

On disjoint ordered pairs of operators

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We obtain sufficient conditions for the nonexistence of non-zero intertwining operators between two nonnormal operators. We say that such pair of operators is disjoint.

1. Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X . For $A \in B(X)$ and for a closed set $\sigma \subset \mathbb{C}$, let

$$X_A(\sigma) = \{x \in X ; (zI-A)f(z) \equiv x \text{ for some analytic function } f : \mathbb{C} \setminus \sigma \rightarrow X \}$$

and for an arbitrary $\sigma \subset \mathbb{C}$, let

$$X_A(\sigma) = \cup \{X_A(\tau) ; \tau \subset \sigma \text{ and } \tau \text{ is closed}\}.$$

The set $X_A(\sigma)$ is called the spectral manifold of A . It is known that $X_A(\sigma)$ is an invariant linear manifold of A and that if $\sigma_1 \subset \sigma_2$ then $X_A(\sigma_1) \subset X_A(\sigma_2)$. And it is clear that $X_A(\sigma) = X_A(\sigma \cap \sigma(A))$, $X_A(\sigma(A)) = X$, $X_A(\emptyset) = \{0\}$ and that $X_A(\sigma) \subset \bigcap_{z \notin \sigma} (A-zI)X$ for any closed set $\sigma \subset \mathbb{C}$.

Clancey [1] proved the following

Proposition. Let T on H be a hyponormal operator (i. e., $T^*T \geq TT^*$) and $\sigma \subset \mathbb{C}$ be a closed set, then $X_T(\sigma) = \bigcap_{z \notin \sigma} (T - zI)H$ and, in particular, $\bigcap_{z \in \sigma(T)} (T - zI)H = \bigcap_{z \in \mathbb{C}} (T - zI)H = X_T(\emptyset) = \{0\}$ where $\sigma(T)$ denotes the spectrum of T .

Corollary 1. If T on H is a subnormal operator with the minimal normal extension N on K , then $\bigcap_{z \in \sigma(N)} (T - zI)H = \{0\}$.

The following theorem is a slight modification of [6].

Theorem 1. For $g \in H^\infty$, let T_g be the analytic Toeplitz operator on H^2 defined by the relation $(T_g f)(z) = g(z)f(z)$. Then $\bigcap_{z \in \delta} (T_g - g(z)I)H^2 = \{0\}$ if δ is an infinite set having a limit point inside $\{z \in \mathbb{C} ; |z| = 1\}$.

2. For $A \in B(X)$ and $B \in B(Y)$, we shall say that the ordered pair (A, B) is disjoint if the only bounded linear operator C mapping X into Y and satisfying the equation $CA = BC$ (i. e., C intertwines A and B) is zero.

Lemma 1. If $CA = BC$ for $C \in B(X, Y)$, then $CX_A(\sigma) \subset X_B(\sigma)$ for

an arbitrary set $\sigma \subset \mathbb{C}$. In particular $CX \subset X_B(\sigma(A))$.

Then, we have only to seek such $\sigma \subset \mathbb{C}$ as $X_A(\sigma)^\sim = X$ and $X_B(\sigma) = \{0\}$, in particular, we may prove $X_B(\sigma(A)) = \{0\}$ in order to show that the pair (A, B) is disjoint.

The following theorem is well known. But we give here a simple proof.

Theorem 2. [7] If $\sigma(A) \cap \sigma(B) = \emptyset$, then the pair (A, B) is disjoint.

Proof. $X_B(\sigma(A)) = X_B(\sigma(A) \cap \sigma(B)) = X_B(\emptyset) = \{0\}$.

Theorem 3. Let T be a subnormal operator on H with the minimal normal extension N on K . If $\sigma(A) \cap \sigma(N) = \emptyset$, then the pair (A, T) is disjoint.

Proof. By the assumption, there is an open set D such that $\sigma(N) \subset D$ and $\sigma(A) \cap D = \emptyset$. Then $X_T(\sigma(A)) \subset X_T(\mathbb{C} \setminus D) = \bigcap_{z \in D} (T - zI)H$
 $\subset \bigcap_{z \in \sigma(N)} (T - zI)H = \{0\}$ by Proposition and by Corollary 1.

Theorem 4. Let T_ϕ be an analytic Toeplitz operator on H^2 .

If $\sigma(T_\phi) \not\subset \sigma(A)$, then the pair (A, T_ϕ) is disjoint.

Proof. It is known that $\sigma(T_\phi)$ is the closure of $\{\phi(z); |z| < 1\}$.

Let $\tau = \{\phi(z); |z| < 1\} \cap [\mathbb{C} \setminus \sigma(A)]$, then τ is either a non-empty open set or a singleton, depending on whether $\{\phi(z); |z| < 1\}$ is an open set or a singleton (that is, whether ϕ is non-constant or constant). In either case, $\delta = \bar{\phi}^{-1}(\tau) \cap \{z \in \mathbb{C}; |z| < 1\}$ is a non-empty open subset in $\{z \in \mathbb{C}; |z| < 1\}$ and hence, by the assumption there is an open set D such that $\tau \subset D$ and that $\sigma(A) \cap D = \emptyset$.

Then $X_{T_\phi}(\sigma(A)) \subset X_{T_\phi}(\mathbb{C} \setminus D) = \bigcap_{z \in D} (T_\phi - zI)H^2$ by Proposition
 $\subset \bigcap_{z \in \tau} (T_\phi - zI)H^2 = \bigcap_{z \in \delta} (T_\phi - \phi(z)I)H^2 = \{0\}$ by Theorem 1.

Corollary 2. [4] Let T_ϕ, T_ψ be two analytic Toeplitz operators on H^2 . If $\{\psi(z); |z| < 1\} \not\subset \sigma(T_\phi)$, then the pair (T_ϕ, T_ψ) is disjoint.

Let A and B are bounded linear operators on two Hilbert spaces H and K respectively.

Lemma 2. [2] Let $CA = BC$ for $C \in B(H, K)$. If C has dense range and if B is hyponormal, then $\sigma(B) \subset \sigma(A)$.

If $CA = BC$ for $C \in B(H, K)$ implies that $CA^* = B^*C$, then $\ker[C]^\perp$ and $\text{range}[C]^\sim$ are reducing subspaces for A and B respectively and it is easily seen that $A|_{\ker[C]^\perp}$ and $B|_{\text{range}[C]^\sim}$ are normal and hence $\sigma(A|_{\ker[C]^\perp}) = \sigma(B|_{\text{range}[C]^\sim})$ by Lemma 2.

$A \in B(H)$ is dominant if there is a number M_λ for each $\lambda \in \mathbb{C}$ such that $\|(A - \lambda I)^* x\| \leq M_\lambda \|(A - \lambda I)x\|$ for all $x \in H$. If there is a constant M such that $M_\lambda \leq M$ for all $\lambda \in \mathbb{C}$, A is called M -hyponormal and if $M = 1$, A is hyponormal.

Theorem 5. Let $A^* \in B(H)$ be M -hyponormal and let $B \in B(K)$ be dominant. If $\sigma(A^{(n)}) \cap \sigma(B^{(n)}) = \emptyset$, then the pair (A, B) is disjoint, where $A^{(n)}$ denotes the normal part of A .

Proof. By [8], $CA = BC$ for $C \in B(H, K)$ implies that $CA^* = B^*C$ and hence, by the arguments after Lemma 2, we have $\sigma(B|_{\text{range}[C]^\sim}) = \sigma(A|_{\ker[C]^\perp}) \cap \sigma(B|_{\text{range}[C]^\sim}) \subset \sigma(A^{(n)}) \cap \sigma(B^{(n)}) = \emptyset$ and $C = 0$.

$A \in B(H)$ is paranormal if $\|Ax\|^2 \leq \|A^2x\| \|x\|$ for all $x \in H$.

If $A^* \in B(H)$ be an isometry and if $B \in B(K)$ be a paranormal contraction (i. e., $\|B\| \leq 1$), then, by [5], it is easily seen

that $CA = BC$ for $C \in B(H, K)$ implies $CA^* = B^*C$. And then

$A|_{\ker[C]^\perp}$ and $B|_{\text{range}[C]^\sim}$ are unitary because

$\sigma(B|_{\text{range}[C]^\sim}) = \sigma(A|_{\ker[C]^\perp}) \subset \sigma(A^{(n)}) = \sigma(A^{(u)})$ by the arguments after Lemma 2, where $A^{(u)}$ denotes the unitary part of A . And

hence we have

Theorem 6. Let $A^* \in B(H)$ be an isometry and let $B \in B(K)$ be a paranormal contraction. If $\sigma(A^{(u)}) \cap \sigma(B^{(u)}) = \emptyset$, then the pair (A, B) is disjoint.

Proof. $\sigma(B|_{\text{range}[C]^\sim}) = \sigma(A|_{\ker[C]^\perp}) \cap \sigma(B|_{\text{range}[C]^\sim})$
 $\subset \sigma(A^{(u)}) \cap \sigma(B^{(u)}) = \emptyset$ and $C = 0$.

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