

AN INVITATION TO THE SIMILARITY PROBLEMS (AFTER PISIER)

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ABSTRACT. This note is intended as a handout for the minicourse given in RIMS workshop “Operator Space Theory and its Applications” on January 31, 2006.

1. THE SIMILARITY PROBLEMS

1.1. The similarity problem for continuous homomorphisms. In this note, we mainly consider *unital* C^* -algebras and *unital* (not necessarily $*$ -preserving) homomorphisms for the sake of simplicity. Let A be a unital C^* -algebra and $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a unital homomorphism with $\|\pi\| < \infty$. We say that π is *similar* to a $*$ -homomorphism if there exists $S \in \text{GL}(\mathcal{H})$ such that $\text{Ad}(S) \circ \pi$ is a $*$ -homomorphism. Here, $\text{GL}(\mathcal{H})$ is the set of invertible element in $\mathbb{B}(\mathcal{H})$ and $\text{Ad}(S)(x) = SxS^{-1}$.

Similarity Problem A (Kadison 1955). Is every continuous homomorphism similar to a $*$ -homomorphism?

We note that a homomorphism π is a $*$ -homomorphism iff $\|\pi\| = 1$, since an element $x \in \mathbb{B}(\mathcal{H})$ is unitary iff $\|x\| = \|x^{-1}\| = 1$. We say A has the *similarity property* (abbreviated as (SP)) if every unital continuous homomorphism from A into $\mathbb{B}(\mathcal{H})$ is similar to a $*$ -homomorphism. Do we really need the assumption that π is continuous? That is another problem. Indeed, the subject of automatic continuity is extensively studied in Banach algebra theory, and it is known that the existence of a *discontinuous* homomorphism from a C^* -algebra into some Banach algebra is independent of (ZFC). As far as the author knows, it is not known whether or not the automatic continuity of a homomorphism between C^* -algebras (say, with a dense image) is provable within (ZFC).

Similarity Problem A is equivalent to several long-standing problems in C^* , von Neumann and operator theories. Among them is the Derivation Problem;

Derivation Problem. Is every derivation $\delta: A \rightarrow \mathbb{B}(\mathcal{H})$ inner?

Let $A \subset \mathbb{B}(\mathcal{H})$ be a (unital) C^* -algebra. A *derivation* $\delta: A \rightarrow \mathbb{B}(\mathcal{H})$ is a linear map which satisfies the derivative identity $\delta(ab) = \delta(a)b + a\delta(b)$. The celebrated theorem of Kadison and Sakai is that every derivation into A'' is inner. We recall

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that $\delta: A \rightarrow \mathbb{B}(\mathcal{H})$ is said to be *inner* if there exists $T \in \mathbb{B}(\mathcal{H})$ such that

$$\forall a \in A \quad \delta(a) = \delta_T(a) := Ta - aT.$$

It is known that every derivation is automatically continuous (Ringrose). We say A has the (DP) if any derivation $\delta: A \rightarrow \mathbb{B}(\mathcal{H})$, for any faithful $*$ -representation $A \subset \mathbb{B}(\mathcal{H})$, is inner.

Theorem 1.1 (Kirchberg 1996). *Let A be a unital C^* -algebra. Then A has the (SP) iff A has the (DP).*

The easier implication (SP) $\Rightarrow$ (DP) (which precedes Kirchberg) follows from the following lemma.

Lemma 1.2. *Let $A \subset \mathbb{B}(\mathcal{H})$ be a unital C^* -algebra and $\delta: A \rightarrow \mathbb{B}(\mathcal{H})$ be a derivation. Then the homomorphism $\pi: A \rightarrow \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ defined by*

$$\pi(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}$$

is similar to a $$ -homomorphism iff δ is inner.*

Proof. We first observe that π is indeed a homomorphism since δ is a derivation. If $\delta = \delta_T$, then we have

$$\pi(a) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & -T \\ 0 & 1 \end{pmatrix}$$

and π is similar to a $*$ -homomorphism $\text{id}_A \oplus \text{id}_A$. We now suppose that $\sigma(a) = S\pi(a)S^{-1}$ is a $*$ -homomorphism. Let $D = S^*S$. Since

$$\|S^{-1}\|^2 \langle D\xi, \xi \rangle = \|S^{-1}\|^2 \|S\xi\|^2 \geq \|\xi\|^2,$$

we have $D \geq \|S^{-1}\|^{-2}$. Since σ is $*$ -preserving, we have

$$D\pi(a) = S^*\sigma(a)S = (S^*\sigma(a^*)S)^* = \pi(a^*)^*D$$

for every $a \in A$. Developing the equation, we get

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ \delta(a^*)^* & a \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$$

Looking at the $(1, 1)$ -entry, we have $D_{11}a = aD_{11}$ for every $a \in A$. Combined with $D_{11} \geq \|S^{-1}\|^{-2}$, this implies that $D_{11}^{-1} \in A'$ with $\|D_{11}^{-1}\| \leq \|S^{-1}\|^2$. Looking at the $(2, 1)$ -entry, we have

$$D_{11}\delta(a) + D_{12}a = aD_{12}.$$

It follows that $\delta = \delta_T$ for $T = -D_{11}^{-1}D_{12}$ with $\|T\| \leq \|S\|^2\|S^{-1}\|^2$. $\square$

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1.2. Known cases and open cases. The important result of Haagerup (1983) is that a continuous homomorphism $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ admitting a finite cyclic subset (i.e., there exists a finite subset $\mathcal{F} \subset \mathcal{H}$ such that $\text{span}\{\pi(a)\xi : a \in A, \xi \in \mathcal{F}\}$ is dense in $\mathcal{H}$), is inner. This does not finish the similarity problem since we cannot decompose a general (non $*$ -preserving) representation into a direct sum of cyclic representations.

Theorem 1.3. *The following C^* -algebras have the (SP).*

- (1) *Nuclear C^* -algebras.*
- (2) *C^* -algebras without tracial states (Haagerup).*
- (3) *Type II_1 factors with the property (Γ) (Christensen).*

We note that one may reduce Similarity problem A (or derivation problem) for C^* -algebras to that for type II_1 factors by considering the second dual, then considering the type decomposition and direct integration. We do not know whether or not the von Neumann algebras $\mathcal{LF}_2$ and $\prod_{n=1}^{\infty} \mathbb{M}_n$ have the (SP). We suspect that $\prod_{n=1}^{\infty} \mathbb{M}_n$ should be a counterexample.

1.3. The similarity problem for group representations. We only consider discrete groups. Let Γ be a discrete group and $C^*\Gamma$ be the full group C^* -algebra. We regard Γ as the corresponding subgroup of unitary elements in $C^*\Gamma$. Every continuous homomorphism $\pi: C^*\Gamma \rightarrow \mathbb{B}(\mathcal{H})$ gives rise to a uniformly bounded (abbreviated as u.b.) representation of Γ on $\mathcal{H}$; $\pi: \Gamma \rightarrow \text{GL}(\mathcal{H})$ is a group homomorphism such that $\|\pi\| := \sup_{s \in \Gamma} \|\pi(s)\| < \infty$ ¹. Obviously, the homomorphism $\pi: C^*\Gamma \rightarrow \mathbb{B}(\mathcal{H})$ is similar to a $*$ -homomorphism iff the representation $\pi|_{\Gamma}$ is unitarizable (i.e., $\exists S \in \text{GL}(\mathcal{H})$ such that $\text{Ad}(S) \circ \pi|_{\Gamma}$ is a unitary representation).

Theorem 1.4 (Dixmier 1950). *Let Γ be an amenable group. Then, every u.b. representation of Γ is unitarizable. More precisely, if $\pi: \Gamma \rightarrow \text{GL}(\mathcal{H})$ is a u.b. representation, then there exists $S \in \text{GL}(\mathcal{H}) \cap \text{vN}(\pi(\Gamma))$ with $\|S\| \|S^{-1}\| \leq \|\pi\|^2$ such that $\text{Ad}(S) \circ \pi$ is unitary.*

Proof. Let Γ be amenable and $\pi: \Gamma \rightarrow \text{GL}(\mathcal{H})$ be a u.b. representation. Let $F_n \subset \Gamma$ be a Følner net. Since π is u.b., the set $|F_n|^{-1} \sum_{s \in F_n} \pi(s)^* \pi(s) \in \text{vN}(\pi(\Gamma))$ has a weak*-accumulation point. Since the accumulation point is positive, we let S be the the square root of it. Then, we have

$$\|S\xi\|^2 = \lim_n \frac{1}{|F_n|} \sum_{s \in F_n} \|\pi(s)\xi\|^2,$$

and hence $\|\pi\|^{-1} \leq S \leq \|\pi\|$ and $\|S\pi(s)\xi\| = \|S\xi\|$ for every $s \in \Gamma$ and $\xi \in \mathcal{H}$. It follows that $\|\text{Ad}(S) \circ \pi\| = 1$ and hence $\text{Ad}(S) \circ \pi$ is unitary. $\square$

¹This notation may cause confusion since the value $\|\pi\|$ is not same as $\|\pi: C^*\Gamma \rightarrow \mathbb{B}(\mathcal{H})\|$.

If one employ the fact that a nuclear C^* -algebra is amenable as a Banach algebra (Haagerup 1983), then we can adopt the above proof to the case of nuclear C^* -algebras. We say Γ is *unitarizable* if every u.b. representation of Γ is unitarizable. Pisier (2004, 2005) proved that if Γ is unitarizable and in addition that the similarity S can be chosen so that (i) $S \in \text{GL}(\mathcal{H}) \cap \text{vN}(\pi(\Gamma))$, or (ii) $\|S\| \|S^{-1}\| \leq \|\pi\|^2$, then Γ is amenable. However, the following is still open.

Similarity Problem B. Is every unitarizable group amenable?

Theorem 1.5. *The free group $\mathbb{F}_\infty$ on countably many generators is not unitarizable.*

Proof. We denote by $|t|$ the word length of $t \in \mathbb{F}_\infty$, by $\mathbb{CF}_\infty$ the space of all finitely supported $\mathbb{C}$ -valued functions on $\mathbb{F}_\infty$, and by $\lambda(s)$ the left translation operator by s on $\ell_\infty \Gamma$ (and its subspaces). Let $B: \mathbb{CF}_\infty \rightarrow \ell_\infty \mathbb{F}_\infty$ be the linear map defined by

$$B\delta_t = \sum \{\delta_{t'} : |t^{-1}t'| = 1, |t'| = |t| + 1\},$$

i.e., $B\delta_t$ is the characteristic function of those points which are just one-step ahead of t (looking from e). Then, for every $s \in \mathbb{F}_\infty$, we have

$$(B\lambda(s) - \lambda(s)B)\delta_t = \begin{cases} 0 & \text{if } |s| \neq |st| + |t^{-1}| \\ \delta_{s(|st|+1)} - \delta_{s(|st|-1)} & \text{if } |s| = |st| + |t^{-1}| \end{cases},$$

where $s(k)$ is the unique element such that $|s(k)| = k$ and $|s| = |s(k)| + |s(k)^{-1}s|$. Hopefully, the figures below explain the above equation. It follows that we may

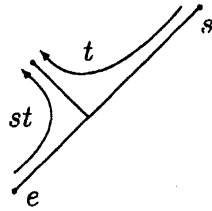


FIGURE 1. $|s| \neq |st| + |t^{-1}|$

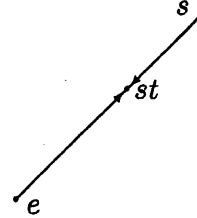


FIGURE 2. $|s| = |st| + |t^{-1}|$

view $D(s) = B\lambda(s) - \lambda(s)B$ as an element in $\mathbb{B}(\ell_2 \mathbb{F}_\infty)$ with $\|D(s)\| = 2$. Thus, $D: \mathbb{F}_\infty \rightarrow \mathbb{B}(\ell_2 \mathbb{F}_\infty)$ is a u.b. derivation; $D(st) = D(s)\lambda(t) + \lambda(s)D(t)$. It is not hard to show that D is not inner, i.e., there is no $B_0 \in \mathbb{B}(\ell_2 \mathbb{F}_\infty)$ such that $B - B_0$ commutes with every $\lambda(s)$ (in $L(\mathbb{CF}_\infty, \ell_\infty \mathbb{F}_\infty)$). We define a u.b. representation $\pi: \mathbb{F}_\infty \rightarrow \mathbb{M}_2(\mathbb{B}(\ell_2 \mathbb{F}_\infty))$ by

$$\pi(s) = \begin{pmatrix} \lambda(s) & D(s) \\ 0 & \lambda(s) \end{pmatrix}.$$

We conclude the proof by using the fact, which is proved in the same way to Lemma 1.2, that π is similar to $*$ -homomorphism only if D is inner. $\square$

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We observe that a subgroup of a unitarizable group is again unitarizable thanks to the fact that the induction of a u.b. representation is again u.b. (and a little more effort). Hence a counterexample (if any) to Similarity Problem B has to be a non-amenable group which does not contain $\mathbb{F}_2$ as a subgroup. Do you think this might be a good time to stop chasing the problem?

2. ISOMORPHIC CHARACTERIZATION OF INJECTIVITY

2.1. A free Khinchine inequality. Let Γ be a discrete group and $\mathcal{L}\Gamma$ be its group von Neumann algebra. By definition, the map

$$\mathcal{L}\Gamma \ni \lambda(f) \mapsto f = \lambda(f)\delta_e \in \ell_2\Gamma$$

is contractive. For which operator space structure on $\ell_2\Gamma$, does the above map completely bounded? We briefly review the column and row Hilbert space structures. Let $\mathcal{H}$ be a Hilbert space. When it is viewed as a column vector space, we say it is a column Hilbert space and denote it by $\mathcal{H}_C$, i.e., $\mathcal{H}_C = \mathbb{B}(\mathbb{C}, \mathcal{H})$ as an operator space. For any finite sequence² $(x_i)_i$ in $\mathbb{B}(H)$ and orthonormal vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$\|(x_i)_i\|_C := \left\| \sum_i x_i \otimes \xi_i \right\|_{\mathbb{B}(H) \otimes \mathcal{H}_C} = \left\| \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \right\| = \left\| \sum_i x_i^* x_i \right\|^{1/2}.$$

Likewise, we define the row Hilbert space as $\mathcal{H}_R = \mathbb{B}(\overline{\mathcal{H}}, \mathbb{C})$, where $\overline{\mathcal{H}}$ is the conjugate Hilbert space of $\mathcal{H}$. For any finite sequence (x_i) in $\mathbb{B}(H)$ and orthonormal vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$, we have

$$\|(x_i)_i\|_R := \left\| \sum_i x_i \otimes \xi_i \right\|_{\mathbb{B}(H) \otimes \mathcal{H}_R} = \left\| (x_1 \ x_2 \ \cdots) \right\| = \left\| \sum_i x_i x_i^* \right\|^{1/2}.$$

We regard the following lemma trivial and use it without referring it.

Lemma 2.1. *For any finite sequences $(a_i)_i$ and $(b_i)_i$ in $\mathbb{B}(\mathcal{H})$, we have*

$$\left\| \sum_i a_i b_i \right\| \leq \|(a_i)_i\|_R \|(b_i)_i\|_C.$$

In particular, $\left\| \sum_i a_i \otimes b_i \right\| \leq \min\{ \|(a_i)_i\|_R \|(b_i)_i\|_C, \|(a_i)_i\|_C \|(b_i)_i\|_R \}.$

We define $\mathcal{H}_{C \cap R} = \{ \xi \oplus \xi \in \mathcal{H}_C \oplus \mathcal{H}_R : \xi \in \mathcal{H} \}.$

Proposition 2.2. *The map*

$$\mathcal{L}\Gamma \ni \lambda(f) \mapsto f \in (\ell_2\Gamma)_{C \cap R}$$

is completely contractive.

²A finite sequence is a sequence of vectors such that all but finitely many are zero

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Proof. We view $\delta_e \in \mathbb{B}(\mathbb{C}, \ell_2 \Gamma)$ and $\delta_e^* \in \mathbb{B}(\overline{\ell_2 \Gamma}, \mathbb{C})$. Since $f = \lambda(f)\delta_e \in \mathbb{B}(\mathbb{C}, \ell_2 \Gamma)$, the above map is a complete contraction into $\mathcal{H}_C$. Since $f = \delta_e^* \overline{\lambda}(f) \in \mathbb{B}(\overline{\ell_2 \Gamma}, \mathbb{C})$, the above map is a complete contraction into $\mathcal{H}_R$ as well. $\square$

We simply write $C \cap R$ for $(\ell_2)_{C \cap R}$ and $\{\theta_i\}$ for a fixed orthonormal basis for $C \cap R$. For instance, we can take $\theta_i = e_{i1} \oplus e_{1i} \in \mathbb{B}(\ell_2) \oplus \mathbb{B}(\ell_2)$. For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{C \cap R} = \left\| \sum_i x_i \otimes \theta_i \right\|_{\mathbb{B}(\mathcal{H}) \otimes (C \cap R)} = \max\{\|(x_i)_i\|_C, \|(x_i)_i\|_R\}.$$

The following is the rudiment of free Khinchine inequalities.

Theorem 2.3 (Haagerup and Pisier 1993). *Let $\mathbb{F}_\infty$ be the free group on countable generators, $\mathcal{S} = \{s_i\} \subset \mathbb{F}_\infty$ be the standard set of free generators and*

$$E_\lambda = \overline{\text{span}}\{s_i\} \subset \mathcal{LF}_\infty$$

be an operator subspace. Then, the map

$$\Phi: C \cap R \ni \theta_i \mapsto \lambda(s_i) \in \mathcal{LF}_\infty$$

is completely bounded with $\|\Phi\|_{\text{cb}} \leq 2$. In particular, the projection Q from $\mathcal{LF}_\infty$ onto E_λ , defined by

$$Q: \mathcal{LF}_\infty \ni \lambda(s) \mapsto \begin{cases} \lambda(s) & \text{if } s \in \mathcal{S} \\ 0 & \text{if } s \notin \mathcal{S} \end{cases},$$

is completely bounded with $\|Q\|_{\text{cb}} \leq 2$.

Proof. For each i , let $\Omega_i^\pm \subset \mathbb{F}_\infty$ be the subsets of all reduced words which begins with respectively $s_i^{\pm 1}$, and $P_i^\pm \in \mathbb{B}(\ell_2 \mathbb{F}_\infty)$ be the orthogonal projection onto $\ell_2 \Omega_i^\pm$. Then, for each i , we have

$$\lambda(s_i) = \lambda(s_i)P_i^- + \lambda(s_i)(1 - P_i^-) = \lambda(s_i)P_i^- + P_i^+ \lambda(s_i).$$

Therefore for any finite sequence $(x_i)_i \subset \mathbb{B}(H)$, we have

$$\left\| \sum_i x_i \otimes \lambda(s_i)P_i^- \right\|_{\mathbb{B}(H \otimes \ell_2 \mathbb{F}_\infty)} \leq \|(x_i)_i\|_R \|(\lambda(s_i)P_i^-)_i\|_C \leq \|(x_i)_i\|_R$$

since $\|(\lambda(s_i)P_i^-)_i\|_C = \|\sum_i P_i^-\|^{1/2} = 1$. Likewise, we have

$$\left\| \sum_i x_i \otimes P_i^+ \lambda(s_i) \right\|_{\mathbb{B}(H \otimes \ell_2 \mathbb{F}_\infty)} \leq \|(x_i)_i\|_C \|(P_i^+ \lambda(s_i))_i\|_R \leq \|(x_i)_i\|_C.$$

It follows that

$$\left\| \sum_i x_i \otimes \lambda(s_i) \right\|_{\mathbb{B}(H \otimes \ell_2 \mathbb{F}_\infty)} \leq 2\|(x_i)_i\|_{C \cap R} = 2\left\| \sum_i x_i \otimes \theta_i \right\|.$$

This means that $\|\Phi\|_{\text{cb}} \leq 2$. The second assertion follows from Proposition 2.2. $\square$

Remark 2.4. The above property of $\mathcal{LF}_\infty$ is related to the fact that $\mathcal{LF}_\infty$ is not injective. We simply write E_n for $(\ell_2^n)_{C \cap R}$. Thus

$$E_n = \text{span}\{e_{i1} \oplus e_{1i} : i = 1, \dots, n\} \subset M_n \oplus M_n.$$

It is known that E_n is far from injective, i.e., any projection from $M_n \oplus M_n$ onto E_n has cb-norm $\geq \frac{1}{2}(\sqrt{n} + 1)$. It follows that if M is an injective von Neumann algebra, then any maps $\alpha: E_n \rightarrow M$ and $\beta: M \rightarrow E_n$ with $\beta \circ \alpha = \text{id}_{E_n}$ satisfy $\|\alpha\|_{\text{cb}} \|\beta\|_{\text{cb}} \geq \frac{1}{2}(\sqrt{n} + 1)$. It is conjectured by Pisier(?) that for any non-injective von Neumann algebra M , there exist sequences of maps $\alpha_n: E_n \rightarrow M$ and $\beta_n: M \rightarrow E_n$ such that $\beta_n \circ \alpha_n = \text{id}_{E_n}$ and $\sup \|\alpha_n\|_{\text{cb}} \|\beta_n\|_{\text{cb}} < \infty$. An affirmative answer would solve several problems around operator spaces (e.g., whether existence of a bounded linear projection from $\mathbb{B}(\mathcal{H})$ onto M implies injectivity of M .) A negative answer would lead to a non-injective type II_1 factor which does not contain $\mathcal{LF}_2$.

2.2. Isomorphic characterization of injective von Neumann algebras. For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{C+R} = \|\Phi: C \cap R \ni \theta_i \mapsto x_i \in \mathbb{B}(\mathcal{H})\|_{\text{cb}}.$$

We say that a von Neumann algebra M has the *property (P)*³ if there exists a constant $C_M > 0$ with the following property; For any finite sequence $(x_i)_i$ in M with $\|(x_i)_i\|_{C+R} \leq 1$, there exist finite sequences $(a_i)_i$ and $(b_i)_i$ in M such that

$$\|(a_i)_i\|_C \leq C_M, \|(b_i)_i\|_R \leq C_M \text{ and } x_i = a_i + b_i \text{ for every } i.$$

Theorem 2.5 (Pisier 1994). *A von Neumann algebra M is injective iff it has the property (P).*

The “if” part requires several lemmas, and we first prove the “only if” part. Let M be an injective von Neumann algebra and consider a complete contraction $\Phi: C \cap R \ni \theta_i \mapsto x_i \in M$. Since M is injective, this map extends to a complete contraction $\tilde{\Phi}: C \oplus R \rightarrow M$, where $C = \overline{\text{span}}\{e_{i1}\}$ and $R = \overline{\text{span}}\{e_{1i}\}$. Then $a_i = \tilde{\Phi}(0 \oplus e_{1i})$ and $b_i = \tilde{\Phi}(e_{i1} \oplus 0)$ satisfies the required condition with $C_M = 1$. We note that $\|(\varphi(a_i))_i\|_C \leq \|\varphi\|_{\text{cb}} \|(a_i)_i\|_C$ for any cb-map φ and any finite sequence $(a_i)_i$. Hence the following is trivial.

Lemma 2.6. *The property (P) inherits to a von Neumann subalgebra which is the range of a completely bounded projection.*

As a corollary to Theorem 2.5, we see that a von Neumann subalgebra $M \subset \mathbb{B}(\mathcal{H})$ which is the range of a completely bounded projection is in fact injective. We observe that by the type decomposition and the Takesaki duality, it suffices to show Theorem 2.5 for a von Neumann algebra of type II_1 .

Let $M \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra. An *M-central state* is a state φ on $\mathbb{B}(\mathcal{H})$ such that $\varphi(uxu^*) = \varphi(x)$ for $u \in M$ and $x \in \mathbb{B}(\mathcal{H})$ (or equivalently

³This nomenclature is nonstandard.

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$\varphi(ax) = \varphi(xa)$ for $a \in M$ and $x \in \mathbb{B}(\mathcal{H})$). Recall that the celebrated theorem of Connes states that a finite von Neumann algebra M is injective iff there exists an M -central state φ such that $\varphi|_M$ is a faithful normal tracial state.

Lemma 2.7. *Let $M \subset \mathbb{B}(\mathcal{H})$. Then, there exists an M -central state if*

$$\left\| \sum_{i=1}^n u_i \otimes \overline{u_i} \right\|_{\mathbb{B}(\mathcal{H} \otimes \overline{\mathcal{H}})} = n$$

for every n and unitary elements $u_1, \dots, u_n \in M$.

Proof. We first recall that $\overline{\mathcal{H}}$ is the complex conjugate Hilbert space of $\mathcal{H}$ and $\overline{x} \in \mathbb{B}(\overline{\mathcal{H}})$ means the element associated with $x \in \mathbb{B}(\mathcal{H})$. We have the canonical identification between the Hilbert space $\mathcal{H} \otimes \overline{\mathcal{H}}$ and the space $\mathcal{S}_2(\mathcal{H})$ of the Hilbert-Schmidt class operators on $\mathcal{H}$, given by $\xi \otimes \overline{\eta} \mapsto \langle \cdot, \eta \rangle \xi \in \mathcal{S}_2(\mathcal{H})$. Under this identification, $\sum a_i \otimes \overline{b_i}$ acts on $\mathcal{S}_2(\mathcal{H})$ as $\mathcal{S}_2(\mathcal{H}) \ni h \mapsto \sum a_i h b_i^* \in \mathcal{S}_2(\mathcal{H})$.

Let $u_1, \dots, u_n \in M$ be unitary elements such that $u_1 = 1$. If $\left\| \sum_{i=1}^n u_i \otimes \overline{u_i} \right\| = n$, then there exists a unit vector $h \in \mathcal{S}_2(\mathcal{H})$ such that $\left\| \sum_{i=1}^n u_i h u_i^* \right\|_2 \approx n$. By uniform convexity, we must have $\|u_i h u_i^* - h\|_2 \approx 0$ for every i . This implies that $\|u_i h^* h u_i - h^* h\|_1 \approx 0$ for every i . It follows that $\varphi(x) = \text{Tr}(h^* h x)$ defines a state on $\mathbb{B}(\mathcal{H})$ such that $\|\varphi \circ \text{Ad}(u_i) - \varphi\|_{\mathbb{B}(\mathcal{H})_*} \approx 0$ for every i . Therefore, taking appropriate limit, we can obtain an M -central state. $\square$

Lemma 2.8 (Haagerup 1985). *Let M be a von Neumann algebra. Assume that there exists a constant $c > 0$ with the following property; For every n , unitary elements $u_1, \dots, u_n \in M$ and every non-zero central projection $p \in M$, we have*

$$\left\| \sum_{i=1}^n p u_i \otimes \overline{p u_i} \right\|_{\mathbb{B}(p \mathcal{H} \otimes \overline{p \mathcal{H}})} \geq c n.$$

Then, M is injective.

Proof. Let $u_1, \dots, u_n \in M$ be unitary elements and $p \in M$ be a non-zero central projection. By assumption, we have

$$\left\| \left(\sum_{i=1}^n p u_i \otimes \overline{p u_i} \right)^k \right\|_{\mathbb{B}(p \mathcal{H} \otimes \overline{p \mathcal{H}})} \geq c n^k$$

for every positive integer k . Therefore, we actually have that

$$\left\| \sum_{i=1}^n p u_i \otimes \overline{p u_i} \right\|_{\mathbb{B}(p \mathcal{H} \otimes \overline{p \mathcal{H}})} \geq \lim_{k \rightarrow \infty} c^{1/k} n = n.$$

By Lemma 2.7, there exists a pM -central state φ_p on $\mathbb{B}(pM)$ for every non-zero central projection $p \in M$. Fix a normal faithful tracial state τ on M . For any finite

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partition $\mathcal{P} = \{p_i\}_i$ of unity by central projections in M , we define the M -central state $\varphi_{\mathcal{P}}$ on $\mathbb{B}(\mathcal{H})$ by

$$\varphi_{\mathcal{P}}(x) = \sum_i \tau(p_i) \varphi_{p_i}(p_i x p_i).$$

Taking appropriate limit of $\varphi_{\mathcal{P}}$, we obtain an M -central state φ on $\mathbb{B}(\mathcal{H})$ such that $\varphi|_M = \tau$. We conclude that M is injective by Connes's theorem. $\square$

For a finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we set

$$\|(x_i)_i\|_{OH} = \left\| \sum_i x_i \otimes \bar{x}_i \right\|_{\mathbb{B}(\mathcal{H} \otimes \overline{\mathcal{H}})}^{1/2}.$$

We note that $\|(x_i)_i\|_{OH} \leq \|(x_i)_i\|_R^{1/2} \|(x_i)_i\|_C^{1/2} \leq \|(x_i)_i\|_{C \cap R}$. Besides those appearing in Lemma 2.1, we have the following mysterious inequality (which manifests the self-dual property of the operator Hilbert spaces).

Lemma 2.9. *For every finite sequences $(a_i)_i$ in $\mathbb{B}(\mathcal{H})$ and $(b_i)_i$ in $\mathbb{B}(\mathcal{K})$, we have*

$$\left\| \sum_i a_i \otimes b_i \right\|_{\mathbb{B}(\mathcal{H} \otimes \mathcal{K})} \leq \|(a_i)_i\|_{OH} \|(b_i)_i\|_{OH}$$

Proof. We may assume that $\mathcal{K} = \overline{\mathcal{H}}$ and use $\bar{b}_i$ in the place of b_i . Identifying $\mathcal{H} \otimes \overline{\mathcal{H}}$ with $\mathcal{S}_2(\mathcal{H})$ as in the proof of Lemma 2.7, we see

$$\left\| \sum_i a_i \otimes \bar{b}_i \right\|_{\mathbb{B}(\mathcal{H} \otimes \overline{\mathcal{H}})} = \sup \left\{ \left| \sum_i \text{Tr}(h a_i k b_i^*) \right| : h, k \in \mathcal{S}_2(\mathcal{H}) \text{ with norm } 1 \right\}.$$

Let $h, k \in \mathcal{S}_2(\mathcal{H})$ with norm 1 be given. Then, we can find decompositions $h = h_1 h_2$ and $k = k_1 k_2$ such that $h_j, k_j \in \mathcal{S}_4(\mathcal{H})$ with norm 1. It follows that

$$\begin{aligned} \left| \sum_i \text{Tr}(h a_i k b_i^*) \right| &= \left| \sum_i \text{Tr}((h_2 a_i k_1)(k_2 b_i^* h_1)) \right| \\ &\leq \text{Tr} \left(\sum_i h_2 a_i k_1 k_1^* a_i^* h_2^* \right)^{1/2} \text{Tr} \left(\sum_i h_1^* b_i k_2^* k_2 b_i^* h_1 \right)^{1/2} \\ &\leq \left\| \sum_i a_i \otimes \bar{a}_i \right\|_{\mathbb{B}(\mathcal{H} \otimes \overline{\mathcal{H}})}^{1/2} \left\| \sum_i b_i \otimes \bar{b}_i \right\|_{\mathbb{B}(\mathcal{H} \otimes \overline{\mathcal{H}})}^{1/2}. \end{aligned}$$

This proves the assertion. $\square$

Lemma 2.10. *For every finite sequence $(x_i)_i$ in $\mathbb{B}(\mathcal{H})$, we have*

$$\|(x_i)_i\|_{C+R} \leq \|(x_i)_i\|_{OH}.$$

Proof. Let $\Phi: C \cap R \ni \theta_i \mapsto x_i \in \mathbb{B}(\mathcal{H})$ and take $z = \sum_i a_i \otimes \theta_i \in \mathbb{B}(\mathcal{H}) \otimes (C \cap R)$. We note that $\|z\| = \|(a_i)_i\|_{C \cap R} \geq \|(a_i)_i\|_{OH}$. Hence, by Lemma 2.9, we have

$$\|(\text{id} \otimes \Phi)(z)\| = \left\| \sum_i a_i \otimes x_i \right\| \leq \|(a_i)_i\|_{OH} \|(x_i)_i\|_{OH} \leq \|(x_i)_i\|_{OH} \|z\|.$$

This implies that $\|(x_i)_i\|_{C+R} = \|\Phi\|_{cb} \leq \|(x_i)_i\|_{OH}$. $\square$

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We have prepared enough lemmas for the proof of Theorem 2.5.

Proof of Theorem 2.5. It is left to show that a finite von Neumann algebra M with the property (P) is injective. To verify the assumption of Lemma 2.8, we give ourselves unitary elements $u_1, \dots, u_n \in M$, a non-zero central projection $p \in M$ and a constant $c > 0$ such that

$$\|(pu_i)_i\|_{OH}^2 \leq cn.$$

Then, by Lemma 2.10 and the property (P), there exist $(a_i)_i$ and $(b_i)_i$ in M such that $\|(a_i)_i\|_C \leq C_M \sqrt{cn}$, $\|(b_i)_i\|_R \leq C_M \sqrt{cn}$ and $pu_i = a_i + b_i$ for every i . We fix a tracial state on pM and denote by $\|\cdot\|_2$ the corresponding 2-norm. It follows that

$$n = \sum_{i=1}^n \|pu_i\|_2^2 \leq 2 \sum_{i=1}^n (\|a_i\|_2^2 + \|b_i\|_2^2) \leq 2(\|(a_i)_i\|_C^2 + \|(b_i)_i\|_R^2) \leq 2C_M^2 cn.$$

Therefore, we have $c \geq (2C_M^2)^{-1}$ and we are done. $\square$

2.3. A characterization of nuclearity. Let A be a (unital) C^* -algebra. We say A has the *strong similarity property* (abbreviated as (SSP)) if for every unital continuous homomorphism $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$, there exists $S \in \text{GL}(\mathcal{H}) \cap \text{vN}(\pi(A))$ such that $\text{Ad}(S) \circ \pi$ is a $*$ -homomorphism.

Theorem 2.11 (Pisier 2005). *A C^* -algebra A is nuclear iff it has the (SSP).*

Proof. As we remarked, the “only if” part follows from Dixmier’s proof + the amenability of nuclear C^* -algebra. To prove the “if” part, let A be a C^* -algebra with the (SSP). By a standard direct sum argument, it is not hard to see that there exists a constant $C > 0$ with the following property; Every unital continuous homomorphism $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ with $\|\pi\| \leq 5^4$, there exists $S \in \text{GL}(\mathcal{H}) \cap \text{vN}(\pi(A))$ with $\|S\| \|S^{-1}\| \leq C$ such that $\text{Ad}(S) \circ \pi$ is a $*$ -homomorphism. Let $A \subset \mathbb{B}(\mathcal{H})$ be a universal $*$ -representation. It suffices to show that A' is injective. Let $(x_i)_i$ be a finite sequence in A' with $\|(x_i)_i\|_{C+R} \leq 1$. Since $\mathbb{B}(\mathcal{H})$ is injective, there exist $(c_i)_i$ and $(d_i)_i$ in $\mathbb{B}(\mathcal{H})$ such that $\|(c_i)_i\|_C \leq 1$, $\|(d_i)_i\|_R \leq 1$ and $x_i = c_i + d_i$ for every i . We define a derivation $\delta: A \rightarrow \mathbb{B}(\mathcal{H}) \bar{\otimes} \mathcal{LF}_\infty$ by

$$\delta(a) = \delta_{\sum c_i \otimes \lambda(s_i)}(a \otimes 1) = \sum_i \delta_{c_i}(a) \otimes \lambda(s_i) \in \mathbb{B}(\mathcal{H}) \otimes E_\lambda \subset \mathbb{B}(\mathcal{H}) \bar{\otimes} \mathcal{LF}_\infty.$$

We recall from the proof of Theorem 2.3 that $\lambda(s_i) = u_i + v_i$ with $\|(u_i)_i\|_C \leq 1$ and $\|(v_i)_i\|_R \leq 1$. Since $\delta_{c_i} = \delta_{-d_i}$ on A , we have $\delta = \delta_B$, where $B = \sum (c_i \otimes v_i - d_i \otimes u_i)$

⁴We can choose any other number that is strictly greater than 1 by scaling the δ later.

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with $\|B\| \leq \|(c_i)_i\|_C \|(v_i)\|_R + \|(d_i)_i\|_R \|(u_i)\|_C \leq 2$. Hence, we have $\|\delta\|_{cb} \leq 4$. We define a homomorphism $\pi: A \rightarrow \mathbb{M}_2(\mathbb{B}(\mathcal{H}) \bar{\otimes} \mathcal{LF}_\infty)$ by

$$\pi(a) = \begin{pmatrix} a \otimes 1 & \delta(a) \\ 0 & a \otimes 1 \end{pmatrix}.$$

By the assumption on the (SSP), there exists an invertible element $S \in \text{vN}(\pi(A))$ with $\|S\| \|S^{-1}\| \leq C$ such that $\text{Ad}(S) \circ \pi$ is a $*$ -homomorphism. By the proof of Lemma 1.2, there exists $T \in \mathbb{B}(\mathcal{H}) \bar{\otimes} \mathcal{LF}_\infty$ with $\|T\| \leq C^2$ such that $\delta(a) = \delta_T(a \otimes 1)$. Let $Q: \mathcal{LF}_\infty \rightarrow E_\lambda$ be the projection appearing in Theorem 2.3. Since $\delta(A) \subset \mathbb{B}(\mathcal{H}) \otimes E_\lambda$ and $\text{id} \otimes Q$ is A -linear, we have

$$\delta(a) = (\text{id} \otimes Q)(\delta(a)) = \delta_{(\text{id} \otimes Q)(T)}(a \otimes 1)$$

for every $a \in A$. We write $(\text{id} \otimes Q)(T) = \sum z_i \otimes \lambda(s_i)$. Then, by Lemma 2.1 and Theorem 2.3, we have

$$\|(z_i)_i\|_{C \cap R} \leq \|(\text{id} \otimes Q)(T)\| \leq \|Q\|_{cb} \|T\| \leq 2C^2.$$

Since $\lambda(s_i)$'s are linearly independent, we have $\delta_{c_i} = \delta_{z_i}$, or equivalently $c_i - z_i \in A'$. Therefore, we have $a_i = c_i - z_i \in A'$ with

$$\|(a_i)_i\|_C \leq \|(c_i)_i\|_C + \|(z_i)_i\|_C \leq 1 + 2C^2,$$

and likewise $b_i = x_i - a_i = d_i + z_i \in A'$ with $\|(b_i)_i\|_R \leq 1 + 2C^2$. We conclude the injectivity of A' by Theorem 2.5. $\square$

We say a group Γ has the (SSP) if for every u.b. representation $\pi: \Gamma \rightarrow \text{GL}(\mathcal{H})$, there exists $S \in \text{GL}(\mathcal{H}) \cap \text{vN}(\pi(\Gamma))$ such that $\text{Ad}(S) \circ \pi$ is a unitary representation.

Corollary 2.12. *A discrete group Γ is amenable iff it has the (SSP).*

Proof. This follows from the fact that Γ is amenable iff $C^*\Gamma$ is nuclear. $\square$

3. SIMILARITY LENGTH OF C^* -ALGEBRAS

The following is the fundamental characterization of a homomorphism which is similar to a $*$ -homomorphism. This has several applications to dilation theory.

Theorem 3.1 (Haagerup, Paulsen). *Let A be a unital C^* -algebra (or just a unital operator algebra), $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a unital homomorphism and $C > 0$ be a constant. Then, $\|\pi\|_{cb} \leq C$ iff there exists $S \in \text{GL}(\mathcal{H})$ with $\|S\| \|S^{-1}\| \leq C$ such that $\|\text{Ad}(S) \circ \pi\|_{cb} = 1$.*

Proof. The “if” part is obvious. To prove the “only if” part, let $A \subset \mathbb{B}(H)$ and $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$ be a homomorphism with $\|\pi\|_{cb} \leq C$. By a Stinespring type theorem, there exist a Hilbert space $\widehat{\mathcal{H}}$, a $*$ -homomorphism $\sigma: \mathbb{B}(H) \rightarrow \mathbb{B}(\widehat{\mathcal{H}})$, and operators $V \in \mathbb{B}(\mathcal{H}, \widehat{\mathcal{H}})$, $W \in \mathbb{B}(\widehat{\mathcal{H}}, \mathcal{H})$ with $\|V\| \|W\| \leq \|\pi\|_{cb}$ such that

$$\forall a \in A \quad \pi(a) = V\sigma(a)W.$$

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Let $\mathcal{K}_1 = \overline{\text{span}}(\sigma(A)W\mathcal{H})$. The subspace $\mathcal{K}_1$ is $\sigma(A)$ -invariant and we may assume that $V = VP_{\mathcal{K}_1}$. Since

$$V\sigma(a)(\sigma(x)W\xi) = \pi(ax)\xi = \pi(a)V\sigma(x)W\xi,$$

we have $V\sigma(a)P_{\mathcal{K}_1} = \pi(a)V$ for every $a \in A$. It follows that $\mathcal{K}_2 = \ker V \subset \mathcal{K}_1$ is also $\sigma(A)$ -invariant. Hence $\mathcal{L} = \mathcal{K}_1 \ominus \mathcal{K}_2$ is “semi-invariant” under $\sigma(A)$, i.e.,

$$\forall a \in A \quad P_{\mathcal{L}}\sigma(a) = P_{\mathcal{L}}\sigma(a)P_{\mathcal{L}}.$$

Consequently, we have

$$\forall a \in A \quad \pi(a) = VP_{\mathcal{L}}\sigma(a)W = VP_{\mathcal{L}}\sigma(a)P_{\mathcal{L}}W.$$

Since $VP_{\mathcal{L}}$ is injective on $\mathcal{L}$ and $VP_{\mathcal{L}}W = \pi(1) = 1$, the operator $S = VP_{\mathcal{L}}$ is a linear isomorphism from $\mathcal{L}$ onto $\mathcal{H}$ with $S^{-1} = P_{\mathcal{L}}W$. We have $\pi = \text{Ad}(S) \circ \sigma$ with $\|S\| \|S^{-1}\| \leq C$ and, since $\mathcal{L} \cong \mathcal{H}$, we are done. $\square$

Corollary 3.2. *A derivation δ is inner iff it is completely bounded.*

By a standard direct sum argument, we obtain the following.

Corollary 3.3. *Let A be a unital C^* -algebra with the (SP). Then, there exists a function f on $[1, \infty)$ such that*

$$\|\pi\|_{\text{cb}} \leq f(\|\pi\|)$$

for every unital continuous homomorphism $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$.

Definition 3.4. Let A be a unital C^* -algebra (or a unital operator algebra). The *similarity length* of A , denoted by $l(A)$, is the smallest integer l with the following property; There exists a constant $C > 0$ such that for any $x \in M_{\infty}(A)$, there exist $\alpha_0, \alpha_1, \dots, \alpha_l \in M_{\infty}(\mathbb{C})$ and $D_1, \dots, D_l \in \text{Diag}_{\infty}(A)$ satisfying

$$x = \alpha_0 D_1 \alpha_1 \cdots D_l \alpha_l$$

and

$$\prod_{m=0}^l \|\alpha_m\| \prod_{m=1}^l \|D_m\| \leq C\|x\|.$$

Here, $M_{\infty}(A) = \bigcup_{n=1}^{\infty} M_n(A)$ and $\text{Diag}_{\infty}(A) \subset M_{\infty}(A)$ is the set of diagonal matrices with entries in A . If there is no l satisfying the above condition, then we set $l(A) = \infty$ by convention.

Theorem 3.5 (Pisier 1999). *Let A be a unital C^* -algebra (or a unital operator algebra) with $\dim(A) > 1$. The following are equivalent.*

- (1) A has the (SP).
- (2) There exist $d > 0$ and $C > 0$ such that $\|\pi\|_{\text{cb}} \leq C\|\pi\|^d$ for every unital continuous homomorphism $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$.
- (3) $l(A) \leq d$.

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The constant d appearing in the conditions (2) and (3) are taken to be same and are possibly non-integer. It follows that the “optimal” function f appearing in Corollary 3.3 is a polynomial of degree $l(A)$. The implication (2) $\Rightarrow$ (1) follows from Theorem 3.1. We do not prove the hard implication (1) $\Rightarrow$ (3), but explain (3) $\Rightarrow$ (2);

$$\|\pi(x)\| = \|\alpha_0 \pi(D_1) \alpha_1 \cdots \pi(D_l) \alpha_l\| \leq \|\pi\|^l \prod_{m=0}^l \|\alpha_m\| \prod_{m=1}^l \|D_m\| \leq C \|\pi\|^l \|x\|$$

for $x = \alpha_0 D_1 \alpha_1 \cdots D_l \alpha_l \in \mathbb{M}_\infty(A)$.

For a unital C*-algebra A with $\dim(A) > 1$, it is known that

- (1) $l(A) = 1 \Leftrightarrow \dim(A) < \infty$ (Exercise),
- (2) $l(A) = 2 \Leftrightarrow A$ is nuclear with $\dim(A) = \infty$ (Pisier 2004),
- (3) $l(A) \leq 3$ if A has no tracial state,
- (4) $l(M) = 3$ if M is a type II_1 factor with the property (Γ) (Christensen 2002),
- (5) $l(A) = \max\{l(I), l(A/I)\}$ for every closed 2-sided ideal $I \triangleleft A$ (Exercise).

It is not known whether there exists a unital C*-algebra with $l(A) > 3$. We note that an affirmative answer to Similarity Problem A would imply that there exists l_0 such that $l(A) \leq l_0$ for every C*-algebra A . We close this note by showing $l(A) \leq 3$ for any C*-algebra A which contains a unital copy of the Cuntz algebra $\mathcal{O}_\infty$. (The case where A has no tracial state is then dealt by passing to the second dual.)

Let $x \in \mathbb{M}_n(A)$ be given. We choose unitary matrices $W_1, W_2 \in \mathbb{M}_n(\mathbb{C})$ with $|W_1(i, j)| = |W_2(i, j)| = n^{-1/2}$ for all i, j (e.g., $W_k(i, j) = n^{-1/2} \exp(2\pi\sqrt{-1}ij/n)$). Let $D_1(i) = S_i^*$ and $D_3(j) = S_j$ for every i, j , where S_i 's are isometries satisfying $S_i^* S_j = \delta_{i,j} I$. For every k , we set

$$\begin{aligned} D_2(k) &= n \sum_{i,j} \overline{W_1(i, k)} S_i x_{i,j} S_j^* \overline{W_2(k, j)} \\ &= n \begin{pmatrix} \overline{W_1(1, k)} S_1 & \cdots & \overline{W_1(n, k)} S_n \end{pmatrix} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \begin{pmatrix} \overline{W_2(k, 1)} S_1^* \\ \vdots \\ \overline{W_2(k, n)} S_n^* \end{pmatrix}. \end{aligned}$$

From the latter expression, we see that $\|D_2(k)\| \leq \|x\|$. We obtained $W_1, W_2 \in \mathbb{M}_n(\mathbb{C})$ and $D_1, D_2, D_3 \in \text{Diag}_n(A) \subset \mathbb{M}_n(A)$ such that

$$\|D_1\| \|W_1\| \|D_2\| \|W_2\| \|D_3\| \leq \|x\|$$

and

$$x = D_1 W_1 D_2 W_2 D_3.$$

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Indeed, we have

$$\begin{aligned}(D_1 W_1 D_2 W_2 D_3)_{i,j} &= \sum_{k=1}^n S_i^* W_1(i, k) D_2(k) W_2(k, j) S_j \\ &= n \sum_{k=1}^n |W_1(i, k)|^2 |W_2(k, j)|^2 x_{i,j} = x_{i,j}.\end{aligned}$$

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