_{H^{-1/2, -1/4}(\Sigma)}^2 \\ & \leq \gamma^2 C_9^2 (\max_{0 \leq k \leq m} \| \{e^k\} \|_{\infty})^2 \| \sum_{j=1}^n \varphi_j \|_{L^2(\Sigma)}^2 \\ & \leq \gamma^2 C_9^2 G_2^2 \Delta t^{2\sigma_\xi} \text{Exp}[2G_2 T \xi] T n \lambda_{\max}(B) . \end{aligned}$$

From Theorem 4.3 and (5.25), the total error satisfies that

$$\begin{aligned} & \| e \|_{H^{-1/2, -1/4}(\Sigma)} \\ & \leq (1 + \frac{\alpha}{\beta}) C_1 h^{s+1/2} \| q \|_{H^s, -1/4(\Gamma)} \\ & + \gamma \sqrt{T \Delta t} \sigma^{-1} \sqrt{n} \{ C_6 \Delta t \sqrt{\lambda_{\max}(B)} \\ & + C_9 G_2 h \sqrt{\frac{\kappa(B)}{\lambda_{\min}(B)}} \text{Exp}[\frac{G_2 T h}{\Delta t \lambda_{\min}(B)}] \} \end{aligned}$$

with $0 \leq s \leq 1$. This completes the proof.

ACKNOWLEDGEMENTS.

The author is most grateful to Dr. M. Costabel of Technische Hochschule Darmstadt for his valuable suggestions. The present investigation started during the author's stay by Professor Dr. W. Wendland of Universität Stuttgart as a scholarship research fellow of the Alexander von Humboldt Foundation. The investigation is also supported financially in part by The Japanese Ministry of Education, Science and Culture, Grant-in-Aid for Scientific Research on Priority Area, and in part by The Central Research Institute of Fukuoka University.

REFERENCES.

Arnold, D.N. and P.J.Noon (1987): Boundary integral equations of the first kind for the heat equation. Proceedings of the 9th International Conference on Boundary Element Methods in Engineering, Stuttgart, Vol.3, pp.213-229, Springer-Verlag, Berlin.

Brebbia, C.A., J.Telles, and L.C.Wrobel (1984): "Boundary Element Techniques - Theory and Application in Engineering". Springer-Verlag, Berlin.

Costabel, M. (1987): Personal communications.

Costabel, M., K.Onishi, and W.Wendland (1987): A boundary element collocation method for the Neumann problem of the heat equation. pp.369-384 in H.W.Engl and C.W.Groetsch (Eds.): "Inverse and Ill-posed Problems", Academic Press, Boston.

Hsiao, G. and W.Wendland (1977): A finite element method for some integral equations of the first kind. Journal of Mathematical Analysis and Applications, Vol.58, pp.449-481.

Hsiao, G. and W.Wendland (1981): Super-approximation for boundary integral methods. Technical Report No.101-A, Applied Mathematics Institute, University of Delaware.

Kantorowitsch, L.W. and G.P. Akilow (1964): "Funktionalanalysis in Normierten Räumen". Akademie-Verlag, Berlin.

Krzyzanski, M. (1971): "Partial Differential Equations of Second Order". Vol.1, PWN - Polish Scientific Publishers, Warszawa.

Lions, J.L. and E. Magenes (1968): "Problèmes aux Limites Non Homogènes et Applications", Vol.2, Dunod, Paris.

Michlin, S.G. (1978): "Partielle Differentialgleichungen in der Mathematischen Physik". Akademie-Verlag, Berlin.

Nedelec, J.C. and J. Planchard (1973): Une methode variationnelle d'elements finis pour la resolution numerique d'un probleme exterieur dans R^3 . Revue Française d'Automatique, Informatique et Recherche Operationelle, R-3, pp.105-129.

Okamoto, H. (1988): Applications of the Fourier transform to the boundary element method (to appear).

Onishi, K. (1982): Application of boundary element method to heat transfer problems. Dissertation, Science University of Tokyo.

Onishi, K. (1987): Galerkin method for boundary integral equations in transient heat conduction. Proceedings of the 9th International Conference on Boundary Element Methods in Engineering, Stuttgart, Vol.3, pp.231-248, Springer-Verlag, Berlin.

Pogorzelski, W. (1966): "Integral Equation and their Applications". Vol.1, Pergamon Press, Oxford.

Richter, G.R. (1978): Numerical solution of integral equations of the first kind with nonsmooth kernels. SIAM Journal of Numerical Analysis, Vol.15, No.3, pp.511-522.

Strang, G. and G. Fix (1973): "An Analysis of the Finite Element Method".
Prentice Hall, Inc., New Jersey.

Wendland, W. (1965): Lösung der ersten und zweiten Randwertaufgaben des
Innen- und Außengebietes für die Potentialgleichung im R^3 durch
Randbelegungen. Dissertation, Technische Universität, Berlin, D83.

Wendland, W. (1982): Boundary element methods and their asymptotic
convergence. Preprint-Nr. 690, Fachbereich Mathematik, Technische Hochschule
Darmstadt.

Yang, D. (1986): Boundary finite element method for heat conduction equation.
Proceedings of the International Conference on Boundary Elements, Beijing,
pp. 219-226, Pergamon Press, Oxford.

Supplement to the References:

Arnold, D. N. and P. J. Noon (1987): Coercivity of the single layer heat
potential, preprint.

Noon, P. J. (1988): The single layer heat potential and Galerkin boundary
element methods for the heat equation. Dissertation, The University of
Maryland.