

Generalized supremum in ordered linear space and facial structure of a convex set

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§1 DEFINITIONS AND BASIC RESULTS

Let E be a linear space over $\mathbb{R}$, and P be a convex cone in E satisfying

- (P1) $E = P - P$,
- (P2) $P \cap (-P) = \{0\}$.

An order relation in E can be defined by $x \leq y \iff y - x \in P$. It can easily be seen that

- (1) $x \leq y$ and $y \leq x \implies x = y$,
- (2) $x \leq y$ and $y \leq z \implies x \leq z$,
- (3) $x \leq y \implies x + z \leq y + z$ for all $z \in E$,
- (4) $0 \leq x$ and $0 \leq \lambda \in \mathbb{R} \implies 0 \leq \lambda x$,
- (5) For every $x \in E$, there exists $x_1, x_2 \in E$ such that $x = x_1 - x_2$, and $0 \leq x_1, x_2$.

Conversely, if an order in E satisfies (1) ~ (5), then $P = \{x \in E \mid 0 \leq x\}$ is a convex cone satisfying (P1) and (P2). A linear space E equipped with such a positive cone P is called a partially ordered linear space, and is sometimes denoted by (E, P) .

Definition. For a subset A of E , the generalized supremum $\text{Sup } A$ is defined to be the set of all minimal elements of $U(A)$, where $U(A)$ is the set of all upper bound of A .

We say in other words that $a \in \text{Sup } A$ if and only if $a \leq b$ whenever $b \in U(A)$ and a, b are comparable. The generalized infimum $\text{Inf } A$ can be defined similarly. In order to distinguish this notion from the least upper bound and the greatest lower bound, we denote the latter ones by $\sup A$ and $\inf A$ respectively. If E is order complete, then $\text{Sup } A = \{\sup A\}$ holds whenever the subset A is upper bounded (i.e., $U(A) \neq \emptyset$). When $E = \mathbb{R}^n$ and P is closed and not a lattice cone, $\text{Sup } A$ becomes a infinite set in most cases. However, it is possibly empty, even when A is upper bounded.

Proposition 1. For $a \in E$ and $\lambda > 0$, we have

- (1) $\text{Sup}(A + a) = \text{Sup } A + a$,
- (2) $\text{Sup } \lambda A = \lambda \text{Sup } A$,
- (3) $\text{Sup } A = -\text{Inf}(-A)$.

Proposition 2. For an arbitrary set $A \subset E$ with $U(A) \neq \emptyset$,

$$\text{Sup } A = \text{Sup}(\text{co}A)$$

holds where $\text{co}A$ is the convex hull of A .

proof. It suffices to show that $U(A) = U(\text{co}A)$. Take $x_0 \in U(A)$ arbitrarily. For $x \in \text{co}A$ there exist some points $x_1, x_2, \dots, x_n$ in A such that $x = \sum_{i=1}^n \lambda_i x_i$ with $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^n \lambda_i = 1$. Hence $x_0 - x = \sum_{i=1}^n \lambda_i (x_0 - x_i) \geq 0$ and we have $x_0 \in U(\text{co}A)$.

When A is a finite set of the form $\{a_1, \dots, a_n\}$, we denote the set of the upper bound of A by $U(a_1, \dots, a_n)$ instead of $U(A)$. With this notation, we define $a \vee b$ ($a, b \in E$) to be the set of all minimal elements of $U(a, b)$. Also $a \wedge b$ can be defined similarly. When (E, P) is a lattice, $a \vee b$ is always a single element which is the minimum of $U(a, b)$.

Proposition 3. For every $a, b, c \in E$ and $\lambda \in \mathbb{R}$,

- (1) $(a + c) \vee (b + c) = (a \vee b) + c$,
- (2) $\lambda a \vee \lambda b = \lambda(a \vee b)$.

Theorem 1. For $a, b \in E$, $a \vee b \neq \emptyset$ implies $a \wedge b \neq \emptyset$ and the converse is also true. Moreover,

$$a + b - (a \vee b) = a \wedge b$$

holds and in particular we have $a \in a_+ + a_-$ where $a_+ = a \vee 0$ and $a_- = a \wedge 0$.

The proof of Theorem 1 can be seen in [6]. Also, some examples in which $a \vee b$ can be empty are shown.

A partially ordered linear space (E, P) is said to be monotone order complete (m.o.c. for short) if every upper bounded totally ordered subset of E has the least upper bound in E . The followings are known.

Proposition 4. In the case $E = \mathbb{R}^d$, (E, P) is m.o.c. if and only if P is closed.

Proposition 5. Suppose that E is a Banach space and P is closed. Let E^* be the topological dual of E and let $P^* = \{x^* \in E^* \mid x^*(x) \geq 0, x \in P\}$. If $P^* - P^* = E^*$, then (E^*, P^*) is m.o.c.

The proof can be done by using Banach Steinhaus theorem, and in [2], one can see some conditions under which $P^* - P^* = E^*$ holds.

A linear topology of (E, P) is called an order continuous topology if every decreasing net $\{a_\lambda\} \subset E$ with $\inf a_\lambda = 0$ converges to 0 by the topology. We consider some further conditions for P ;

(P3) P is closed with respect to an order continuous topology,

(P4) For every decreasing net $\{a_\lambda\}$ in P , $\inf a_\lambda = a$ implies $a \in P$.

Note that (P3) implies (P4).

Theorem 2. Suppose that a partially ordered linear space (E, P) is monotone order complete and P satisfies (P3) or (P4). Then for every subset A of E ,

$$U(A) = (\text{Sup } A) + P$$

holds. In particular, $a \vee b \neq \emptyset, a \wedge b \neq \emptyset$ for every $a, b \in E$, and $U(a, b) = (a \vee b) + P$.

proof. It suffices to show that $U(A) \subset (\text{Sup } A) + P$. For an arbitrary $x \in U(A)$, the section $U(A)_x = \{y \in U(A) \mid y \leq x\}$ is a nonempty convex set in E . If $T \subset U(A)_x$ is a totally ordered subset, then by monotone order completeness, there exists a greatest lower bound z_0 of T . Since $T \subset U(A) = \bigcap_{y \in A} (y + P)$, (P4) yields $z_0 \in U(A)_x$. Hence by Zorn's lemma, $U(A)_x$ has at least a minimal element y_0 . It is easy to see that y_0 is also a minimal element of $U(A)$, and it means that $x \in (\text{Sup } A) + P$. The second statement of the theorem is obvious. Indeed, $U(a, b)$ is always nonempty because $P - P = E$. Hence it is sufficient to use the first statement. Q.E.D.

Corollary 1. Suppose that (E, P) satisfies the hypotheses in Theorem 2 and let A be a subset of E . If $\text{Sup } A$ consists of a single element a , then a is the least upper bound of A .

Corollary 2. *For every subset A of E , $U(L(U(A))) = U(A)$ holds where $L(U(A))$ denotes the lower bound of $U(A)$. Moreover, if (E, P) satisfies the hypotheses in Theorem 2, then we have $\text{Sup Inf Sup } A = \text{Sup } A$.*

Next we give another sufficient condition for the same results by considering the faces of P . Moreover, we will give an example which shows that each of the two conditions does not imply the other.

§2 FACES OF THE POSITIVE CONE

Let (E, P) be a partially ordered linear space, and suppose that P is algebraically closed, that is, every straight line of E meets P by a closed interval. A point x of a convex subset $A \subset E$ is called an algebraic interior point of A if for every $z \in E$, there exists $\lambda > 0$ such that $x + \lambda z \in A$. Algebraic exterior points are defined similarly, and we denote the algebraic interior (exterior) of A by $\text{int}A$ ($\text{ext}A$) respectively. Moreover, $\partial A = (\text{int}A \cup \text{ext}A)^c$ is called the algebraic boundary of A . A convex subset C of P is called an exposed face of P if there exists a supporting hyperplane H of P such that $C = P \cap H$. By $\mathfrak{F}(P)$, we denote the set of all exposed faces of P . For $C \in \mathfrak{F}(P)$, $\dim C$ is defined as the dimension of $\text{aff}C$ where $\text{aff}C$ denotes the affine hull of C . The following theorem is a fundamental result, and is also useful when we intend to determine the set $a \vee b$ explicitly.

Theorem 3. *Suppose that P is algebraically closed and $\text{int } P \neq \emptyset$. If $\dim C \leq 1$ for every $C \in \mathfrak{F}(P)$, then*

$$a \vee b = \partial U(a) \cap \partial U(b)$$

holds for every incomparable pair $a, b \in E$.

In the case when a linear topology is given in E , the assertion of Theorem 3 can be translated into the terms of topology and is still valid.

Lemma 1. *If $0 \leq x \leq y$ and $y \in \partial P$, then $x \in \partial P$.*

proof. Suppose that $x \in \text{int } P$ and put $z = 2y - x$, then $z = y + (y - x) \in P + P = P$. Since P is convex and $x \in \text{int } P$, $y = \frac{1}{2}(x + z) \in \text{int } P$. This contradicts the assumption.

proof of Theorem 3. Let x_0 be an element of $a \vee b$, and suppose that $x_0 \in \text{int } U(a)$. Then there exists $\lambda > 0$ such that $c \stackrel{\text{def}}{=} (1 - \lambda)x_0 + \lambda b \in U(a)$. It is easy to see that $c \in U(a) \cap U(b) = U(a, b)$ and $c \not\leq x_0$. This contradicts the fact that x_0 is a minimal element of $U(a, b)$, and hence $a \vee b \subset \partial U(a) \cap \partial U(b)$.

Conversely, take $x_0 \in \partial U(a) \cap \partial U(b)$ arbitrarily and suppose that $y_0 \leq x_0$, $y_0 \in U(a, b)$. Since $a \leq y_0 \leq x_0$, it follows by Lemma 1 that

$$y_0 \in [a, x_0] \subset \partial U(a),$$

where $[a, x_0] = \{x \in E \mid a \leq x \leq x_0\}$ is an order interval. Obviously every order interval is a convex set. Similarly we have

$$y_0 \in [b, x_0] \subset \partial U(b),$$

and hence

$$[a, x_0] \cap \text{int } U(a) = \emptyset, \quad [b, x_0] \cap \text{int } U(b) = \emptyset,$$

while $\text{int } U(a)$ and $\text{int } U(b)$ are both assumed to be nonempty. Applying the separation theorem, we can find hyperplanes H_1, H_2 of E such that

- (1) H_1 separates $[a, x_0]$ and $U(a)$ and,
- (2) H_2 separates $[b, x_0]$ and $U(b)$.

Since $[a, x_0] \subset U(a)$ and $[b, x_0] \subset U(b)$, we can see that $[a, x_0] \subset U(a) \cap H_1$ and $[b, x_0] \subset U(b) \cap H_2$. By the condition of $\mathfrak{F}(P)$, these two faces are actually half lines. On the other hand, a, b , and x_0 cannot be in any single straight line because a and b are not comparable. Hence $[a, x_0]$ and $[b, x_0]$ are respectively included in two different lines, and in particular, both x_0 and y_0 belong to the intersection of those two lines. This means $x_0 = y_0$ and so $x_0 \in a \vee b$. Q.E.D.

Lemma 2. *Suppose that the positive cone P is algebraically closed and $\text{int } P \neq \emptyset$. Then $\partial U(a) \cap \partial U(b) \neq \emptyset$ for every incomparable pair $a, b \in E$.*

proof. We can take an element $x \in U(a) \cap U(b)$. Indeed, $b - a$ can be written in the form $p - q$ with $p, q \in P$, and so $a + p = b + q \in U(a) \cap U(b)$. Since $a \notin U(b)$, and $U(b)$ is algebraically closed, there exists $\lambda_0 \in [0, 1)$ such that $\lambda_0 = \max\{\lambda > 0 \mid x + \lambda(a - x) \in U(b)\}$. Obviously, $z_0 \stackrel{\text{def}}{=} x + \lambda_0(a - x) \in U(a) \cap \partial U(b)$. Next we take $\lambda_1 = \max\{\lambda \mid z_0 + \lambda(b - z_0) \in U(a)\}$ similarly. Then $z_1 \stackrel{\text{def}}{=} z_0 + \lambda_1(b - z_0) \in \partial U(a)$. Moreover, since $b \leq z_1 \leq z_0 \in \partial U(b)$, it follows by Lemma 1 that $z_1 \in \partial U(b)$.

Applying Theorem 3 and Lemma 2, we can obtain the following.

Corollary 3. *Under the hypotheses in Theorem 3, $a \vee b \neq \emptyset$ holds for every $a, b \in E$. Moreover when a and b are not comparable, we have*

$$U(a, b) = (a \vee b) + P.$$

proof. The first statement of the theorem follows immediately from Theorem 3 and Lemma 2. To see the latter, it is sufficient to show $U(a, b) \subset (a \vee b) + P$. For an arbitrary element $x \in U(a, b)$, we can choose z_1 as in the proof of Lemma 2. Then $z_1 \leq x$ and $z_1 \in \partial U(a) \cap \partial U(b)$. Hence by Theorem 3, $z_1 \in a \vee b$, and this means that $x \in (a \vee b) + P$.

Theorem 4. *Under the hypotheses in Theorem 3,*

$$U(A) = (\text{Sup } A) + P$$

holds for every subset $A \subset E$. In particular, the conclusions in Corollary 1 and Corollary 2 are valid.

Remark. The hypotheses of this theorem can be somewhat weakened. Moreover, using this theorem, we can simplify the proof of Lemma 2 and can obtain the second statement of Corollary 3 directly.

Lemma 3. *If $x \in \partial U(A)$ for a subset A of E , then $U(A)_x \subset \partial U(A)$ where $U(A)_x = \{y \in U(A) \mid y \leq x\}$.*

proof. Let y be an arbitrary point in $U(A)_x$. Since $x \in \partial U(A)$ there exists a point $z \in E$ such that $\{x + tz \mid t > 0\} \cap U(A) = \emptyset$. By the definition of $U(A)$, $U(A) + P = U(A)$, and this yields $\{y + tz \mid t > 0\} \cap U(A) = \emptyset$. This means that $y \in \partial U(A)$.

proof of Theorem 4. Let x_0 be an arbitrary point in $U(A)$. Since P is algebraically closed, P can not include any straight line. Indeed if $\{x + ty \mid t \in \mathbb{R}\} \subset P$ for some $y \neq 0$, then $\{ty \mid t \in \mathbb{R}\} \subset P \cup \partial P = P$ and this contradicts (P2). Hence for a positive element $x \neq 0$, there exists $t_1 = \max\{t \geq 0 \mid x_0 - tx \in U(A)\}$. If we put $x_1 = x_0 - t_1 x$, then $x_1 \in \partial U(A)$ and it follows from Lemma 3 that $U(A)_{x_1} \subset \partial U(A)$. Since $U(A)_{x_1}$ is a convex set and $\text{int } U(A) \neq \emptyset$, we can apply the separation theorem and there exists a hyper plane H which separates $U(A)_{x_1}$ and $U(A)$. $U(A)_{x_1} \subset (x_1 - P) \cap H$ and this is a straight half line by the assumption. Moreover, since $U(A)$ can not include the whole straight line, $U(A)_{x_1}$ is the form $\{\lambda x_1 + (1 - \lambda)z \mid 0 \leq \lambda \leq 1\}$ where $z \leq x_1$. Clearly, z is a minimal element of $U(A)$ and $z \leq x_0$, and this completes the proof. Q.E.D.

§3 EXAMPLES

Let E be the space of all symmetric matrices of $M_2(\mathbb{R})$, and let P be the set of all positive semi definite matrices in E . Then (E, P) is m.o.c., but it is not a lattice. E and P can be identified with $\mathbb{R}^3$ and

$$P = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 \leq xy, 0 \leq x, 0 \leq y\}$$

respectively. It is easy to see that every exposed face of the positive cone P is 1-dimensional except the trivial face $\{0\}$, and P satisfies the condition in Theorem 3. Hence, by some simple calculations, we can determine the set $a \vee b$ for incomparable pair $a, b \in E$.

Next we investigate the relation between the condition of Theorem 2 and that of Theorem 3. For a partially ordered linear space (E, P) , we say that the positive cone P satisfies condition $(\mathfrak{F})$ when $\dim C \leq 1$ for every $C \in \mathfrak{F}(P)$. In finite dimensional cases, P does not satisfy the condition $(\mathfrak{F})$ when P is a closed convex cone generated by a finite set. On the other hand, such a positive cone satisfies monotone order completeness. This means that monotone order completeness does not imply the condition $(\mathfrak{F})$. Now we show an example in order to see the converse implication is also not true.

Let E be the linear space consisting of all sequences $x = (x_1, x_2, \dots)$ ($x_i \in \mathbb{R}$) such that $x_i = 0$ except for finite number of $i = 1, 2, \dots$. We define

$$P = \{x = (x_1, x_2, \dots) \mid x_1 \geq \left(\sum_{i=2}^{\infty} x_i^2\right)^{\frac{1}{2}}\}.$$

Then it is easy to see that P is algebraically closed and $\text{int } P \neq \emptyset$. Indeed $(1, 0, 0, \dots) \in \text{int } P$. Let $C \in \mathfrak{F}(P)$ and let $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots)$ be two points in $C \setminus \{0\}$. Since $x, y \in \partial P$, $x_1^2 = \sum_{i=2}^{\infty} x_i^2$, and $y_1^2 = \sum_{i=2}^{\infty} y_i^2$. By the convexity of C , we also have $\frac{1}{2}(x + y) \in \partial P$, and hence $(x_1 + y_1)^2 = \sum_{i=2}^{\infty} (x_i + y_i)^2$. By simple calculation, we obtain $x = \lambda y$ for some $\lambda > 0$. This means that $\dim C = 1$, and that P satisfies the condition $(\mathfrak{F})$. Thus Theorem 3 and Theorem 4 are applicable in this case.

We will show that (E, P) is not m.o.c. We define a sequence $\{a_n\} \subset E$ by

$$a_n = \left(\frac{1}{2^n}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{2^n}, 0, 0, \dots\right) \quad (n = 1, 2, \dots).$$

Then we have $a_1 \geq a_2 \geq a_3 \geq \dots$. Moreover, since $(\frac{1}{2})^2 + (\frac{1}{4})^2 + (\frac{1}{8})^2 + \dots = \frac{1}{3}$, we can see that $(-\sqrt{\frac{1}{3}}, 0, 0, \dots)$ is a lower bound of $\{a_n\}$. Let $b = (b_1, b_2, \dots, b_i, 0, 0, \dots)$

be an arbitrary lower bound of $\{a_n\}$. Then an element of the form $c = (b_1 + \lambda, b_2, b_3, \dots, b_i, \mu, 0, 0, \dots)$ always satisfies $b \not\leq c$ when $\lambda > 0$. It is easy to see that we can choose λ and μ such that c is also a lower bound of $\{a_n\}$. This means that the greatest lower bound of $\{a_n\}$ does not exist, and (E, P) is not m.o.c..

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