

A GENERALIZATION OF THE SIZES OF DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS TO G-FUNCTION THEORY

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This is a summary about “a generalization of the sizes of differential equations and its applications to G-function theory” [5].

Let K be an algebraic number field of a finite degree. We consider a linear differential equation:

$$(1) \quad \frac{d}{dx}y = Ay, \quad (A \in M_n(K(x))).$$

Let us define the sizes and the global radii regarding differential equation (1).
For a place v of K we put

$$\begin{cases} |p|_v := p^{-\frac{d_v}{d}} & \text{if } v \mid p \quad (p : \text{prime}), \\ |\xi|_v := |\xi|^{\frac{d_v}{d}} & \text{if } v \nmid \infty \quad (\xi \in K), \end{cases}$$

where $d = [K : \mathbb{Q}]$ and $d_v = [K_v : \mathbb{Q}_p]$.

We define a pseudo valuation on $M_{n_1, n_2}(K)$: for $M = (m_{i,j})_{j=1, \dots, n_2}^{i=1, \dots, n_1} \in M_{n_1, n_2}(K)$,

$$|M|_v := \max_{\substack{i=1, \dots, n_1 \\ j=1, \dots, n_2}} |m_{i,j}|_v.$$

For $Y_i \in M_{n_1, n_2}(K)$, we consider the Laurent series $Y = \sum_{i=-N}^{\infty} Y_i x^i \in M_{n_1, n_2}(K((x)))$ with $N \in \mathbb{N} \cup \{0\}$.

We write $\log^+ a := \log \max(1, a)$ ($a \in \mathbb{R}$). André's symbol $h_{\cdot, \cdot}(\cdot)$ in [1] is defined by

$$\begin{aligned} h_{v,0}(Y) &:= \max_{i \leq 0} \log^+ |Y_i|_v, \\ h_{v,m}(Y) &:= \frac{1}{m} \max_{i \leq m} \log^+ |Y_i|_v \quad (m \neq 0). \end{aligned}$$

Definition 2. (Cf. [1]) We define the size of $Y \in M_{n_1, n_2}(K((x)))$ as

$$\sigma(Y) := \overline{\lim}_{m \rightarrow \infty} \sum_v h_{v, m}(Y)$$

and the global radii of Y as

$$\rho(Y) := \sum_v \overline{\lim}_{m \rightarrow \infty} h_{v, m}(Y),$$

where $\sum_v$ means that v ranges over all places of K .

The following definition coincides with the one in [6] in the case of $Y \in K[[x]]$.

Definition 3. We call $Y \in M_{n_1, n_2}(K((x)))$ with $\sigma(Y) < \infty$ a matrix of G -functions.

For $f = f(x) = \sum_{i=0}^N f_i x^i \in K[x]$ and for every place v of K , the Gauss absolute value is defined by $|f|_v := \max_{i=0, \dots, N} |f_i|_v$.

For every place v with $v \nmid \infty$ and for $f, g \in K[x]$ with $g \neq 0$, the Gauss absolute value is extended to $K(x)$ by

$$\left| \frac{f}{g} \right|_v := \frac{|f|_v}{|g|_v}.$$

We also define a pseudo valuation on $M_n(K(x))$: for $M = (m_{i,j})_{i,j=1, \dots, n} \in M_n(K(x))$,

$$|M|_v := \max_{i,j=1, \dots, n} |m_{i,j}|_v.$$

Suppose that $A \in M_n(K(x))$. A sequence $\{E_i\}_{i=0,1, \dots} \subset M_n(K(x))$ is defined by

$$E_0 := I$$

and recursively for $i = 1, 2, \dots$,

$$E_{i+1} := \frac{1}{i+1} \left(\frac{d}{dx} E_i + E_i A \right).$$

For this sequence $\{E_i\}_{i=0,1, \dots} \subset M_n(K(x))$ and for every place $v \nmid \infty$, we put

$$h_{v,0}(\{E_i\}) := \log^+ |E_0|_v,$$

$$h_{v,m}(\{E_i\}) := \frac{1}{m} \max_{i \leq m} \log^+ |E_i|_v \quad (m = 1, 2, \dots).$$

Definition 4. We define the size of A as

$$\sigma(A) := \overline{\lim}_{m \rightarrow \infty} \sum_{v \nmid \infty} h_{v,m}(\{E_i\})$$

and the global radii of A as

$$\rho(A) := \sum_{v \nmid \infty} \overline{\lim}_{m \rightarrow \infty} h_{v,m}(\{E_i\}),$$

where $\sum_{v \nmid \infty}$ means that v ranges over all finite places of K .

Definition 5. We call $\frac{d}{dx} - A$ with $\sigma(A) < \infty$ *G-operator* and $\frac{d}{dx} - A$ with $\rho(A) < \infty$ *the Arithmetic type*.

According to these notations, we state known results:

Theorem 6. (Cf. [1], [2], [3]) Suppose that $A \in M_n(K(x))$ and suppose that A has at most the simple pole at $x = 0$. For a solution, y , of differential equation (1), let y belong to $K[[x]]$ and its entries be linear independent over $K(x)$. Then the following five assertions are equivalent:

- (6.1) $\sigma(y) < \infty$,
- (6.2) $\sigma(A) < \infty$,
- (6.3) $\sigma(A^*) < \infty$,
- (6.4) $\rho(A) < \infty$,
- (6.5) $\rho(A^*) < \infty$

where $A^* = -{}^tA$. Moreover they imply

- (6.6) $\rho(y) < \infty$.

Theorem 6 is the main theorem in [1]. Before stating André results, we need a definition.

After a transformation of differential equation (1), there exists the unique matrix solution of differential equation (1), Yx^C with $Y \in Gl_n(K[[x]])$, $Y|_{x=0} = I$, where C is the residue of A at $x = 0$. This $Y \in Gl_n(K[[x]])$ is called *the normalized uniform part of the solution* of differential equation (1).

He proved Theorem 6 by using the following:

Theorem 7. (Cf. [1]) Suppose that $A \in M_n(K(x))$ and suppose that A has at most the simple pole at $x = 0$. let $Y \in Gl_n(K[[x]])$ be the normalized uniform part of differential equation (1). Let differential equation (1) be Fuchsian and let all eigenvalues of the residue matrix of A at $x = 0$ be rational numbers. Then

- (7.1) $\sigma(A) < \infty$ if and only if $\rho(A) < \infty$,
- (7.2) $\rho(A) < \infty$ implies $\rho(Y) < \infty$,
- (7.3) $\rho(Y) < \infty$ implies $\sigma(Y) < \infty$.

i.e.,

$$\sigma(A) < \infty \text{ implies } \sigma(Y) < \infty.$$

Now for a differential equation

$$(8) \quad \frac{d}{dx}X = AX - XB, \quad (A, B \in M_n(K(x))),$$

we introduce its new size $\sigma(A, B)$ of differential equation (8).

Let us define another sequence $\{F_i\}_{i=0,1,\dots} \subset M_n(K(x))$ as

$$F_0 := I$$

and recursively for $i = 1, 2, \dots$,

$$F_{i+1} := \frac{1}{i+1} \left(\frac{d}{dx} F_i - A F_i + F_i B \right).$$

Definition 9. We define the size of A and B as

$$\sigma(A, B) := \overline{\lim}_{m \rightarrow \infty} \sum_{v \nmid \infty} h_{v,m}(\{F_i\})$$

and the global radii of A and B as

$$\rho(A, B) := \sum_{v \nmid \infty} \overline{\lim}_{m \rightarrow \infty} h_{v,m}(\{F_i\}).$$

Namely $\sigma(A) = \sigma(0, A)$.

This size $\sigma(A, B)$ has the following properties:

Theorem 10. (Cf. [5]) For any $A, B, C \in M_n(K(x))$ and any $T \in Gl_n(K(x))$, the followings hold:

$$(10.1) \quad \sigma(A, A) = 0,$$

$$(10.2) \quad \sigma(A, B) = \sigma(T[A], T[B]),$$

$$(10.3) \quad \sigma(A, B) \leq \sigma(A, C) + \sigma(C, B).$$

Here $T[A] = TAT^{-1} + (\frac{d}{dx}T)T^{-1}$.

An application of Theorem 10 as the converse proposition of Theorem 7 is following:

Theorem 11. (Cf. [5]) Let $A \in M_n(K(x))$ and let Y be the normalized uniform part of the solution of differential equation (1). Let $u \in \mathcal{O}_K[x]$ be a common denominator of A , where $\mathcal{O}_K$ denotes the integer ring of K . Let $s := \max(\deg u, \deg(uA))$. Suppose that

$$\mathcal{E} := \{\text{Eigenvalues of the residue of } A\} \subset \mathbb{Q}.$$

Then

$$(11.1) \quad \begin{aligned} \sigma(A) \leq & 9n^4(s+1)\sigma(Y) + 3 \log N_{\mathcal{E}} + 3 \sum_{\substack{p|N_{\mathcal{E}} \\ p:\text{prime}}} \frac{\log p}{p-1} \\ & + (s+1)h_{\infty}(u) + \log(s+1) + 3(n-1), \end{aligned}$$

where $h_{\infty}(u) := \frac{1}{m+1} \sum_{v|\infty} \max_{i \leq m} \log^+ |u_i|_v$ and $N_{\mathcal{E}} \in \mathbb{N}$ is a common denominator of $\mathcal{E}$. i.e.,

$$\sigma(Y) < \infty \text{ implies } \sigma(A) < \infty.$$

Remark 12. The same result on the finiteness by another method was published [4].

From Theorem 7, Theorem 11 and the uniqueness of the normalized uniform part, we summarize them as follows:

Theorem 13. *Under the assumptions of Theorem 7, the following eight assertions are equivalent:*

- (13.1) $\sigma(Y) < \infty,$
- (13.2) $\sigma(A) < \infty,$
- (13.3) $\rho(Y) < \infty,$
- (13.4) $\rho(A) < \infty,$
- (13.5) $\sigma(Y^{-1}) < \infty,$
- (13.6) $\sigma(A^*) < \infty,$
- (13.7) $\rho(Y^{-1}) < \infty,$
- (13.8) $\rho(A^*) < \infty,$

where $A^* = -{}^tA$. More precisely

- (13.9) $\sigma(A) = \sigma(A^*),$
- (13.10) $\rho(A) = \rho(A^*).$

Remark 14. Equation (13.10) is derived using a different method in [1].

REFERENCES

- [1] Y. André, *G-functions and Geometry*, Max-Planck-Institut., Bonn, 1989.
- [2] E. Bombieri, *On G-functions*, Recent progress in analytic number theory **2** (1981), Academic Press, New York, 1 – 67.
- [3] D. V. Chudnovsky, G. V. Chudnovsky, *Applications of Padé approximations to diophantine inequalities in values of G-functions*, Lect. Notes in Math. **1135** (1985), Springer-Verlag, Berlin, Heidelberg, New York, 9 – 51.
- [4] B. Dwork, G. Gerotto, F. J. Sullivan, *An Introduction to G-functions*, Annals of Math. Studies **133** (1994), Princeton University Press, Princeton, New Jersey.
- [5] M. Nagata, *A generalization of the sizes of differential equations and its applications to G-function theory* (1994), Preprint series in Math. (Tokyo Inst. Tech.).
- [6] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss., Phys. Math. Kl. nr.1 (1929).

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