

Strongly starlikeness criteria for certain analytic functions

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Abstract

Let $\mathcal{U}_3(\lambda)$ be the subclass of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ which was introduced by S. Ponnusamy (Appl. Math. Lett. 24(2011), 381 - 385). For $f(z) \in \mathcal{U}_3(\lambda)$, some condition for the domain of $|z|$ such that $f(z)$ is strongly starlike of order γ in $\mathbb{U}$.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ which are univalent in $\mathbb{U}$.

Obradović and Ponnusamy [2] define the class $\mathcal{U}(\lambda)$ of $f(z) \in \mathcal{A}$ satisfying the condition

$$(1.2) \quad \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda \quad (z \in \mathbb{U})$$

for some real $\lambda > 0$.

The condition (1.2) is equivalent to

$$\left| z^2 \left(\frac{1}{f(z)} - \frac{1}{z} \right)' \right| < \lambda \quad (z \in \mathbb{U}).$$

Ponnusamy [3] introduces the class $\mathcal{U}_3(\lambda)$ of function $f(z) \in \mathcal{U}(\lambda)$ for which $a_3 - a_2^2 = 0$.

For some real $\gamma \in (0, 1]$, a function $f(z) \in \mathcal{A}$ is called strongly starlike of order γ if

$$(1.3) \quad \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi\gamma}{2} \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{SS}(\gamma)$ the set of all strongly starlike functions of order γ in $\mathbb{U}$.

2010 *Mathematics Subject Classification*: Primary 30C45.

Key Words and Phrases: Analytic function, Strongly starlike of order γ , Schwarz lemma.

Ponnusamy [3] has shown the following theorem.

Theorem 1.1 *Let $f(z) \in \mathcal{U}_3(\lambda)$, $\gamma \in (0, 1]$, and*

$$\lambda_*(\gamma, |a_2|) = \frac{-2(1 + 2\cos\frac{\pi\gamma}{2})|a_2| + 2\sin\frac{\pi\gamma}{2}\sqrt{5 + 4\cos\frac{\pi\gamma}{2} - 4|a_2|^2}}{5 + 4\cos\frac{\pi\gamma}{2}}.$$

Then $f(z) \in \mathcal{SS}(\gamma)$ for $0 < \lambda \leq \lambda_(\gamma, |a_2|)$.*

The aim of this paper is to derive a condition for the domain of $f(z) \in \mathcal{U}_3(\lambda)$ to be in the class $\mathcal{SS}(\gamma)$.

2 Main Result

Suppose that $f \in \mathcal{U}_3(\lambda)$. Then a simple calculation shows that

$$\begin{aligned} (2.1) \quad & -z \left(\frac{z}{f(z)} \right)' + \left(\frac{z}{f(z)} \right) = \left(\frac{z}{f(z)} \right)^2 f'(z) \\ & = 1 + A_3 z^3 + \cdots = 1 + \lambda w(z), \quad w(z) \in \mathcal{B}_3, \end{aligned}$$

where $\mathcal{B}_3$ denotes the set of all analytic functions $w(z)$ in $\mathbb{U}$ such that $w(0) = w(0)' = w(0)'' = 0, w'''(0) \neq 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. From (2.1), we easily have the following representation for $\frac{z}{f(z)}$:

$$(2.2) \quad \frac{z}{f(z)} - 1 = -a_2 z - \lambda \int_0^1 \frac{w(tz)}{t^2} dt.$$

Since $w(z) \in \mathcal{B}_3$, from the Schwarz lemma

$$(2.3) \quad |w(z)| \leq |z|^3$$

holds true. Thus, we have that

$$(2.4) \quad \left| \frac{z}{f(z)} - 1 \right| \leq |z| \left(|a_2| + \frac{\lambda}{2} |z|^2 \right), \quad z \in \mathbb{U}.$$

We have the following result.

Theorem 2.1

If $f(z) \in \mathcal{U}_3(\lambda)$, then $f(z) \in \mathcal{SS}(\gamma)$ for $|z| < \min\{1, r_0\}$, where $r_0 = \sqrt{R_0}$ for the positive root R_0 of the equation

$$(2.5) \quad \lambda^2 \left(\frac{5}{4} + \cos\frac{\pi\gamma}{2} \right) X^3 + \lambda |a_2| \left(1 + 2\cos\frac{\pi\gamma}{2} \right) X^2 + |a_2|^2 X + \cos^2\frac{\pi\gamma}{2} - 1 = 0.$$

Proof

Suppose that $f(z) \in \mathcal{U}_3(\lambda)$. Then we can see from (2.1) and (2.3) that

$$(2.6) \quad \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| = \lambda |w(z)| \leq \lambda |z|^3.$$

Therefore, it follows from (2.4) and (2.6) that

$$(2.7) \quad \begin{aligned} \left| \arg \frac{zf'(z)}{f(z)} \right| &\leq \left| \arg \left(\left(\frac{z}{f(z)} \right)^2 f'(z) \right) \right| + \left| \arg \frac{z}{f(z)} \right| \\ &\leq \arcsin(\lambda |z|^3) + \arcsin \left(|z| \left(|a_2| + \frac{\lambda}{2} |z|^2 \right) \right) \\ &= \arcsin \left(\lambda |z|^3 \sqrt{1 - |z|^2 \left(|a_2| + \frac{\lambda}{2} |z|^2 \right)^2} + \sqrt{1 - \lambda^2 |z|^6} |z| \left(|a_2| + \frac{\lambda}{2} |z|^2 \right) \right). \end{aligned}$$

Now, we have to find the range of $|z|$ for $f(z) \in \mathcal{SS}(\gamma)$ such that

$$(2.8) \quad \arcsin \left(\lambda |z|^3 \sqrt{1 - |z|^2 \left(|a_2| + \frac{\lambda}{2} |z|^2 \right)^2} + \sqrt{1 - \lambda^2 |z|^6} |z| \left(|a_2| + \frac{\lambda}{2} |z|^2 \right) \right) < \frac{\pi\gamma}{2},$$

which is equivalent to

$$(2.9) \quad \lambda |z|^3 \sqrt{1 - |z|^2 \left(|a_2| + \frac{\lambda}{2} |z|^2 \right)^2} + \sqrt{1 - \lambda^2 |z|^6} |z| \left(|a_2| + \frac{\lambda}{2} |z|^2 \right) < \sin \frac{\pi\gamma}{2}.$$

Putting

$$(2.10) \quad F(X) = \lambda^2 \left(\frac{5}{4} + \cos \frac{\pi\gamma}{2} \right) X^3 + \lambda |a_2| \left(1 + 2 \cos \frac{\pi\gamma}{2} \right) X^2 + |a_2|^2 X - \sin^2 \frac{\pi\gamma}{2}$$

and

$$(2.11) \quad G(X) = \lambda^2 \left(\frac{5}{4} - \cos \frac{\pi\gamma}{2} \right) X^3 + \lambda |a_2| \left(1 - 2 \cos \frac{\pi\gamma}{2} \right) X^2 + |a_2|^2 X - \sin^2 \frac{\pi\gamma}{2},$$

(2.9) can be written as $F(X)G(X) > 0$ with $X = |z|^2$. Since $F(0) < 0$ and $G(0) < 0$, $F(X)G(X) > 0$ is equivalent to $F(X) < 0$ and $G(X) < 0$. Comparing the coefficients of $F(X)$ and $G(X)$, we easily find the inequality $G(X) < F(X)$.

In order to find the condition of $|z|$ such that $f(z) \in \mathcal{U}_3(\lambda)$ to be in $\mathcal{SS}(\gamma)$, we consider the condition for $F(X) < 0$.

Since

$$F'(X) = 3\lambda^2 \left(\frac{5}{4} + \cos \frac{\pi\gamma}{2} \right) X^2 + 2\lambda |a_2| \left(1 + 2 \cos \frac{\pi\gamma}{2} \right) X + |a_2|^2 > 0$$

and $\lim_{X \rightarrow \infty} F'(X) = \infty$,

$F(X)$ is an increasing function for X . Thus $F(X)$ has a positive root $R_0 > 0$. Therefore, for $|z| < \min\{1, R_0\}$, inequality (2.9) holds.

Remark

Substituting $X = 1$ and solving the equation (2.5) as the equation of λ , we have $\lambda_*(\gamma, |a_2|)$ of Theorem 1.1.

3 Example

We give an example which shows the existence of r_0 satisfying Theorem 2.1. Since $F(X)$ has a unique solution for $0 < X < 1$ if $F(1) > 0$, we consider a condition of $|a_2|$ for $F(1) > 0$. From

$$F(1) = \lambda^2 \left(\frac{5}{4} + \cos \frac{\pi\gamma}{2} \right) + \lambda |a_2| \left(1 + 2 \cos \frac{\pi\gamma}{2} \right) + |a_2|^2 - \sin^2 \frac{\pi\gamma}{2} > 0,$$

we have

$$\left(|a_2| + \frac{\lambda (1 + 2 \cos \frac{\pi\gamma}{2})}{2} \right)^2 > \sin^2 \frac{\pi\gamma}{2} (1 - \lambda^2),$$

This gives us that

$$(3.1) \quad |a_2| > \sin \frac{\pi\gamma}{2} \sqrt{1 - \lambda^2} - \frac{\lambda (1 + 2 \cos \frac{\pi\gamma}{2})}{2} \quad (0 < \lambda < 1).$$

If $|a_2|$ satisfies the condition (3.1), $F(X)$ has a positive root for $0 < X < 1$.

Let us take $\lambda = \frac{1}{2}$ and $\gamma = \frac{2}{3}$. Then $|a_2| > \frac{1}{4}$ from (3.1). Thus, we may take $|a_2| = 1$ and

$$(3.2) \quad F(X) = 7X^3 + 16X^2 + 16X - 12.$$

Since $F(0) = -12 < 0$, and $F(1) = 27 > 0$, $F(X)$ has a real positive root $0 < X < 1$. Actually, the root r_0 of (3.2) satisfies $0.47605 < r_0 < 0.47615$.

References

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