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Introductory Lectures on Group Representations

by

Takeshi HIRAI (Kyoto University)

Nobuhiko TATSUUMA (Kyoto University)

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M. Morimoto, K. Shinoda

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Foreword

In 1981, an exchange program of mathematicians started between Japan and the Philippines under the auspices of JSPS and NSDB. (Later NSDB was reorganized as DOST.) In 1987, Sophia University became the Japanese core university to coordinate the JSPS-DOST exchange program in breeder sciences, that is, mathematics, physics, chemistry, and molecular biology.

Under this exchange program, about forty Japanese mathematicians have visited the Philippines and almost the same number of the Filipino mathematicians have come to Japan to conduct joint researches in several branches of mathematics. During these visits, several introductory lectures were given by Japanese specialists. We were asked to consider the possibility to collect and publish these lectures because they would be profitable for new students to acquire a general idea of mathematics research and for new lecturers to help to prepare for introductory lectures.

Now to respond these suggestions, we decided to publish the series "JSPS – DOST Lecture Notes in Mathematics". This is the first volume of this series consisting of two parts written by Professor T. Hirai and Professor N. Tatsuuma. We are grateful for their careful preparation of the manuscript in spite of our early deadline. We are sure this lecture note will serve as a good introduction for those students who want to study the representation theory of groups.

Sophia University, Tokyo
March 1994

M. Morimoto and K. Shinoda

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PREFACE

Here is a note of my introductory lectures on the theory of group representations, which were given at Dept. of Math. Fac. of Sci. Kyoto Univ. on the occasion of three monthes stay of Prof. T. Rapanut from Univ. of the Philippines (=UP) College Baguio. At that time I had only a handwritten manuscript for myself, but later I typed it out and added several pages to give some fundamental definitions, so as to distribute its copies to the participants of the summer school for mathematics teachers of high schools held at our Department on 1988.

In 1989, when I visited UP one month under the DOST program, I delivered its copies to Prof. Rapanut herself and to my host scientist Prof. R. Felix, to whom I express my hearty thanks to their warm hospitality on that occasion.

Takeshi HIRAI

February 17, 1994
in Kyoto

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Atmosphere in the theory of group representations

--- invariant differential operators and representations of the Lorentz group and the three dimensional rotation group ---

By Takeshi HIRAI (Kyoto University)

1988.6 (added on July 20)

The subjects of research which can be included under the name of "theory of group representations", are very diverse. I would like explain how the theory comes into naturally in our sight.

§1. How and why do we arrive to the theory of group representations ?

== A group G is a totality of actions on an object, satisfying certain conditions.

== This action gives us linear representations of the group by several kinds of linearizations.

(I) First linearization. Let X be an object on which G acts. Consider

1) a vector space of functions on X, or more generally

2) a vector space of certain sections of a vector bundle on X such as tangent bundle, cotangent bundle or induced bundles.

Example 1. A group G acts on G itself from the left and also from the right: for $g \in G$,

$$G \ni h \longmapsto gh \in G \quad (\text{resp.} \quad G \ni h \longmapsto hg^{-1} \in G).$$

We take as spaces of functions on G, $L^p(G, d_\ell g)$, $L^p(G, d_r g)$ ($p = 1, 2$), $C(G)$ etc. Here $d_\ell g$ (resp. $d_r g$) denotes a left (resp. right) invariant measure (= Haar measure) on G, and $C(G)$ denotes the space of all continuous functions on G.

(II) <u>Second linearization</u>. This appears as a linear approximation of a non-linear object on which G acts.

Here are some leading ideas for the theory of group representations.

(1°) In case of a finite group, almost all things about G is already contained in the (right) regular representation $(R_g, L^2(G))$, where

$$(R_g f)(h) = f(hg) \quad (g, h \in G, f \in L^2(G)).$$

(2°) In general case, many informations about the object X are contained in or absorbed into the representations constructed in (I), (II), if G acts on X transitively preserving the structure of X. For instance, this is the case if X is a manifold with some structure such as Riemannian or Hermitian symmetric, and every element $g \in G$ preserves the manifold structure.

Example 2. Let X be a complete Riemannian manifold with constant negative curvature, G the motion group of X. Then the geodesic flow on X is realized on a space of spherical tangent bundle on X by an action of a one-parameter subgroup $\{g_t; -\infty < t < \infty\}$ in G. Some important properties such as spectral type (actually countable Lebesgue), ergodicity and mixing property can be treated using group representation theory for G. For the case $\dim X = 2$, see [3].

== Such vague ideas or intensions as those stated above, are the fundamental sentiment of our group-representation-people.

Thus stated our sentiment, the core of our reseaches in the theory of group representations are:

1) Construction of irreducible (unitary) representations and classification of them.

2) To construct or find interesting representations relating also with another or other branches of mathmatics or physics. To study their mutual relations such as intertwining relations (or intertwinig operators), decomposition into irreducibles, mutual imbeddings etc.

On the other hand, the range of the theory of group representations is still diverging. For instance, for myself, current subjects of my reseach are:

1) Construction of irreducible unitary representations (= IURs) of certain infinite discrete groups, such as the infinite

permutation group S_∞, the infinite wreath products of finite groups.

2) Classification and construction of IURs of Lie super-algebras (with H.Furutsu).

3) Ergodicity of product measures under S_∞ (with N.Obata).

Further, even just surrounding me, many people, N.Tatsuuma, T.Nomura, H.Yamashita and so on are working in different directions. Other people graduated here are working also on the following subjects:

== Kac-Moody groups (their construction and their linear representations),

== IURs of Chevalley groups over a local field or a finite field,

== Relations with number theory.

§2. Invariant differential operators (first part).

2.1. Vibration of a string. Let us consider a string extended from $x = 0$ to $x = \pi$. Let $\rho(x)$ be the density of the string, $\mu(x)$ that of tension, and $f(x,t)$ that of outer force at the time t. Then we know in [1, Chap.IV, §10] that the equation for the vibration $u(x,t)$ of the string is given by

$$(2.1) \qquad \rho u_{tt} - \mu u_{xx} + f(x,t) = 0.$$

Consider the case where the force $f = 0$ and $\mu/\rho = c^2$, constant, then the Eq.(2.1) turns out to

$$(2.2) \qquad u_{xx} = c^2 u_{tt}.$$

Note that this equation is invariant under translations of the variables x and t. Putting $c = 1$, we look for a solution of the form $u(x,t) = v(x)g(t)$, then

$$\frac{v''(x)}{v(x)} = \frac{\ddot{g}(t)}{g(t)} = -\lambda \text{ (put)}.$$

We see that λ should be constant and get

$$(2.3) \qquad v''(x) + \lambda v(x) = 0, \qquad \ddot{g}(t) + \lambda g(t) = 0.$$

In case of fixed ends $v(0) = v(\pi) = 0$, we get eigenvalues $\lambda = 1$, 2^2, ..., n^2, ..., and a general solution $u(x,t)$ of the Eq.(2.2) has a formal expansion as

$$u(x,t) = \sum_n \sin nx\,(a_n \cos nt + b_n \sin nt),$$

$$(a_n,\ b_n \text{ constants}).$$

We can discuss the convergence of this expansion if necessary.

2.2. Vibration of a membrane. Now consider a membrane on a domain in (x,y). Let $\rho(x,y)$, $\mu(x,y)$ and $f(x,y,t)$ be similar as above. Then the vibration $u(x,y,t)$ of the menbrane is controlled by the equation

$$\mu\cdot\Delta u - \rho u_{tt} = f(x,y,t),$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Consider the case $f = 0$ and $\rho/\mu = c^2$, constant, then we get

$$\Delta u = c^2 u_{tt}. \tag{2.4}$$

Note that this equation is invariant under translation of variables (x,y,t) and moreover invariant under rotations of variables (x,y).

Consider a membrane on the unit disk. Introduce the polar coordinates (r,θ): $x = r\sin\theta$, $y = r\cos\theta$ $(0\leqq r\leqq 1,\ 0\leqq\theta\leqq 2\pi)$, then

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Consider a solution of the form $u = v(r,\theta)g(t)$, then we have similar equations as (2.3) with eigenvalues λ:

$$\Delta v + \lambda v = 0, \qquad c^2\ddot{g} + \lambda g = 0. \tag{2.5}$$

Note that the 1st equation is invariant under Euclidean motion group of the (x,y)-plane.

Further look for a solution of the form $v(r,\theta) = f(r)h(\theta)$, then for an integer n

$$h(\theta) = a \cos n\theta + b \sin n\theta,$$
$$r^2 f'' + rf' + (r^2\lambda - n^2)f = 0.$$

For the last equation, put $y = f$, $\xi = kr$ with $k^2 = \lambda$, then we get the Bessel's equation

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right)y = 0.$$

Summary for §§2.1-2.2. In the fundamental solutions given above, there appear functions $\sin nx$, $\sin nt$, $\cos nt$, $f(r)\sin n\theta$, $f(r)\cos n\theta$ with Bessel's function $f(r)$. These phenomena are not accidental but have intimate relation with irreducible unitary representations (= IURs) of groups which make the corresponding equations invariant. These special functions are essentially matrix elements of IURs of respective groups, or more exactly their real or complex parts.

§3. Invariance of differential operators (2nd part).

3.1. Case of linear transformation groups.

Let a group G act on $X = \mathbf{R}^n$ as linear transformations: for $g = (g_{ij})_{1\leq i,j\leq n} \in G$, $x = (x_i)_{1\leq i\leq n} \in \mathbf{R}^n$,

$$(gx)_i = \sum_{1 \leq j \leq n} g_{ij} x_j \quad (1 \leq i \leq n).$$

Take a function space $\underline{F}$ on X, e.g., $\underline{F} = C^\infty(X)$, and put

$$(3.1) \qquad (L_g f)(x) = f(g^{-1}x).$$

Then $L_e = I$, $L_{g_1} L_{g_2} = L_{g_1 g_2}$ $(g_1, g_2 \in G)$, where e denotes the identity element in G, and I the identity operator. Thus the correspondence $G \ni g \longmapsto L_g$, gives a linear representation of G on $\underline{F}$. As we see visually the transformation L_g on f is naturally induced from the action of g on the base space X on which the function f grows like a forest on the earth.

Now consider a constant coefficient differential operator $D = P(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$, then as a change of variables we have

Lemma 3.1. Put $L_g f$ as $(L_g f)(x) = f(g^{-1}x)$, then

$$(3.2) \quad (D(L_g f))(x) = D(f(g^{-1}x)) = [P((\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n})g^{-1})f(y)]\Big|_{y=g^{-1}x}.$$

Proof. Put $y = g^{-1}x$, then $x = gy$ and $\frac{\partial x_i}{\partial y_j} = g_{ij}$. Hence we get in the form of matrix multiplication

$$\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_n}\right) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\cdot g. \qquad \text{Q.E.D.}$$

Let us introduce the definition of invariance under g.

Definition 3.2. A differential operator D on X is said to be invariant under a transformation g on X if D commutes with the transformation L_g on the function space $\underset{\sim}{F}$.

Then we get from (3.2) the following

Theorem 3.3. Let D be a constant coefficient differential operator $D = P(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$ with a polynomial P. Then D is invariant under G if and only if the polynomial P is invariant in the sense that

$$P(x_1, x_2, \ldots, x_n) = P((x_1, x_2, \ldots, x_n)g) \quad (g \in G),$$

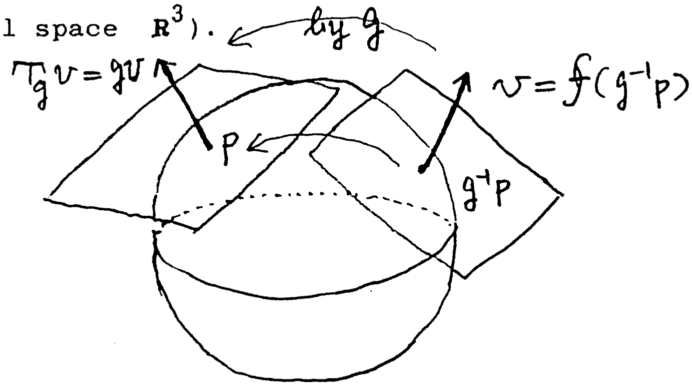
that is, P is invariant under changes of variables $x \longmapsto {}^t g^{-1} x$ for any $g \in G$.

3.2. Case of vector valued functions on $\underset{\sim}{X}$.

Let W be a G-module, that is, we are given a representation $G \ni g \longmapsto T_g \in \mathfrak{L}(W) \equiv \{\text{all continuous linear operators on } W\}$: $T_e = I$, $T_{g_1}T_{g_2} = T_{g_1 g_2}$ $(g_1, g_2 \in G)$. Take a vector space $\underset{\sim}{F}$ of W-valued functions **f** on X. We define an action of G on it by

$$(3.3) \qquad (U_g\mathbf{f})(x) = T_g(\mathbf{f}(g^{-1}x)) \quad (g \in G).$$

This definition of G-action is very natural as you can see from the picture below in the case of G = SO(3), 3-dimensional ratation group acting on the 2-dimensional unit sphere $X = S^2 \subset \mathbf{R}^3$, and $W = \mathbf{R}^3$ on which G acts naturally (in this example, X is no longer equal to the total space $\mathbf{R}^3$).



We can verify easily $U_e = I$, $U_{g_1}U_{g_2} = U_{g_1g_2}$ $(g_1, g_2 \in G)$.

The action U on $\underset{\sim}{F}$ is nothing but the tensor product of two G-modules (T, W) and (L, $\underset{\sim}{F}$), where $L_g\mathbf{f}(x) = \mathbf{f}(g^{-1}x)$ $(g \in G)$ as in the case of usual scalar valued functions.

Let us now consider a system of first order homogeneous equations on $\mathbf{f} \in \underset{\sim}{F}$:

$$(3.4) \quad D\mathbf{f} = 0, \text{ with } D = L_1 \frac{\partial}{\partial x_1} + L_2 \frac{\partial}{\partial x_2} + \dots + L_n \frac{\partial}{\partial x_n} + \kappa,$$

where $L_1, L_2, \dots, L_n$ are constant matrices of type $N \times N$, N =

dim W, and $\varkappa$ is a constant.

We define the invariance of differential operator D under $g \in G$ by $D \circ L_g = L_g \circ D$ and then the invariance of the system of equations (3.4) by the invariance of D itself when $\varkappa \neq 0$. The case $\varkappa = 0$ should be treated more carefully.

Let us write down a necessary and sufficient condition for D to be invariant under G. Calculating $U_g \circ D \circ U_g^{-1}$, we get the following

Theorem 3.4. A differential operator $D = L_1 \frac{\partial}{\partial x_1} + L_2 \frac{\partial}{\partial x_2} + \dots + L_n \frac{\partial}{\partial x_n} + \varkappa$ is invariant under G if and only if

$$\sum_j g_{ij} T_g L_j T_g^{-1} = L_i \quad (1 \leqq i \leqq n,\ g \in G).$$

Proof. This follows from the fact that for $y = gx$,

$$\frac{\partial}{\partial x_i} = \sum_j g_{ji} \frac{\partial}{\partial y_j}. \qquad \text{Q.E.D.}$$

§4. Maxwell's equation for electromagnetic field.

4.1. Maxwell's equation in a vacuum and its invariance.

An electromagnetic field on the Minkowski space $X = \mathbf{R}^4 = \{(x,y,z,t)\}$ of space-time is given by a pair $\{V(x,y,z,t), \mathbf{A}(x,y,z,t)\}$ of scalar field V and a vector potential $\mathbf{A}$, where $\mathbf{A}$ is an $\mathbf{R}^3$-valued function on X: $\mathbf{A} = {}^t(A_x, A_y, A_z)$, denoting by ${}^t(\dots)$ the transpose of a matrix or a numerical vector. Then the

Maxwell's equation in a vacuum of the eletromagnetic field is given as follows. Assume an additional condition as

(4.1) $\quad \operatorname{div} \mathbf{A} + \varepsilon_0\mu_0\dot{V} = 0, \quad$ where $\quad \operatorname{div} \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$

Then the Maxwell's equation in the case $\rho_m = 0$, $\mathbf{J}_m = 0$, is given by

(4.2) $\quad \Box V = -\rho_0/\varepsilon_0, \qquad \Box \mathbf{A} = -\mu_0\mathbf{J}_0,$

with $\quad \Box = \Delta - \varepsilon_0\mu_0\cdot\frac{\partial^2}{\partial t^2} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ (d'Alembertian),

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Laplacian in (x,y,z)).

Here $\rho_m(\cdot)$, $\mathbf{J}_m(\cdot)$ denote respectively magnetic density and magnetrocurrent density, ε_0, μ_0 are constants with $\varepsilon_0\mu_0 = 1/c^2$ (called permitivity and permiability respectively), ρ_0 and $\mathbf{J}_0$ denote respectively electric density and electric current density. Further, in the static case where V, ρ_0 and $\mathbf{J}_0$ do not depend on t, we have the following equation:

(4.3) $\quad \Delta V = -\rho_0/\varepsilon_0; \quad \Box\mathbf{A} = -\mu_0\mathbf{J}_0, \quad \operatorname{div} \mathbf{A} = 0.$

Introduce the polar co-ordinates (r,θ,φ) as

$$x = r \sin\theta\cdot\cos\varphi,\ y = r \sin\theta\cdot\sin\varphi,\ z = r\cos\theta,$$

then Δ is written as

$$(4.4)\qquad \Delta = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cdot\Delta_{S^2},$$

with
$$\Delta_{S^2} = \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\varphi^2}.$$

The differential operator Δ_{S^2} is equal to a constant multiple of the Laplace operator on the unit sphere $\mathbf{S}^2$ in $\mathbf{R}^3$ considered as a Riemannian manifold with the metric $d\theta^2 + \sin^2\theta\cdot d\varphi^2$. Note that on $\mathbf{R}^3$,

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\cdot d\varphi^2).$$

Now consider functions on X with values in a 4-dimensional vector space $W_1 \cong \mathbf{R}^4$ on which the Lorentz group $\mathfrak{L}_4$ acts covariantly:

$$\mathbf{a} \equiv {}^t(A_x,\ A_y,\ A_z,\ \tfrac{1}{c}V) \equiv {}^t(A_1,\ A_2,\ A_3,\ A_4),$$
$$\mathbf{j} \equiv {}^t(J_{0x},\ J_{0y},\ J_{0z},\ c\rho_0).$$

Then the equations (4.2) and (4.1) are rewritten respectively as

$$\Box \mathbf{a} = -\mu_0 \mathbf{j},$$

(4.5)

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right)\mathbf{a} \equiv \sum_{i=1}^{4} \frac{\partial A_i}{\partial x_i} = 0,$$

where $(x_1, x_2, x_3, x_4) = (x, y, z, ct)$.

The Lorentz group $\mathfrak{L}_4$ is defined as a connected component of the identity element of the group of 4×4 matrices $g = (g_{ij})_{1\leq i,j\leq 4}$ leaving the Minkowski's quadratic form $dx_1^{\,2} + dx_2^{\,2} + dx_3^{\,2} - dx_4^{\,2} = dx^2 + dy^2 + dz^2 - c^2dt^2$ invariant. Denote by I_{31} the diagonal matrix $\mathrm{diag}(1,1,1,-1)$ with diagonal elements 1, 1, 1, -1. Put

$$(4.6) \qquad SO(3,1) = \{g = (g_{ij})_{1\leq i,j\leq 4};\ gI_{31}{}^tg = I_{31}\},$$

and denotes its connected components of the identity by $SO_0(3,1)$. Then $\mathfrak{L}_4 = SO_0(3,1)$. The actions of $g \in \mathfrak{L}_4$ on $p = {}^t(x_1, x_2, x_3, x_4) \in X$ and also on $w = {}^t(w_1, w_2, w_3, w_4) \in W_1$ are given respectively by $p \longmapsto gp$ and $w \longmapsto T_g w \equiv gw$. The action on $\mathbf{a}$ and $\mathbf{j}$ of $g \in \mathfrak{L}_4$ is given as in (3.3) by

$$(U_g\mathbf{a})(p) = T_g(\mathbf{a}(g^{-1}p)), \quad p \in X.$$

We assert that the equation (4.5) is invariant (or rather better to say covariant) under $\mathfrak{L}_4$ in the following sense.

(1) For the first equation, the differential operator $\Box$ is itself invariant under $\mathfrak{L}_4$, because

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2} \quad \text{and} \quad gI_{31}{}^t g = I_{31} \quad (g \in \mathfrak{L}_4).$$

Therefore, under the action of $g \in \mathfrak{L}_4$, we get

$$\square(U_g \mathbf{a}) = U_g(\square \mathbf{a}),$$

on the left hand side of the equation, whereas $U_g \mathbf{j}$ on the right hand side. They are consistent with each other.

(2) For the second equation, take for instance the spaces

$$\mathbf{F} = C^\infty(X) \equiv \{C^\infty\text{-functions on } X\},$$
$$\mathbf{F}_1 = W_1 \otimes_{\mathbf{R}} \mathbf{F} \equiv \{W_1\text{-valued } C^\infty\text{-functions on } X\},$$

and consider a map from $\mathbf{F}_1$ to $\mathbf{F}$ as

$$D: \mathbf{F}_1 \ni \mathbf{a} \longmapsto \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right)\mathbf{a} \in \mathbf{F}.$$

On the spaces $\mathbf{F}_1$ and $\mathbf{F}$, we have the representations U_g and L_g of $\mathfrak{L}_4$. The map D intertwines them, that is,

$$D(U_g \mathbf{a}) = L_g(D\mathbf{a}) \quad (g \in G),$$

because the first order differential operators are transformed contravariantly to T_g under $p \longmapsto g^{-1}p$. We call this property the invariance of the second equation $\mathbf{D}\mathbf{a} = 0$ in (4.5).

We remark here that the operator U_g on $\mathbf{F}_1 = W_1 \otimes_{\mathbf{R}} \mathbf{F}$ is the tensor product of T_g on W_1 and L_g on $\mathbf{F}$, so that the representation U is nothing but the tensor product of T and L.

4.2. Application of representations of the rotation group to solve the Maxwell's equation.

Lut us consider the Maxwell's equation (4.3) in the static case. The equation on $\mathbf{A}$ is reduced to two equations, inhomogeneous one without time parameter t and homogeneous one:

$$(4.7) \qquad \Delta A = f_0, \qquad \operatorname{div} A \equiv \sum_{i=1}^{3} \frac{\partial A^i}{\partial x_i} = 0,$$

$$(4.8) \qquad \Box \mathbf{A} = 0, \qquad \operatorname{div} \mathbf{A} = 0,$$

where $f_0 = -\mu_0 \mathbf{J}_0$, a known W_0-valued function, $W_0 = \mathbf{R}^3$, and

$$A = A(x,y,z) = {}^t(A^1(x,y,z),\ A^2(x,y,z),\ A^3(x,y,z)).$$

for $(x,y,z) = (x_1,x_2,x_3)$. Further, introducing complex valued functions, we consider a solution of (4.8) of the form

$$\mathbf{A}(x,y,z,t) = A(x,y,z)\cdot e^{ikct} \quad \text{with } A(x,y,z) \text{ as above,}$$

then (4.8) turns out to

$$(4.9) \qquad (\Delta + k^2)A = 0, \qquad \operatorname{div} A = 0.$$

Now let us consider The Eq.(4.9). The 3-dimensional rotation group SO(3) is defined as

$$SO(3) = \{h = (h_{ij})_{1 \leqq i, j \leqq 3};\ hI_3{}^t h = I_3,\ \det h = 1\},$$

where I_3 denotes the identity matrix of degree 3. An element $h \in SO(3)$ acts on A by

$$(U_h^0 A)(q) = h(A(h^{-1}q)), \quad q \in \mathbf{R}^3.$$

The Eq.(4.9) is rotation-invariant in the sense that

$$\Delta(U_h^0 A) = U_h^0(\Delta A), \quad \operatorname{div}(U_h^0 A) = L_h(\operatorname{div} A).$$

Denote by $\mathbf{S}(k)$ the space of all the solutions of (4.9). Then it follows from the invariance above that $\mathbf{S}(k)$ is invariant under SO(3), that is, if $A \in \mathbf{S}(k)$, then $U_h^0 A \in \mathbf{S}(k)$ too.

On the other hand, the differential equation $(\Delta + k^2)A = 0$, is elliptic and therefore every $A \in \mathbf{S}(k)$ is real analytic. Introduce a scalar product in $\mathbf{S}(k)$ as

$$\langle A, B \rangle = \int_{\omega \in S^2} \{\sum_{i=1}^{3} A^i(\omega)\overline{B^i(\omega)}\}\, d\omega, \quad A, B \in \mathbf{S}(k),$$

where $d\omega$ denotes an SO(3)-invariant measure on S^2: $d\omega = \text{const}\cdot\sin\theta\, d\theta\, d\varphi$ in the polar co-ordinates $\omega = (\theta,\varphi)$, $r = 1$. Then $\mathbf{S}(k)$ becomes a pre-Hilbert space and the operators U_h^0 are unitary in the sense that

$$\langle U_h^0 A,\ U_h^0 B\rangle = \langle A,\ B\rangle, \quad A,\ B \in \mathbf{S}(k).$$

We can decompose this unitary representation U^0 on $\mathbf{S}(k)$ of SO(3) into an orthogonal direct sum of irreducible ones (actually the space $\mathbf{S}(k)$ is known to be complete). Knowing these facts, we can make elements of irreducible representations in $\mathbf{S}(k)$ of SO(3) play the role of fundamental solutions of the equation, like $\sin nx$, $\sin n\theta\cdot f(r)$ etc. in §2. Any solution can be expanded as a linear combination of these fundamental solutions. We see in [2, §8] that, using a moving frame for the bundle space W_0 at each point $q \in \mathbf{R}^3$ and also matrix elements of irreducible unitary representations of the group SO(3), the separation of variables can be achieved as in §2. Thus the problem is essentially reduced to solve ordinary differential equations in the variable r. This is the advantage of the "invariant" method using representation theory of SO(3).

We can apply also the "invariant" method to the inhomogeneous equation (4.7).

Remark 4.1. We may also utilize the 3-dimensional Euclidean motion group M_3, since the Eq.(4.9) is also invariant under M_3. Here M_3 is defined as the group of transformations on $\mathbf{R}^3$ given by $(h,q_0) \in SO(3) \ltimes \mathbf{R}^3$ as

$$\mathbf{R}^3 \ni q \longmapsto hq + q_0 \in \mathbf{R}^3,$$

and the action of (h,q_0) on A is given by

$$(U^0_{(h,q_0)}A)(q) = h(A((h,q_0)^{-1}q)), \quad q \in \mathbf{R}^3.$$

Utilization of representations of M_3 is rather delicate because M_3 is no longer compact contrary to $SO(3)$.

§5. Irreducible representations of the rotation group.

Since the present text becomes already sufficiently long, I should content myself with referring a classical paper [2] or a good text book [4] for this subject.

Added on July 20. === The earlier version of this text ended by the above sentence. However a friend of mine recomended me to write down some explicit informations about the subject of this section. So I add here the least minimum. ===

5.1. Covering map from SU(2) onto SO(3).

The 3-dimensional rotation group $SO(3)$ has $SU(2)$ as its (two-fold) covering group. A covering map π, which is a group homomorphism, from $SU(2)$ onto $SO(3)$ is given as follows.

We make $SL(2,\mathbf{C})$ act on the complex plane as a group of fractional linear trnasformations: for $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbf{C})$,

$$(5.1) \qquad \mathbf{C} \ni \zeta \longmapsto \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \in \mathbf{C}.$$

We denote ζ' by $g\zeta$, then $(gh)\zeta = g(h\zeta)$ $(g,\ h \in SL(2,\mathbf{C}))$ as is easily proved. To be more precise, we should take the projective complex plane $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$, since the denominator $\gamma\zeta + \delta$ may become zero. The identity transformation $\zeta \longmapsto \zeta$, is realized by two matrices $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $-I_2$. Note that an element g of the subgroup $SU(2)$ of $SL(2,\mathbf{C})$ is of the form

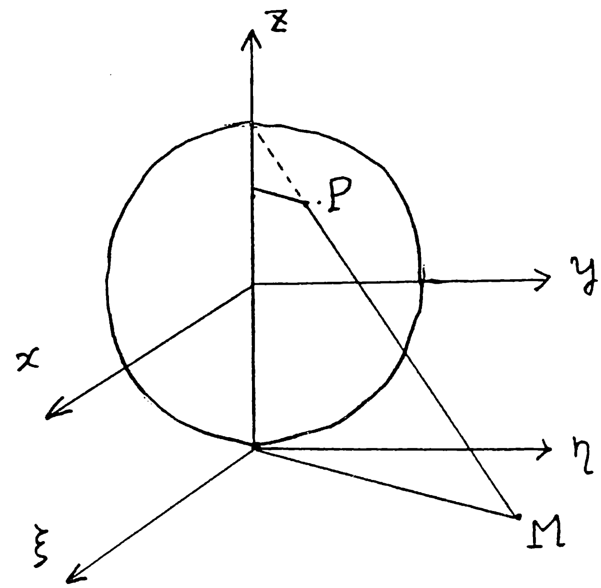
$$(5.2) \quad g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \ (\text{i.e., } \gamma = -\bar{\beta},\ \delta = \bar{\alpha}) \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1.$$

Now consider a stereographic projection from the unit sphere

$$(5.3) \qquad |x|^2 + |y|^2 + |z|^2 = 1 \quad \text{in} \quad \mathbf{R}^3$$

onto $\mathbf{C} \cup \{\infty\}$ given by

$$(5.4) \quad \zeta = \xi + i\eta = 2\cdot\frac{x + iy}{1 - z} \ \left(= 2\cdot\frac{1 + z}{x - iy}\right) \quad (i = \sqrt{-1}).$$



$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$M : \zeta = \xi + i\eta$

Then we can prove by calculations that a fractional linear transformation in (5.1) coming from g in (5.2) corresponds to a rotation $\pi(g)$ on the sphere and naturally that on the whole space $\mathbf{R}^3$. For example,

$$\text{for} \quad g = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}, \qquad \pi(g) = g_3(\varphi) \equiv \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{for} \quad g = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad \pi(g) = g_1(\theta) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

A general form of $\pi(g)$ for g in (5.2) is given by

$$\pi(g) = \begin{pmatrix} \frac{1}{2}(\alpha^2-\beta^2+\bar{\alpha}^2-\bar{\beta}^2) & \frac{i}{2}(\alpha^2+\beta^2-\bar{\alpha}^2-\bar{\beta}^2) & -\alpha\beta-\bar{\alpha}\bar{\beta} \\ \frac{i}{2}(-\alpha^2+\beta^2+\bar{\alpha}^2-\bar{\beta}^2) & \frac{1}{2}(\alpha^2+\beta^2+\bar{\alpha}^2+\bar{\beta}^2) & i(\alpha\beta-\bar{\alpha}\bar{\beta}) \\ \alpha\bar{\beta}+\bar{\alpha}\beta & i(\alpha\bar{\beta}-\bar{\alpha}\beta) & \alpha\bar{\alpha}-\beta\bar{\beta} \end{pmatrix}$$

As a conclusion, the kernel of π, $\mathrm{Ker}(\pi)$, is given as $\mathrm{Ker}(\pi)$

$= \{\pm I_2\}$, and therefore $SU(2)/\{\pm I_2\} \cong SO(3)$ through π.

5.2. Euler angles. A rotation expressed by the matrix $g_3(\varphi)$ (resp. $g_1(\theta)$) is the rotation of angle φ (resp. θ) arround the z-axis (resp. x-axis). Any rotation $g' \in SO(3)$ is expressed as a product of $g_3(\varphi_1)$, $g_1(\theta)$ and $g_3(\varphi_2)$ as

$$(5.5) \quad g' = g_3(\varphi_1)g_1(\theta)g_3(\varphi_2), \qquad 0 \leqq \varphi_1 \leqq 2\pi,\ 0 \leqq \theta \leqq \pi,\ 0 \leqq \varphi_2 \leqq 2\pi.$$

The angles $(\varphi_1, \theta, \varphi_2)$ are called the Euler angles of the rotation g', and by (5.5) we can introduce on $SO(3)$ global co-ordinates valid except a set of lower dimension. The decomposition (5.5) can be proved by purchasing movements of a unit tangent vector on the sphere under g' and also under the right hand side of (5.5).

The decomposition corresponding to (5.5) of $g \in SU(2)$ in (5.2) is given by

$$(5.6) \qquad g = u(\varphi_1)v(\theta)u(\varphi_2)$$

with $$u(\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}, \quad v(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Recall that $\pi(u(\varphi)) = g_3(\varphi)$, $\pi(v(\theta)) = g_1(\theta)$. Here $(\varphi_1, \theta, \varphi_2)$ is determined by

$$\cos\frac{\theta}{2} = |\alpha|, \quad \sin\frac{\theta}{2} = |\beta|,$$
$$\frac{1}{2}(\varphi_1+\varphi_2) = \arg(\alpha), \quad \frac{1}{2}(\varphi_1-\varphi_2+\pi) = \arg(\beta).$$

5.3. <u>Irreducible</u> <u>representations</u> <u>of</u> SL(2,C).

Finite-dimensional and holomorphic irreducible representations of the group SL(2,C) are parametrized, up to equivalence, by non-negative half integers $\ell \in (1/2)Z_{\geqq 0} = \{0,\ 1/2,\ 1,\ 3/2,\ \ldots\}$. (Note that this group has many infinite-dimensional irreducible representations by unitary operators on Hilbert spaces.)

A representation $(T_\ell,\ \mathbf{P}_\ell)$ corresponding to a parameter ℓ is given as follows. Let $\mathbf{P}_\ell$ be the vector space over $\mathbf{C}$ consisting of polynomials in ζ with complex coefficients of degree $\leqq 2\ell$. Then $\mathbf{P}_\ell$ has dimension $2\ell+1$ and a basis of it is given by

$$(5.7)\quad f_p(\zeta) = \zeta^{\ell-p}, \qquad p \in \Omega_\ell \equiv \{p \in (1/2)\mathbf{Z};\ -\ell \leqq p \leqq \ell,\ \ell-p \in \mathbf{Z}\}.$$

The operator $T_\ell(g)$ for $g \in SL(2,\mathbf{C})$ is given by the following formula: let $g^{-1} = \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$.

$$(5.8)\quad (T_\ell(g)f)(\zeta) = (\gamma\zeta+\delta)^{2\ell}\cdot f(g^{-1}\zeta) = (\gamma\zeta+\delta)^{2\ell}\cdot f\left(\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}\right).$$

In particular,

$$(T_\ell(g)f_q)(\zeta) = (\gamma\zeta+\delta)^{2\ell}\cdot\left(\frac{\alpha\zeta+\beta}{\gamma\zeta+\delta}\right)^{\ell-q} = (\gamma\zeta+\delta)^{\ell+q}(\alpha\zeta+\beta)^{\ell-q}.$$

Expanding the right hand side, we get

$$(T_\ell(g)f_q)(\zeta) = \sum_{p\in\Omega_\ell} a_{pq}(g) f_p(\zeta) \qquad \text{or} \qquad T_\ell(g)f_q = \sum_{p\in\Omega_\ell} a_{pq}(g) f_p.$$

Thus the linear transformation $T_\ell(g)$ is expressed with respect to the basis $\{f_q\}_{q\in\Omega_\ell}$ by a $(2\ell+1)\times(2\ell+1)$ matrix

$$(5.9) \qquad (a_{pq}(g))_{p,q\in\Omega_\ell}.$$

Note that $g^{-1} = u(-\varphi)$ for $g = u(\varphi)$ and so $T_\ell(u(\varphi))f_q = e^{iq\varphi}\cdot f_q$. Therefore the matrix $(a_{pq}(u(\varphi)))_{p,q}$ is a diagonal matrix with diagonal elements $e^{i\ell\varphi}$, $e^{i(\ell-1)\varphi}$, ..., $e^{-i(\ell-1)\varphi}$, $e^{-i\ell\varphi}$.

5.4. <u>Irreducible representations of</u> SU(2) <u>and those of</u> SO(3).

Restrict the representation $(T_\ell, \mathbf{P}_\ell)$ of $SL(2,\mathbf{C})$ to its subgroup $SU(2)$, then we see that it remains still irreducible. We denote this irreducible representation of $SU(2)$ again by the same simbol T_ℓ. Any irreducible representation of $SU(2)$ is finite-dimensional (since $SU(2)$ is compact) and is equivalent to T_ℓ for some ℓ.

If ℓ is an integer, then, for $g' = -I_2$ in the center of $SU(2)$, we have $T_\ell(-I_2) = I$, the identity operator on $\mathbf{P}_\ell$. If ℓ is not an integer, i.e., $\ell = 1/2, 3/2, 5/2, \ldots$, then $T_\ell(-I_2) = -I$. Therefore, according as ℓ is an integer or not, T_ℓ of $SU(2)$ gives a one-valued or a two-valued irreducible representation of $SO(3)$ through $\pi\colon SU(3) \longrightarrow SO(3) \cong SU(2)/\{\pm I_2\}$. They exhaust irreducible representations of $SO(3)$ up to equivalence.

5.5. Matrix lements of irreducible representations of SO(3).

Now let $\ell \geqq 0$ be an integer for simplicity and denote T_ℓ by T'_ℓ when it is considered as a representation of $SO(3)$: $T'_\ell(g') = T_\ell(g)$ for $g' = \pi(g)$, $g \in SU(2)$. Then the space $\mathbf{P}_\ell$ contains $f_0(\xi) = \xi^\ell$, which is invariant under $g_3(\varphi)$ since $T'_\ell(g_3(\varphi))f_0 \equiv T_\ell(u(\varphi))f_0 = f_0$. Consider the matrix elements $a_{pq}(g)$ as functions in $g' = \pi(g)$, and denote it by $a'_{pq}(g')$. Then we get

$$(5.10)\quad a'_{pq}(g_3(\varphi_1)g_1(\theta)g_3(\varphi_2)) = e^{ip\varphi_1}\cdot a'_{pq}(g_1(\theta))\cdot e^{iq\varphi_2}.$$

The function $a'_{pq}(g_1(\theta))$ can be calculated explicitly and be expressed using Legendre's functions in the variable $\cos\theta$ or $\sin\theta$. In particular,

$$a_{00}(g_3(\theta)) = \mathrm{const}\cdot P_\ell(\cos\theta),$$

where P_ℓ is the Legendre's polynomial of degree ℓ:

$$P_\ell(\mu) = \frac{(-1)^\ell}{2^\ell\cdot \ell!}\cdot\frac{d^\ell}{d\mu^\ell}(1-\mu^2)^\ell.$$

Finally we remark the following. Consider

$$a'_{p0}(g_3(\varphi)g_1(\theta)g_3(\varphi')) = a'_{p0}(g_3(\varphi)g_1(\theta)) \ (= F_p(\theta,\varphi)\ (\mathrm{put}))$$

as a function in (θ,φ). Further consider (θ,φ) as the co-ordinates of a point on the unit sphere S^2 in $\mathbf{R}^3$, as in §4.1. Then all the functions F_p, $p \in \Omega_\ell$, give a complete system of linearly independent eigenfunctions with eigenvalue $-\ell(\ell+1)$, for the invariant differential operator Δ_{S^2} in (4.4), Laplacian on S^2:

$$\Delta_{S^2}(F) = -\ell(\ell+1)\cdot F \quad \text{or} \quad \{\Delta_{S^2} + \ell(\ell+1)\}F = 0 \quad (F \in C^\infty(S^2)).$$

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Appendix 1. Fundamental definitions.

Here we give the exact definitions for some fundamental things.

1. Definition of a group. We call G a group if it is a set equipped with an operation $G \times G \ni (g,h) \longmapsto gh \in G$ which satisfies the following axioms.

(i) There holds the associative law: $(gh)k = g(hk)$ $(g, h, k \in G)$.

(ii) There exists an element $e \in G$ such that $eg = ge = g$ for any $g \in G$.

(iii) For every $g \in G$, there exists an element $h \in G$ such that $hg = gh = e$.

The element e in (ii) is unique and called the identity element of G, and the element h for g in (iii) is also unique and called the inverse of g and denoted by g^{-1}.

2. Action of a group. Let G be a group and X a set. Then we say that G acts on X if, for every $g \in G$, there corresponds a transformation on X, denoted as $x \longmapsto gx$ $(x \in X)$, which satisfies the following

$$\text{(A1.1)} \qquad \begin{aligned} ex &= x \quad (e = \text{the identity element of } G), \\ (gh)x &= g(hx) \quad (g, h \in G, x \in X). \end{aligned}$$

3. Linear representation of a group.

Let W be a vector space over a scalar field K $(= \mathbf{R}$ or $\mathbf{C})$. Assume that, for every $g \in G$, there corresponds a linear transformation T_g on W which satisfies

$$(\text{A1.2})\qquad \begin{aligned} T_e &= I \quad (\text{the identity operator on } W),\\ T_{gh} &= T_g T_h \quad (g,\ h \in G),\end{aligned}$$

that is, $G \ni g \longmapsto T_g \in GL(W)$, the group of all invertible linear transformations on W, is a group homomorphism. If W is a topological vector space, we usually assume every T_g is continuous and also assume a certain continuity on the correspondence $G \ni g \longmapsto T_g \in GL(W)$. We sometimes call W, equipped with T, a G-module over K.

We say (T, W) is <u>irreducible</u> if W has no non-trivial invariant subspace.

4. <u>Equivalence of two representations, tensor products</u>.

Let (T^i, W_i), $i = 1, 2$, be two representations of a group G. Then we say that they are mutually <u>equivalent</u> if there exists an invertible linear operator A of W_1 onto W_2 which intertwines T^1 with T^2, that is, $A \cdot T_g^1 = T_g^2 \cdot A$ for $g \in G$.

The tensor product $U = T^1 \otimes T^2$ of two representations T^1 and T^2 is defined on the space $W = W_1 \otimes_K W_2$ by

$$U_g = T_g^1 \otimes T_g^2 \quad (g \in G).$$

5. <u>Matrix groups (linear groups) and Lorentz groups</u>.

Let I_{mn} be a diagonal matrix of type $(m+n)\times(m+n)$ with diagonal elements $1, 1, \ldots, 1$ (m-times), $-1, -1, \ldots, -1$ (n-times). Then the group $O(m,n)$ and $SO(m,n)$ are defined as follows:

$$\text{(A1.3)}\quad \begin{aligned} O(m,n) &= \{g \in GL(m+n,\mathbf{R});\ gI_{mn}{}^{t}g = I_{mn}\}, \\ SO(m,n) &= \{g \in O(m,n);\ \det g = 1\}, \end{aligned}$$

and $SO_0(m,n)$ denotes the connected component of the identity of $SO(m,n)$. When $n = 0$, we get orthogonal groups $O(m)$ and $SO(m)$. In (A1.3), $GL(m+n,\mathbf{R})$ denotes the group of all $(m+n)\times(m+n)$ matrices with determinant $\neq 0$, and so $GL(m+n,\mathbf{R}) \cong GL(\mathbf{R}^{m+n})$.

The (homogeneous) Lorentz group is $O(3,1)$ and the proper Lorentz group $\mathfrak{L}_4$ is $SO_0(3,1)$. The inhomogeneous Lorentz group $\mathfrak{L}_4'$ is the semidirect product of the linear group $\mathfrak{L}_4$ and the group $\mathbf{R}^4 \cong$ the group of all the translations in the Minkowski space X. The group $\mathfrak{L}_4'$ acts on X as follows: for $(g,q) \in \mathfrak{L}_4 \ltimes \mathbf{R}^4$,

$$X \ni x = {}^{t}(x_1,x_2,x_3,x_4) \longmapsto x' = gx + q \in X.$$

Let us represent $x \in X$ by a vector $\underset{=}{x} = {}^{t}(x_1,x_2,x_3,x_4,1)$, then the above transformation is expressed in the matrix multiplication form as

$$(g,q):\ \underset{=}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} \longmapsto \underset{=}{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ 1 \end{pmatrix} = \left(\begin{array}{cccc|c} & & & & q_1 \\ & & & & q_2 \\ & (g_{ij}) & & & q_3 \\ & & & & q_4 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ 1 \end{pmatrix},$$

where $g = (g_{ij})_{1\leq i,j\leq 4}$ and $q = {}^{t}(q_1,q_2,q_3,q_4)$. Denote by

$T_{(g,q)}$ the above 5×5 matrix, then the correspondence $(g,q) \longmapsto T_{(g,q)}$ gives a faithful linear representation of $\mathfrak{T}_4'$ on $\mathbf{R}^4$.

Appendix 2. Actions of the symmetric groups.

Let $X_n = \{1, 2, \ldots, n\}$ and $G = \mathbf{S}_n$ be the group of all permutations on X_n, called n-th symmetric group:

$$(\sigma\tau)(i) = \sigma(\tau(i)) \qquad \text{for } \sigma, \tau \in \mathbf{S}_n, \ i \in X_n. \tag{A2.1}$$

Every element $\sigma \in \mathbf{S}_n$ is expressed by

$$\begin{pmatrix} 1 & 2 & 3 & \cdots\cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots\cdots & \sigma(n) \end{pmatrix},$$

and the product $\sigma\tau$ of $\sigma, \tau \in \mathbf{S}_n$ can be calculated using this expression.

Now consider the vector space V_n consisting of all real-valued functions on X_n. Then $V_n \cong \mathbf{R}^n$ by the correspondence Φ: $V_n \ni \varphi \longmapsto x = (x_i)_{1\leqq i\leqq n} \in \mathbf{R}^n$ with $x_i = \varphi(i)$ $(i \in X_n)$. The linear representation of $\mathbf{S}_n$ on V_n canonically induced from the action on X_n is

$$(\sigma\varphi)(i) = \varphi(\sigma^{-1}(i)) \qquad (i \in X_n), \tag{A2.2}$$

and through Φ, it is transformed on $\mathbf{R}^n$ as

$$(\sigma x)_i = x_{\sigma^{-1}(i)} \qquad (i \in X_n). \tag{A2.3}$$

In this way, to an element $\sigma \in \mathbf{S}_n$, there corresponds an $n\times n$ matrix $\pi(\sigma)$ in $GL(n,\mathbf{R})$ given by

$$\text{(A2.4)} \quad \pi(\sigma) = (g_{ij}), \quad g_{ij} = \begin{cases} 1 & \text{if } j = \sigma^{-1}(i), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathbf{S}_n \ni \sigma \longmapsto \pi(\sigma) \in GL(n,\mathbf{R})$ is a matrix representation of $\mathbf{S}_n$, and $\mathbf{S}_n$ acts on $X = \mathbf{R}^n$ by linear transformations. This action on X induces in its turn a linear representation of $\mathbf{S}_n$ on each ($\mathbf{S}_n$-) invariant vector space $\mathbf{F}$ consisting functions on $X = \mathbf{R}^n$. For instance, take $\mathbf{F} = C(\mathbf{R}^n)$, $C^\infty(\mathbf{R}^n)$, $P(\mathbf{R}^n) \equiv$ the space of all polynomials on $\mathbf{R}^n$, and so on. Then L_σ on $\mathbf{F}$, $\sigma \in \mathbf{S}_n$, is given by

$$(L_\sigma f)(x) = f(\sigma^{-1}x) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),$$

that is, L_σ is nothing but a permutation of the variables x_1, x_2, $\ldots$, x_n.

We call a function f <u>symmetric</u> if $L_\sigma f = f$ for any $\sigma \in \mathbf{S}_n$. Denote by $D = D(x_1, x_2, \ldots, x_n) \in P(\mathbf{R}^n)$ the difference product:

$$D = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Then $L_\sigma D = \mathrm{sgn}(\sigma) D$ $(\sigma \in \mathbf{S}_n)$, where $\mathrm{sgn}(\sigma)$ is the sign of σ. A function f is called <u>alternating</u> if $L_\sigma f = \mathrm{sgn}(\sigma) f$ for any $\sigma \in \mathbf{S}_n$. We know the following

Theorem A2.1. (i) The space $P(\mathbf{R}^n)$ contains as its proper subspace a direct sum of the space of symmetric polynomials $S(\mathbf{R}^n)$ and that of alternating polynomials $A(\mathbf{R}^n)$.

(ii) The space $S(\mathbf{R}^n)$ is generated freely by the following fundamental symmetric polynomials: for $0 \leqq k \leqq n$,

$$P_k = \sum x_{i_1} x_{i_2} \dots x_{i_k} ,$$

where the sum runs over all $1 \leqq i_1 < i_2 < \dots < i_k \leqq n$.

(iii) Every element in $A(\mathbf{R}^n)$ is a product of D and a Q in $S(\mathbf{R}^n)$.

We remark that the assertion (i) says that the representation $(L, P(\mathbf{R}^n))$ contains the direct sum $(\mathbf{1}, S(\mathbf{R}^n)) \oplus (\mathrm{sgn}, A(\mathbf{R})$ as its subrepresentation, where $\mathbf{1}$ denotes the trivial representation of $\mathbf{S}_n$ or its multiple. Moreover it is known that any other irreducible representations are contained also in $P(\mathbf{R}^n)$ modulo equivalence.

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Part II

General theory of unitary representation of locally compact groups

by Nobuhiko Tatsuuma

Preface

This presentation is an introductory short communication for the theory of unitary representations of general locally compact groups and the duality theorem for such groups without proof. We start from the preliminary definitions about locally compact groups.

The Pontryagin duality theorem for abelian locally compact groups and the Tannaka duality theorem for compact groups are famous, and have so many applications in wide fields. But there exists a theory which concludes these two duality theorems, duality theorem for genaral locally compact groups. This duality theorem shows that the so-called regular represantation has complete information of the base group.

contents

General theory of unitary representations of locally compact groups

N.Tatsuuma

26th April 1988

§0 Purpose of this talk

Definition 1. G :locally compact group is a topological group[(1)] with compact[(2)] neighborhood of the unit[(3)] in G.

(1) Topological group is a group with topology under which the group operations are continuous, i.e.

$$G\times G \ni (g_1, g_2) \to g_1^{-1}g_2 \in G \text{ ; continuous.}$$

(2) We assume T_2 (i.e. Hausdorff) separating property for the definition of "compact".

(3) By the group structure, this is equivalent to the existence of compact neighborhoods for any point in G.

Definition 2. $D \equiv \{\underline{H}, U_g\}$; unitary representation of G is a strong continuous homomorphism[(3)] $G \ni g \to U_g \in U(\underline{H})$ from G to the group $U(\underline{H})$ of all unitary operators on a Hilbert space[(1)] $\underline{H}$.

(1) $\underline{H}$; Hilbert space is a complete topological vector space[(a)] on the field of complex numbers (which is denoted by **C**) with inner product[(b)] $\langle\ ,\ \rangle$.

(a) Vector space with topology τ under which the operations "addition +" and "scalar multiplication ·" are continuous.

$$\mathbf{C}\times\underline{H}\times\mathbf{C}\times\underline{H} \ni (a,u,b,v) \to a\cdot u + b\cdot v \in \underline{H} \text{ : continuous.}$$

(b) $\langle\ ,\ \rangle$ is a positive sesqui-linear form.

$\underline{H}\times\underline{H} \ni (u,v) \to \langle u,v\rangle \in \mathbf{C}$ such that $\langle u,u\rangle > 0$ for all $u(\neq 0) \in \underline{H}$ and τ is given by the norm $\|u\| = (\langle u,u\rangle)^{1/2}$, and $\underline{H}$ is complete.

(2) Put

$B(\underline{H}) \equiv \{A; \text{bounded(i.e.continuous) linear operator on } \underline{H}\}$,

$U(\underline{H}) \equiv \{A \in B(\underline{H}); AA^* = A^*A = I \text{(identity operator), i.e. unitary}\}$.

A^*: the conjugate of A defined by $\langle A^*u, v\rangle = \langle u, Av\rangle$ for all $u, v \in \underline{H}$.

(3) The strong topology ν on $B(\underline{H})$ is given by the family of seminorms $\{\ \nu_u\ ;\ \nu_u(\cdot) \equiv \|\cdot u\|,\ u \in \underline{H}\ \}$.

<u>THE PURPOSE OF THIS TALK</u>. Based on such definitions, to investigate properties of unitary representations of locally compact groups.

§1. **Properties of locally compact groups.**

a) HAAR MEASURE.

<u>Theorem(A.Weil)</u>. 1) For any locally compact group G, there exists a right-invariant(Haar) measure μ_r.

2) Right Haar measure is unique up to constant.

3) There exists a continuous real positive character

Δ_G: $G \ni g \to \Delta_G(g) \in \mathbf{R}^+$, i.e.

$\Delta_G(g_1g_2) = \Delta_G(g_1)\cdot\Delta_G(g_2)$ for all $g_1, g_2 \in G$ and

$\mu_r(gE) = \Delta_G(g)\mu_r(E)$ for all E : measurable set, $g \in G$.

4) $d\mu_r(g^{-1}) = \Delta_G(g^{-1})d\mu_r(g)$ becomes a left-invariant measure.

(<u>definition</u>. $\int_G f(g)\, d\mu_r(g^{-1}) \equiv \int_G f(g^{-1})\, d\mu_r(g)$.)

Hereafter we denote shortly $d\mu_r(g) \equiv d_r g$.

<u>Definition 3.</u> G is <u>unimodular</u> when $\Delta_G \equiv 1$.

H ($\subset G$) closed subgroup, $X \equiv H\backslash G$: factor space.

Then X becomes a locally compact space under quotient topology. Denote $G \ni g \to \pi(g) \equiv \tilde{g} = Hg \in H\backslash G = X$ (canonical map).

Definition 4. μ on X is quasi-invariant iff

$\mu(\cdot g) \sim$ (equivalent) $\mu(\cdot)$ for all $g \in G$.

μ on X is relative invariant iff

$\exists$ character $\tilde{\Delta}$ on G, $\mu(\cdot g) = \tilde{\Delta}(g)\mu(\cdot)$ for all $g \in G$.

μ on X is invariant iff $\mu(\cdot g) = \mu(\cdot)$ for all $g \in G$.

Theorem(A.Weil). 1) For all $X = H\backslash G$, $\exists$ quasi-invariant measure μ.

2) All quasi-invariant measure on X are mutually equivalent (with Raikov's proof).

3) $\exists$ Invariant measure on X iff $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$.

4) $\exists$ Relative invariant measure on X iff

$(\Delta_G|_H)/\Delta_H$ extendable to a continuous character on G.

Lemma(F.Bruhat). For any continuous real character χ on H, there exists a continuous(C^∞ for Lie group) function ψ s.t.

$$\psi(hg) = \chi(h)\psi(g) \quad \text{for all } h \in H, \ g \in G.$$

Proposition. 1) For the case of $\chi(h) = (\Delta_H(h)/\Delta_G(h))$, a measure μ on X is given by the followings.

$$\mu(\tilde{f}) (\equiv \int_X \tilde{f}(\tilde{g})\, d\mu(\tilde{g})) = \int_G f(g)\chi(g)\, d_r g,$$

here $\tilde{f}(\tilde{g}) \equiv \int_H f(hg)\, d_r h$,

for $f \in C_0(G)$: continuous functions with compact supports.

2) μ is a quasi-invariant measure on $X = H\backslash G$.

3) $w(\tilde{g}_1, g_0)$ $(\equiv d\mu(\tilde{g}_1 \cdot g_0)/d\mu(\tilde{g}_1)) = \psi(g_1 g)/\psi(g_1)$.

(Caution!!) In the definition of Haar measure, the following properties are important.

1) It is a regular measure, defined on the Borel field

generated by all relatively compact open set in G.

2) Every open measurable set has positive measure.

3) Every compact set has finite measure.

Example 1. For non σ compact locally compact group G, if all open set in G is measurable , invariant regular measure does not exist.

Example 2. $\mathbf{R}^d$ (discrete additive group), put

$\mu_1(E) = 0$ (for countable set E), $= \infty$ (others).

$\mu_2(E) = \#E$.

Then both are invariant ~~regular~~ measures.

Example 3. On $\mathbf{R}$ (ordinary additive group). Consider

$\mu_1(E) = \#E$

μ_2: ordinary Lebesgue measure.

Then both are invariant regular measures.

Theorem(Y.Yamasaki). For infinite dimensional Hilbert space (additive group), no translation quasi-invariant measure exists.

Theorem(Weil's inverse Theorem). Let G be a group, $\mathbf{B}$ Borel structure on G , s.t.

1) the map $G\times G \ni (g_1, g_2) \to g_1^{-1}g_2 \in G$ is $\mathbf{B}$-measurable.

2) the map $G\times G \ni (g_1, g_2) \to (g_1g_2, g_2) \in G\times G$ is $\mathbf{B} \times \mathbf{B}$-measurable.

If there exists a G-invariant measure μ on $(G, \mathbf{B})$ then

1) $\exists\ \tilde{G}$:a locally compact group which contains G densely.

2) For $\tilde{\mu}$;Haar measure on $\tilde{G}$, $\tilde{\mu}|G = c\cdot\mu$, c : constant.

b) STRUCTURAL THEOREM.

Theorem(D.Montgomery & L.Zippin). For all connected locally compact group G and all neighborhood V of e, $\exists$compact normal subgroup $H \subset V$ s.t. G/H is a Lie group.

{e}		{e}

$$\begin{array}{ccccccccc}
 & & \{e\} & & & & \text{Infinite} & & \\
 & & | & & & & | & & \\
\{e\} \to & \text{Connected Solvable Lie Group} \to & \text{Connected Lie Group} & \to \text{Connected Semisimple Lie Group} \to \{e\} & & & \text{Infinite Discrete Group} & & \\
 & & | & & & & | & & \\
\{e\} \to & & \text{Connected Locally Compact Group} & \to & \text{LOCALLY COMPACT GROUP} & \to & \text{Totally Disconnected Group} & \to & \{e\} \\
 & & | & & & & | & & \\
 & & \text{Connected Compact Group} \sim \text{Proj}(\text{Compact Lie Group}) & & & & \text{Compact Totally Disconnected Group} \sim \text{Proj}(\text{Finite Group}) & & \\
 & & | & & & & | & & \\
 & & \{e\} & & & & \{e\} & &
\end{array}$$

§2. **Unitary Representation**

a) CONTINUITY.

Only here, we consider non-unitary representations.

Let E be a locally convex topological vector space,

$\underline{B}^{\times}(E) \equiv \{ A$; bounded, inverse bounded operators on E $\}$.

Call a <u>representation</u> $\{ E , A_g \}$ of G on E ,

$G \ni g \to A_g \in \underline{B}^{\times}(E)$: weak continuous group homomorphism.

<u>Proposition</u>.(See G.Warner : Harmonic Analysis on Semi Simple Lie Groups 1. (1972) Springer P.237 Prop4.2.2.1)

For $G \ni g \to A_g \in \underline{B}^{\times}(E)$ group homomorphism on separable Banch space E. The following (1),(2),(3) are mutually equivalent.

(1) The map $G \times E \ni (g, v) \to A_g v \in E$ is continuous.

(2) For all $v \in E$, the map $G \ni g \to A_g v \in E$ is continuous.

(3) For all $v \in E$, $\varphi \in E^*$ (dual space of E), the map

$$G \ni g \to \langle A_g v, \varphi \rangle \in \mathbf{C} \text{ is continuous.}$$

Analogously, we can obtain

<u>Proposition(no references)</u>. For $G \ni g \to A_g \in \underline{B}^{\times}(E)$: group homomorphism, assume that,

1) E is a reflexive locally convex topological vector space,

2) for all $v \in E$, $\varphi \in E^*$, $G \ni g \to \langle A_g v, \varphi \rangle \in \mathbf{C}$ is locally bounded and measurable,

3) there exists a neighborhood V of e in G,s.t. for all $v \in E$, $\{ A_g v \ ; \ g \in V \}$ spans a separable subspace of E.

Then for all $v \in E$, the map $G \ni g \to A_g v \in E$ is continuous. (we call 2)+3) <u>strongly measurable</u>.)

These properties come from same reason as the following famous fact, which shows the relation between the topology and Haar measure of G.

<u>Proposition</u>. Put $\mathfrak{L}^p(G) \equiv \{ f;$ measurable function on G,

$$\text{s.t. } \|f\| \equiv \int_G |f(g)|^p d_r g)^{1/p} < \infty \}, \ (1 \leq p < \infty).$$

(Precisely we must take equivalence classes "up to measure zero".)

Consider the right translation,

$$\mathfrak{L}^p(G) \ni f \to R_{g_0} f(g) \equiv f(gg_0).$$

Then, for all $f \in \mathfrak{L}^p(G)$, the map $G \ni g \to R_g f \in \mathfrak{L}^p(G)$ is continuous.

Definition 5. $\{ \mathfrak{L}^p(G), R_g \}$ $(1 \leq p < \infty)$ is called regular representation of G. Hereafter we restrict this word to the case p = 2. In this case $\mathfrak{L}^2(G)$ is a Hilbert space.

Example. As an example of non continuous unitary represe-ntation, here we quate a "non-measurable" character on **R**, which is constructed using "Hamel basis". (Precise discussion is omitted.)

2) COMPLETELY REDUCIBLE PROPERTY, IRREDUCIBILITY.

Go back to unitary case, let $\mathbf{D} = \{ \underline{H}, U_g \}$ be a unitary representation of G.

Proposition. Let $\underline{H}_1$ be a G-invariant closed subspace in $\underline{H}$ i.e. for all $g \in G$, $U_g \underline{H}_1 \subset \underline{H}_1$.

Then $\underline{H}_1^{\perp} \equiv \{ v \in \underline{H} \ ; \ \langle v, u \rangle = 0$ for all $u \in \underline{H}_1 \}$ is G-invariant, too.

(Remark) This property comes from only the $*$-invariant property of the family of operators $\{ U_g ; g \in G \}$.

Example. $G \equiv \mathbf{R} \ni t \to \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in M(\mathbf{C}^2)$ (matrices on $\mathbf{C}^2$) : non-unitary representation. $\underline{H}_1 \equiv \{ \begin{pmatrix} y \\ 0 \end{pmatrix} ; y \in \mathbf{C} \}$ is G-invariant, but $\underline{H}_1^{\perp} \equiv \{ \begin{pmatrix} 0 \\ x \end{pmatrix} ; x \in \mathbf{C} \}$ is not G-invariant.

Definition 6. $\mathbf{D} = \{ \underline{H}, U_g \}$; irreducible iff no G-invariant closed subspace in $\underline{H}$ exists except $\{0\}$ and $\underline{H}$.

Definition 7. For a family $\{ \mathbf{D}_\alpha \equiv \{ \underline{H}_\alpha, U_g^\alpha \} \}$ of unitary representations of G, we call $\sum_\alpha^\oplus \mathbf{D}_\alpha \equiv \{ \sum_\alpha^\oplus \underline{H}_\alpha, \sum_\alpha^\oplus U_g^\alpha \}$

the <u>(discrete) direct sum</u> of $\{D_\alpha\}$'s.

<u>Corollary</u>. All finite dimensional unitary representations have irreducible decompositions. (In general, not unique !!.)

<u>Definition 8</u>. For $D= \sum_\alpha^\oplus D_\alpha$, D_α are called components of D.

<u>Example</u>(∞-dimensional case). $G = \underline{T}$ (1-dimensional torus) $\cong \{ e^{\sqrt{-1}\theta} ; -\pi < \theta \leqq \pi \}$. By Fourier expansion,

$$\underline{R}(\text{regular representation}) \equiv \{ L^2(T), R_g \} = \sum_{j=-\infty}^{\infty} {}^\oplus \{C, e^{\sqrt{-1}j\theta} \}$$

(direct sum of 1-dimensional i.e. irreducible unitary representations of G.)

Extending the notion of "direct sum", we can define "direct integral" or "continuous direct sum", $\int_X D_x \, d\nu(x)$

Here $D_x = \{ \underline{H}_x, U_g^x \}$ ($x \in X$) are unitary representations.

I am sorry that precise discussions must be omitted, because there exist some complicated situations and many pathological phenomena. And I quote here the NOMURA's talk (not reproduced here). But we have to state,

<u>Theorem(S.Teleman)</u>. (Rev. Roum. Math. pures et Appl. <u>21</u> (1976) pp 465-486.)

All D(unitary representation of G) is decomposed to a continuous direct sum of irreducible representations,

$$D \cong \int_X D_x \, d\nu(x) \quad ,$$

(D_x: irreducible unitary representations of G.)

<u>Example</u>. If ν is a point mass, $\int_X D_x \, d\nu(x) = \sum_{x \in [\nu]}^{\oplus} D_x$

($[\nu]$: support of ν), that is a discrete direct sum.

Remark. Such a decomposition is not unique **essentially**.

Example(H.Yoshizawa). For $G=F_2$(discrete free group with 2-generators), there exist two irredusible decompositions,

$$\int_X D_x \, d\nu(x) \cong \int_Y D_y \, d\tau(y),$$

for which D_x are not equivalent to D_y for all pairs (x, y).

An important conclusion from the existence of irreducible decomposition of the regular representation $\underline{R}$, is obtained.

Theorem(I.M.Gelfand & A.Raikov). For any locally compact group G , there exist sufficiently many irreducible unitary representations. (cf. NOMURA's talk)

Example(as a remark). Let $D \equiv \{\underline{H}, T_g\}$ be a fixed unitary representation of G, (X, μ) a measure space. Put $D_x \equiv (\underline{H}_x, T_g^x) = D$ (for all $x \in X$) and $\mathcal{L}_X^2(\underline{H}, \mu)$($\underline{H}$-valued $\mathcal{L}^2$-functions)$\sim \int \underline{H}_x d\mu(x)$ ($\underline{H}_x \sim \underline{H}$) and consider U_g on it as $U_g \cong \int_X T_g^x \, d\mu(x)$.

On the other hand, take a C.O.N.S. $\{\varphi_\alpha\}$ in $\mathcal{L}_X^2(\mu)$ and consider closed subspaces $\underline{H}_\alpha \equiv \{v\varphi_\alpha(x);\ v \in \underline{H}\}$ in $\mathcal{L}_X^2(H, \mu)$. Put $D_\alpha \equiv \{ \underline{H}_\alpha, \Psi_{g|\underline{H}_\alpha} \} \sim D$. Then we get

$$\int_X D_x \, d\mu(x) \cong \sum_\alpha^{\oplus} D_\alpha \ , \text{ symbolically } \int_X D \, d\mu(x) \sim \sum^{\oplus} D.$$

3) SCHUR'S LEMMA.

Definition 9. For $D_j \equiv \{\underline{H}_j, U_g^j\}$ $(j = 1,2)$(unitary representations) of G , $A \in \mathbf{B}(\underline{H}_1, \underline{H}_2)$ (bounded operators from $\underline{H}_1$ to $\underline{H}_2$) is an intertwining operator (between D_1 and D_2) iff

$$A \cdot U_g^1 = U_g^2 \cdot A \quad \text{for all } g \in G.$$

Notation. $I(D_1, D_2) \equiv \{A \in B(\underline{H}_1, \underline{H}_2)$; intertwining operator between D_1 and D_2).

Theorem(Schur's lemma). A unitary representation, $D=\{\underline{H},U_g\}$ is irreducible iff $I(D,D)=\{cI \ ; \ c\in C\}$(scalar operator).

Remark. 1) In this LEMMA, the assumption of "boundedness" of $I(D,D)$ can be loosen to "closedness".

2) This LEMMA depends only on the $*$-invariant property of $\{U_g\}$.

Example. $G\equiv\{g=\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}; \ a\in R^+, \ b\in R\}$. Consider the nonunitary 2-dimensional representation

$$D \ ; \ g \to \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \quad \text{on } C^2.$$

Then $I(D,D)=\{cI\}$, but D is not irreducible.

Corollary. Let $D_j=\{\underline{H}_j, \ U_g^j\}$ be two mutually equivalent irreducible unitary representations of G. Then there exists unique surjective isometric operator U_0 up to constant from $\underline{H}_1$ to $\underline{H}_2$, such that

$$I(D_1,D_2)=\{cU_0 ; \ c\in C\}.$$

4) CONJUGATE REPRESENTATION.

For instance, let $D\equiv\{\underline{H},T_g\}$ be a finite dimensional unitary representation of G, $\{v_j\}_{j=1,..,n}$ C.O.N.S.(complete orthonormal system) in $\underline{H}$. Then T_g is represented by a matrix

$$T_g\equiv(T_{ij}(g))_{i,j}=(\langle T_g v_i, \ v_j\rangle)_{i,j}.$$

Consider a homomorphism: $G\ni g \to \overline{(T_g)}\equiv(\overline{T_{ij}(g)})$,

("-"means complex conjugate). Then this map gives a unitary representation on the same space $\underline{H}$, too.

Example. 1) $G=R\ni t \to e^{\sqrt{-1}t} \in C$. We take the conjugation $: t \to e^{-\sqrt{-1}t}$.

Then $\{C, e^{-\sqrt{-1}t}\}$ is a 1-dimensional representation.

2) $G=Rot(3)$ operates on $\mathbf{R}^3$ as rotations, we can extend it naturally on $\mathbf{C}^3$ and get a unitary representation. This representation is obviously self-conjugate.

Definition 10(G.W.Mackey's definition). The map $\underline{H}$(Hilbert space)$\ni v \to \langle\ .,\ v\ \rangle \in \underline{H}^*$(dual as a Banach space) is conjugate linear, and the family of operators,

$\{U_g^*;\underline{H}^* \ni \langle\ .,\ v\ \rangle \to \langle\ .,\ U_g v\ \rangle \in H^*\}$ gives a unitary representation $D^* \equiv \{\underline{H}^*,\ U_g^*\}$ of G.

We call D^* conjugate representation of D. And the map

$\underline{H} \ni v \overset{*}{\to} v^* \equiv \langle\ .,\ v \rangle \in \underline{H}^*$, conjugation map.

Example. 1) On the regular representation $\underline{R}$, the conjugation map is, $*\ :\ \mathcal{L}^2(G) \ni f \to \bar{f} \in \mathcal{L}^2(G)$ so, $\underline{R} \sim \underline{R}^*$.

2) For $G = SL(2,\mathbf{R})$, for the representations in discrete series, $(D_n^+)^* \sim D_n^-$. etc.

§3 Tensor product.

Example. $\underline{H}_1, \underline{H}_2$: finite dimensional vector spaces.

(Form 1:1st version) $\underline{H}_1 \otimes \underline{H}_2 \equiv \{\sum_{j=1}^{N} v_j^1 \otimes v_j^2;\ v_j^1 \in \underline{H}_1,\ v_j^2 \in \underline{H}_2\}$.

(Form 2: 2nd version) Take basis $\{u_p^1\}$ in $\underline{H}_1$, $\{u_q^2\}$ in $\underline{H}_2$ and dual basis $\{\hat{u}_p^1\}$ in $\underline{H}_1^*$(i.e. $\langle\ u_p^1,\ \hat{u}_q^1 \rangle = \delta_{pq}$). Define a linear maps φ_{pq} as $\hat{u}_p^1 \to u_q^2$, $\hat{u}_\ell^1 \to 0$ $(p \neq \ell)$ in $\mathcal{L}(\underline{H}_1^*, \underline{H}_2)$. We connect the above two versions by the map,

$$\underline{H}_1 \otimes \underline{H}_2 \ni \sum_{j,k}^{N} a_{jk} u_j^1 \otimes u_k^2 \to \sum_{j,k}^{N} a_{jk} \varphi_{jk} \in \mathcal{L}(\underline{H}_1^*, \underline{H}_2).$$

In the case of ∞-dimensional Hilbert spaces $\underline{H}_1$, $\underline{H}_2$, we can define in the analogous way.

(1st version) $\underline{H}_1 \otimes_0 \underline{H}_2 \equiv \{\sum_j^N v_j^1 \otimes v_j^2;\ v_j^1 \in \underline{H}_1,\ v_j^2 \in \underline{H}_2,\ N \in \infty\}$.

Put $\langle (\sum_j^N v_j^1 \otimes v_j^2), (\sum_k^M u_k^1 \otimes u_k^2) \rangle \equiv \sum_j^N \sum_k^M \langle v_j^1, u_k^1 \rangle \langle v_j^2, u_k^2 \rangle$.

This gives an inner product and defines norm $\|\cdot\|$. Let $\underline{H}_1 \otimes \underline{H}_2$ be the completion of $\underline{H}_1 \otimes_0 \underline{H}_2$ with respect to $\|\cdot\|$.

(2nd version) Consider

$\underline{H}_1 \otimes_0 \underline{H}_2 \ni \sum_{j,k}^{M,N} a_{jk} u_j^1 \otimes u_k^2 \to \sum_{j,k}^{M,N} a_{jk} \varphi_{jk} \in \mathcal{L}(\underline{H}_1^*, \underline{H}_2)$ (bounded operators).

Here φ_{pq}: $\hat{u}_p^1 \to u_q^2$, $\hat{u}_\ell^1 \to 0$ ($p \neq \ell$). (Well-defined !!.)

Obviously, rank $\varphi_{p,q} = 1$, so for all $\varphi \in \mathrm{Image}(\underline{H}_1 \otimes_0 \underline{H}_2)$, $\mathrm{rank}(\varphi) < \infty$. This shows, as the completion of $\mathrm{Image}(\underline{H}_1 \otimes \underline{H}_2)$, $\mathrm{Image}(\underline{H}_1 \otimes \underline{H}_2) \subset HS(\underline{H}_1^*, \underline{H}_2)$ (Hilbert-Schmidt operators).

Moreover all rank 1 operators are in $\mathrm{Image}(\underline{H}_1 \otimes \underline{H}_2)$, so $HS(\underline{H}_1^*, \underline{H}_2) = \mathrm{Image}(\underline{H}_1 \otimes \underline{H}_2)$!!. This concludes that $\underline{H}_1 \otimes \underline{H}_2 \sim HS(\underline{H}_1^*, \underline{H}_2)$, with the norm $\|\varphi\| = (\sum_p \|\varphi(\hat{u}_p)\|^2)^{1/2}$.

(Independent of the choice of C.O.N.S. !!.)

We define for $A \in \mathbf{B}(\underline{H}_1)$, $B \in \mathbf{B}(\underline{H}_2)$ (bounded operators),

$(A \otimes B)(\sum_j v_j^1 \otimes v_j^2) \equiv \sum_j (Av_j^1) \otimes (Bv_j^2)$. (Well-defined !!.)

<u>Lemma</u>. For $U_j \in \mathbf{U}(\underline{H}_j)$ (unitary operators),

$$U_1 \otimes U_2 \in \mathbf{U}(\underline{H}_1 \otimes \underline{H}_2)$$

<u>Definition 11</u>. For two unitary representations $\mathbf{D}_j = \{\underline{H}_j, U_g^j\}$ of G_j ($j = 1,2$), we call

$D_1 \hat{\otimes} D_2 \equiv (\underline{H}_1 \otimes \underline{H}_2,\ U^1_{g_1} \otimes U^2_{g_2})$ (unitary representation of $G_1 \otimes G_2$),

the <u>outer tensor product</u> of D_1 and D_2.

<u>Proposition</u>. If D_j are irreducible unitary representations of $G_j (j=1,2)$, then $D_1 \hat{\otimes} D_2$ is irreducible.

<u>Example</u>. Let $G_1 = G_2 = G = \{$Yoshizawa group$\}$ (i.e. discrete group with 2-generators). On $\underline{H} \equiv \mathfrak{L}^2(G)$, consider

$$\underline{R} = (\underline{H},\ R_g) : R_{g_0} f(g) \equiv f(g g_0),\ \underline{L} = (\underline{H},\ L_g) : L_{g_0} f(g) \equiv f(g_0^{-1} g).$$

For $G \times G \ni (g_1, g_2)$, define a representation of $G \times G$,

$$\mathfrak{L}^2(G) \ni f \to R_{g_1} L_{g_2} f \in \mathfrak{L}^2(G).$$

This representation is irreducible, but not the form of outer tensor product of some representations of G.

In general, we use the word "tensor product" for following "inner tensor product".

<u>Definition 12</u>. D_j ($j = 1,2$) ; unitary representations of the same group G. We call <u>(inner) tensor product</u>, the representation $D_1 \otimes D_2 \equiv D_1 \hat{\otimes} D_2 |_{\Delta G}$.

Here $\Delta G \equiv \{(g, g) \in G \times G\ ;\ g \in G\}$ (the diagonal group).

<u>Example</u>. If D_1 is irreuducible and D_2 is 1-dimensional representation of G. Then $D_1 \otimes D_2$ is irreducible.

<u>Proposition</u>. $D_1 \otimes D_2^* \supset I$ (contains as a discrete component), if and only if $\exists$finite dimensional mutually equivalent components in both D_1, D_2 .

<u>Corollary</u>. If D_2 is finite dimensional unitary representation, $[D_1, D_2] = [D_1 \otimes D_2^*, I]$. Here $[D_1, D_2] = \dim I[D_1, D_2]$.

§ 4 . Induced representation (of $\mathcal{L}^2$-type)

$H(\subset G)$: closed subgroup. $D \equiv \{\underline{H}, T_h\}$: unitary representation of H. Fix μ ; a quasi-invariant measure on $X \equiv H\backslash G$. Consider $\underline{H}$- valued functions on G.

$\tilde{H} \equiv \{f$; $\underline{H}$-valued function on G. (1)(2)(3) holds$\}$.

(1) $f(hg) = T_h f(g)$ for all $h \in H$, $g \in G$.

(2) f ; strongly measurable.

(3) $\|f\| \equiv (\int_X \|f(g)\|_{\underline{H}}^2 d\mu(\tilde{g}))^{1/2} < \infty$.

$\tilde{H}$ is a Hilbert space with $\langle f_1, f_2 \rangle \equiv \int_X \langle f_1(g), f_2(g) \rangle \, d\mu(\tilde{g})$.

Operators $(U_{g_1} f)(g) \equiv w(g, g_1)^{1/2} f(gg_1)$, ($w(g, g_1)$: weight function for μ) gives a unitary representation of G .

Defionition 13. $\mathrm{Ind}_H^G(\{\underline{H}, T_h\}) \equiv \{\tilde{H}, U_g\}$.

We call this induced representation (representation induced from D).

Example. For any representation $D = \mathrm{Ind}_G^G D$.

Example. $\underline{R}$ (regular representation) $= \mathrm{Ind}_{\{e\}}^G 1$.

Here 1 shows the trivial representation of subgroup $\{e\}$.

Theorem(Step theorem). $G \supset H_1 \supset H_2$: closed subgroups.

D : unitary representation of H_2. Then

$$\mathrm{Ind}_{H_2}^G D \cong \mathrm{Ind}_{H_1}^G (\mathrm{Ind}_{H_2}^{H_1} D).$$

Corollary. $\underline{R}_G = \mathrm{Ind}_H^G \underline{R}_H$.

Theorem. Let $D_j = \mathrm{Ind}_{H_j}^G D_j$ of closed subgroups $H_j (j=1,2)$ of G respectively. Assume $H_1\backslash G/H_2$ is countably separated. Then

$D_1 \otimes D_2 \cong$

$$\int_{H_1\backslash G/H_2} \mathrm{Ind}^G_{H_1\cap g^{-1}H_2g}((D_1\otimes gD_2g^{-1})|_{H_1\cap g^{-1}H_2g})\ d\tilde{\mu}(\tilde{g}).$$

Example. Put $H_2 = G$ then $(\mathrm{Ind}^G_{H_1} D_1)\otimes D_2 \cong \mathrm{Ind}^G_{H_1}(D_1\otimes D_2|_{H_1})$.

$\underline{R} \otimes D_2 \cong \mathrm{Ind}^G_{\{e\}} 1\otimes D_2 \cong \mathrm{Ind}^G_{\{e\}}(1\otimes D_2|_{\{e\}}) \cong [\dim D_2]\underline{R}$.

(Here we remark that $1\otimes D_2|_{\{e\}} \cong \sum^{\oplus}_{\dim D} 1$).

§5. Compact group.

Many results in the representation theory of compact groups are considered as direct extensions of one of finite groups. This comes from the only reason that the function "constant 1" is contained in $\mathfrak{L}^2(G)$, that is the same, the total mass of whole group is finite.

We shall state here such results.

Theorem. Any continuouse representation of a compact group by bounded operators on a Hilbert space, is equivalent to a unitary representation.

Theorem. Any irreducible unitary representation of a compact group is finite dimensional.

Theorem. 1) Any unitary representation of compact group is decomposed to a discrete direct sum of its irreducible components.

2) (Orthogonal relations.) If we take a C.O.N.S. in representation spaces for each irreducible representations and represent these representation operators by unitary matrices as $U_g(\rho) \equiv (u^{\rho}_{ij}(g))_{ij}$, then

$$\int_G u_{ij}^{\rho}(g)\cdot u_{k\ell}^{\rho}(g)\ d_rg = (\delta(\rho,\sigma)\cdot\delta^k_i\cdot\delta^{\ell}_j)/\dim(\rho).$$

Here $\delta(\rho,\sigma)=1$ for $\rho\sim\sigma$, and $=0$ otherwise.

Proposition. For a compact group, $\underline{R}\cong\sum^{\oplus}_{D\in G^{\wedge}}\{\dim D\}D$.

Proposition(Frobenius's reciprocity). For irreducible representations ω of G and ρ of closed subgroup H of G,

$[\mathrm{Ind}_H^G\rho,\ \omega]\cong[\omega|_H,\ \rho]$. Here $[D,\ \omega]=\dim I[D,\ \omega]$.

Proposition. For three irreducible representations ρ,τ,σ of G, $\mathrm{Max}([\sigma\otimes\tau,\omega],[\tau\otimes\omega,\sigma],[\omega\otimes\sigma,\tau])\leqq(\dim\sigma)(\dim\omega)/(\dim\tau)$.

§6 Duality theorem for locally compact groups.

1) ABELIAN GROUP A .

$\hat{A}\equiv\{\ \chi\ ;$ continuous unitary character on A $\}$

i.e. $\chi(a_1a_2)=\chi(a_1)\cdot\chi(a_2)$ for all $a_1,a_2\in A$.

$|\chi(a)|\equiv 1$ for all $a\in A$.

$\hat{A}$ becomes an abelian locally compact group, too, by $(\chi_1\cdot\chi_2)(a)\equiv\chi_1(a)\cdot\chi_2(a)$ for all $a\in A$ (multiplication), $\chi\to\chi_0$ is uniform convergence on any compact set in A (topology).

We call this group $\hat{A}$ "The dual group of A" .

Consider $\hat{\hat{A}}$ =(the dual group of $\hat{A}$), then naturally,

$A\ni a\to\{a(\chi)\equiv\chi(a)\}$(for all $\chi\in\hat{A}$) $\in\hat{\hat{A}}$ gives an imbedding.

Theorem(L.Pontrjagin's duality).

1) $\hat{\hat{A}}=A$ as topological groups (by the above imbedding).

2) For all $B(\subset A)$ closed subgroup, put $\mathring{B}\equiv\{\chi\in\hat{A};\ \chi(B)=1\}$, then the set {closed subgroup of A} corresponds to the set {closed subgroup of $\hat{A}$ } one to one way by $B\longleftrightarrow\mathring{B}$ and

$\hat{B}\sim\hat{A}\ /\ \mathring{B}$, $(\mathring{B})^{\wedge}\sim A\ /\ B$.

2) COMPACT GROUP K.

$\hat{K} \equiv \{\rho;$ (equivalence class of) all irreducible repres. of K$\}$.

For all $(\rho, \sigma) \in \hat{K}\times\hat{K}$, let $\rho\otimes\sigma = \tau_1 \oplus \tau_2 \oplus \ldots \oplus \tau_n$ $(\tau_j \in \hat{K})$.

$\hat{K}$ can be considered as a discrete space.

$\hat{\hat{K}} \equiv \{\tilde{T} = \{T(\rho)\};$ operator field over $\hat{K}$ s.t. (1)(2) hold$\}$.

(1) $T(\rho)$; unitary matrix on the space of representation ρ.

(2) $T(\rho)\otimes T(\sigma) = T(\tau_1)\oplus \ldots \oplus T(\tau_n)$ for all $(\rho, \tau) \in \hat{K}\times\hat{K}$.

Then $\hat{\hat{K}}$ becomes a compact group by

$\tilde{T}_1 \cdot \tilde{T}_2 \equiv \{T_1(\rho)\cdot T_2(\rho)\}$ for $\tilde{T}_j = \{T_j(\rho)\}$ ($j = 1,2$)(group operation).

$T \to T_0$ iff $T(\rho) \to T_0(\rho)$ for all $\rho \in \hat{K}$ (topology).

Take an imbedding $K \ni k \to \{k(\rho)(\equiv U_k(\rho))\} \in \hat{\hat{K}}$.

<u>Theorem(T.Tannaka's duality theorem)</u>.

$\hat{\hat{K}} = K$ as topological groups (by the above imbedding).

(Remark) The 1-1 corresponding between the sets

$\{$Normal subgroups L in K$\}$ and $\{\rho \in \hat{K}; \rho|_L = 1\}$(closed under tensor product and *) is easily shown.

3) GENERALIZATION TO LOCALLY COMPACT GROUPS G.

$\hat{G} \equiv \{\rho;$ (equivalent class of) irred. unitary repres. of G$\}$.

Consider irreducible decompositions of tensor products,

$U(\rho\otimes\sigma)U^{-1} = \int_{\hat{G}} \omega \, d\nu(\omega)$ $(\rho, \sigma, \omega \in \hat{G})$.

Here U is the operator of equivalence.

On $\hat{G}$, we can consider the "Mackey-Borel structure". Put

$\hat{\hat{G}} \equiv \{\tilde{T} = \{T(\rho)\};$ operator field over $\hat{G}$ s.t.(1)(2)(3) hold$\}$

(1) $T(\rho)$; unitary operator on the space of ρ .

(2) $U(T(\rho)\otimes T(\sigma))U^{-1} = \int_{\hat{G}} T(\omega)\, d\nu(\omega)$

for all pairs $\rho, \sigma \in \hat{G}$, and all irreducible decompositions.

(3) $\{T(\rho)\}$:Mackey-Borel measurable. (Precise definition is omitted.)

Consider a structure of topological group on $\hat{G}$,

$\tilde{T}_1 \cdot \tilde{T}_2 \equiv \{T_1(\rho) \cdot T_2(\rho)\}$ for $T_j = \{T_j(\rho)\}(j=1,2)$.(multiplication)

The topology "comes from the weak topology on $\underline{R}$". (omitted)

We can give an imbeddig , $G \ni g \rightarrow \tilde{U}_g \equiv \{ U_g(\rho) \} \in \hat{\hat{G}}$,also.

<u>THEOREM(duality for locally compact group)</u>.

$\hat{\hat{G}} = G$ as topological groups.

(Remark) 1) The property 2) in the Pontrjagin's duality can be extended to general case under adequate interpretations. But we have a pathological result as "for a closed normal subgroup H in G such that G/H is non-amenable, G/H corresponds to many closed <u>subgroups</u> in $\hat{G}$."

2) The assumption "$T(\rho)$ is unitary" in our duality theorem can be loosen to "$T(\rho)$ is closed" under some additional conditions.

3) The assumption (2) $U(T(\rho)\otimes T(\sigma))U^{-1} = \int T(\omega)d\nu(\omega)$ contains "measurability" of $T(\omega)$ in its definition. That is, it contains partly the assumption (3).

4) In the proof of duality, the regular representation $\underline{R}$ of G plays very important role.

<u>A counter(?) example to duality theorem</u>.

Consider non-abelian group G of 8 order (#G = 8).

From $$8 = \dim(\mathfrak{L}^2(G)) = \sum_{\tau\in \hat{G}} (\dim \tau)^2 = 1^2+1^2+1^2+1^2+2^2,$$

it is concluded directly

$\hat{G} = \{1, \chi_1, \chi_2, \chi_3, \rho\}$ (here χ_j;character, ρ; 2-dim.)

$\underline{R} \cong I \oplus \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus [2]\rho$.

Now we calculate the tensor product table of $\hat{G}$.

At first, since G is non-abelian, the kernel of character χ_j must be have 4-elements. This concludes directly that, $(\chi_j)^2 = 1$ for all j. And $\chi_j \cdot \chi_k = \chi_\ell$ for all different j,k,ℓ.

Next ρ is only non-1-dimensional irreducible representation of G, and $\chi_j \otimes \rho$ are all irreducible 2-dimensional, so

$\chi_j \otimes \rho \cong \rho$ (j=1,2,3)

Lastly we consider $D \cong \rho \otimes \underline{R}$ in two ways.

1. $D \cong \rho\otimes(1\oplus\chi_1\oplus\chi_2\oplus\chi_3\oplus[2]\rho) \cong \rho\oplus\rho\oplus\rho\oplus\rho\oplus[2](\rho\otimes\rho) \cong [4]\rho\oplus[2](\rho\otimes\rho)$.
2. $D \cong [2]\underline{R} \cong [2](1\oplus\chi_1\oplus\chi_2\oplus\chi_3\oplus[2]\rho) \cong [2](1\oplus\chi_1\oplus\chi_2\oplus\chi_3)\oplus[4]\rho$.

Comparing above two results, we get, $\rho \otimes \rho \cong 1\oplus\chi\oplus\chi\oplus\chi$, so obtain the complete table of tensor products of $\hat{G}$ uniquely.-

It is remarkable that to get this table we use only the order of G, that is, we get same table for two mutually non isomorphic 8-order groups.

IT SEEMS TO US THAT THIS RESULT CONTRADICTS TO OUR DUALITY THEOREM (AND ALSO TO TANNAKA'S DUALITY THEOREM).

We leaves to solve this question to readers, it will be obtained by considering the correspondence of vectors in the decompositions of tensor products.

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