数理解析研究所講究録
第 1617 巻 2008 年 187-192



# Base change lift type spinor L-function of $GSp_2(\mathbb{Q})$

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In [4], we happened to consturct Siegel modular cuspform $F$ and non-cuspform $E$ of degree 2 having the same spinor L-function: $L^{spin}(s,E)=L^{spin}(s,F)$, which is equal to the Hasse-Weil zeta function of hyper-elliptic curve $y^2=x^5-x$. The CAP representation has a L-function of a non-cuspidal one, but, our phenomenon is not the case. In this article, we consider the problem 'What type of spinor $L$-function is related to cuspform and non-cuspform, simultaneously?'. To do it, we will classify the spinor $L$-functions of non-cuspforms. The classical 'Zharkovskaya relation' describes L-function of Siegel non-cuspform by that of the elliptic modular form obtained by the Siegel operator, as follows. If $E\in M_\kappa(Sp_2(\mathbb{Z}))$ is an eigenform, then it holds

$$L^{spin}(s,F)=L(s,\Phi(E))L(s-\kappa+2,\Phi(E)).$$

where the elliptic modular eigenform $\Phi(E)\in M_\kappa(SL_2(\mathbb{Z}))$ is

$$\Phi(E)(z)=\lim_{t\to\infty}E(\begin{bmatrix} z & 0\\ 0 & it\end{bmatrix}),\ z\in\mathfrak{H}. \tag{1}$$

We will generalize her relation for non-holomorphic and non-full modular cases. Let $N_1,N_2$ be the unipotent radicals of the two parabolic subgroups, such as

$$N_1(\mathbb{Q})=\left\{\begin{bmatrix}1 & & * & *\\ & 1 & * & *\\ & & 1 & \\ & & & 1\end{bmatrix}\right\},\ N_2(\mathbb{A})=\left\{\begin{bmatrix}1 & & & *\\ * & 1 & * & *\\ & & 1 & *\\ & & & 1\end{bmatrix}\right\}\subset Sp_2(\mathbb{Q}).$$

If $E$ is not cuspidal, then

$$\int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})}E(ug)du\neq 0$$

for $i=1$ or 2 where $dh$ is a suitable Haar measure $du$. We label the former case as (CASE 1), and the latter as (CASE 2). In the both cases, we obtain automorphic forms on $GL_2(\mathbb{A})$ by

$$\int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})}E(ue_i(g)h)du,$$

for some $h\in Sp_2(\mathbb{A})$. Here we write

$$e_1(g)=\begin{bmatrix}{}^tg^{-1} & \\ & g\end{bmatrix},e_2(g)=\begin{bmatrix}a & & b & \\ & \det(g) & & \\ c & & d & \\ & & & 1\end{bmatrix}\in GSp_2(\mathbb{A})$$

*This work was supported by Grant-in-Aid for JSPS Fellows.

for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{A})$. So, after the original Siegel operator (1),

**DEFINITION 1** *We define 'Siegel operator along $N_i$' at $h \in Sp_2(\mathbb{A})$*

$$\Phi_i(E)(g;h) = \int_{N_i(\mathbb{Q})\backslash N_i(\mathbb{A})} E(ue_i(g)h)du,$$

*where du is the Haar measure so that $vol(N_i(\mathbb{Q})\backslash N_i(\mathbb{A})) = 1$.*

Remark that the Siegel operator (1) is equal to $\Phi_2$ at $h = 1$, and that holomorphic $E$ is cuspidal iff $\Phi_2(E) = 0$. Let $\psi$ be the standard additive character on $\mathbb{Q}\backslash\mathbb{A}$ so that $\psi_\infty(x) = \exp(2\pi ix), x \in \mathbb{R}$. For automorphic form $F$ and $T = {}^tT \in M_2(\mathbb{Q})$, $F_T$ denote the fourier coefficient;

$$F_T(g) = \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \psi(-tr(S\cdot T))F(\begin{bmatrix} 1 & S \\ & 1 \end{bmatrix}g)dS.$$

**(CASE 1)** Suppose that irrducible $\pi \in \widehat{Sp_2(\mathbb{A})}$ is in the (CASE.1). Take $E \in \pi$, so that $f = \Phi_1(E)(*;1) \neq 0$. If an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(\mathbb{A}))$ is an extension of $E$, then there exists $\delta \in \widehat{(\mathbb{Q}\backslash\mathbb{A})^\times}$ such as

$$F_0(e_1(g)\begin{bmatrix} 1_2 & \\ & t\cdot 1_2 \end{bmatrix}) = \delta(t)F_0(e_1(g)). \tag{2}$$

Since $E_0, \widetilde{E_0}$ and $f$ have the informations of L-paramerters of themselves, by comparing the action of Hecke operator on them, we can obtain the following.

**PROPOSITION 1** *Let $S$ be the collection of bad primes of $E$. With the assumption as above, $f$ is an eigenform at every $p \notin S$, and the stnadard L-function $L_S^{st}(s,E)$ is written as*

$$L_S^{st}(s,E) = \zeta_S(s)L_S(s-1,f)L_S(s+2,f,w_f). \tag{3}$$

*If an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(\mathbb{A}))$ is an extension of $E$, then*

$$L_S^{spin}(s,\widetilde{E},\delta^{-1}) = \zeta_S(s)L_S(s-3,w_f^{-1})L_S(s-1,f). \tag{4}$$

*Here $L(s,f,w_f)$ means the $w_f$-twist of $L(s,f,w_f)$, and so on.*

**(CASE 2)** Suppose that irreducible $\pi \in \Pi(Sp_2(\mathbb{A}))$ is in the (CASE.2) and take $E \in \pi$ so that $\Phi_2(E)(*;1) \neq 0$. Let $(\kappa_1,\kappa_2)$ with $\kappa_1 \geq \kappa_2$ be the highest weight of $E$, and $E$ is an eigenvector with rsepect to $\mathcal{Z}(\mathfrak{sp}(2,\mathbb{R}))$, the center of the Lie algebra $\mathfrak{sp}(2,\mathbb{R})$. Then, for a certain $T_a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{Q})$, $E_{T_a}$ is not zero. $E_{T_a}$ has the property

$$E_{T_a}(\begin{bmatrix} 1 & & x & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}e_2(g)) = \psi(ax)E_{T_a}(e_2(g)),\ x \in \mathbb{A}. \tag{5}$$

There exists a unique $\xi \in \widehat{\mathcal{Z}(\mathfrak{sl}(2,\mathbb{R}))}$ such as $E_{T_a}(e_2(z * g)) = \xi(z)E_{T_a}(e_2(g))$ for $z \in \mathcal{Z}(\mathfrak{sl}(2,\mathbb{R}))$. Indeed, $\mathcal{Z}(\mathfrak{sp}(2,\mathbb{R})$ is generated by two elements $L_1, L_2$ as in [3].

In particular, since $L_1$ acts on $E_{T_a}$ as an element of $\mathcal{Z}(\mathfrak{sl}(2,\mathbb{R}))$, $E_{T_a}$ is an eigenvector with respect to $\mathcal{Z}(\mathfrak{sl}(2,\mathbb{R}))$. Thus $f(g) = \Phi_2(E)(g;1) = \sum_{b\in\mathbb{Q}} E_{T_b}(e_2(g)) \in \mathcal{A}(SL_2(K_\mathbb{A}))$ is of weight $\kappa_1$ and corresponds to $\xi$. From $E_{T_a}$, for some $\chi \in \widehat{\mathbb{Q}^\times\backslash\mathbb{A}^\times}$, cut out a nonzero $\chi$-section, that is,

$$E_{T_a}^{(\chi)}(e_2(g)\begin{bmatrix} 1 & & & \\ & y & & \\ & & 1 & \\ & & & y^{-1}\end{bmatrix}) = E_{T_a}^{(\chi)}(\begin{bmatrix} 1 & & & \\ & y & & \\ & & 1 & \\ & & & y^{-1}\end{bmatrix}e_2(g)) = \chi(y)E_{T_a}^{(\chi)}(e_2(g)).$$

Further, if an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(K_\mathbb{A}))$ is an extension of $E$, we cut out a nonzero $\omega$-section

$$\widetilde{E}_{T_a}^{(\chi,\omega)}(e_2(zg)) = \omega(z)\widetilde{E}_{T_a}^{(\chi,\omega)}(e_2(g)) \;\; z \in \mathbb{A}^\times.$$

Remark that $f^{(\chi)}(g) = \sum_{b\in\mathbb{Q}} E_{T_b}^{(\chi)}(e_2(g))$ belongs to $\mathcal{A}(SL_2(K_\mathbb{A}))$, and that $f^{(\chi,\omega)}(g) = \sum_{b\in\mathbb{Q}} E_{T_b}^{(\chi,\omega)}(e_2(g))$ to $\mathcal{A}(GL_2(\mathbb{A}))$.

**PROPOSITION 2** *With the assumptions as above, at $p \notin S$, $f^{(\chi)}$ is an eigenform such as*

$$L_S^{st}(s,E) = L_S(s-2,\chi)L_S(s+2,\chi^{-1})L_S^{st}(s,f^{(\chi)}). \tag{6}$$

*If an eigenform $\widetilde{E} \in \mathcal{A}(GSp_2(\mathbb{A}))$ is an extension of $E$, then $f^{(\chi,\omega)}$ has the following properties:*

*i) the central character of $f^{(\chi,\omega)}$ is $|\cdot|^2\chi\omega$.*

*ii) If $\chi_p(p) \neq -p^{-2}$, then $f^{(\chi,\omega)}$ is also an eigenform at p, such as*

$$L^{spin}(s,\widetilde{E})_p = L(s,f^{(\chi,\omega)})_p L(s-2,f^{(\chi,\omega)},\chi)_p; \tag{7}$$

*iii) Otherwise, $f^{(\chi,\omega)}$ is not an eigenform in general.*

*However, in the case iii), instead of $f^{(\chi,\omega)}$, we can take an eigenform $\widetilde{f}' \in \mathcal{A}_{\kappa_1}(GL_2(\mathbb{A}))$ having the same central character and satisfies (7) at every $p \notin S$.*

**REMARK 1** *The above $\chi$ is determined uniquely by $\pi$, and $\omega$ is by extended $\widetilde{\pi}$ which contains $\widetilde{E}$, indeed.*

Summing up the above results, we can give the following answer to the first problem.

**THEOREM 1** *Suppose that a non CAP-type spinor L-function of $\Pi(GSp_2(\mathbb{A}))$ is related to a cuspform and a non-cuspform, simultaneously. Then it is a Base change lift type, i.e., $L(s,\sigma)L(s,\sigma,\chi_{E/\mathbb{Q}})$ for $\sigma \in \Pi(GL_2(\mathbb{A}))$ and quadratic character $\chi_{E/\mathbb{Q}}$ associated to an extention $E/\mathbb{Q}$.*

**PROOF.** Suppose that cuspidal $\pi$ and non-cuspidal $\tau$ have an identical spinor $L$-function up to finitely many primes. We can assume $\pi$ and $\tau$ are unitalized. In the (CASE.1) of $\tau$, by Proposition 1, we can write

$$L_S^{spin}(s,\pi,\omega_\tau^{-1}) = L_S^{spin}(s,\tau,\omega_\tau^{-1}) = \zeta_S(s-z_0)L_S(s+z_0,\omega_\sigma^{-1})L_S(s,\sigma) \tag{8}$$

for a certain $z_0 \in \mathbb{C}$, where $\sigma \in \Pi(GL_2(\mathbb{A}))$ is related to $\tau$ and unitalized. According to Jacquet, Shalika [1], Shahidi [7], $L_S(s,\sigma)$ and $L_S(s,\sigma,\omega_\sigma)$ does not vanish in the region $\mathrm{Re}(s) \geq 1$. Hence the right hand side of (8) has a pole at $1+z_0$, or its $\omega_\sigma$-twist has a pole at $1-z_0$. By lemma 3.1 of Piatetski-Shapiro [6], $\pi$ is written as $\pi_1 \otimes (\mu \circ \nu)$ by the similtude norm $\nu$ of $GSp(2)$, certain $\mu \in \widehat{\mathbb{Q}^\times \backslash \mathbb{A}^\times}$ and $\pi_1 \in \Pi(GSp_2(\mathbb{A}))$ with $\omega_\pi = 1$. And by Thoerem 2.2 of [6], we conclude the spinor $L$-function is related to some CAP representation associated to Siegel parabolic subgroup.

In the (CASE.2) of $\tau$, $L_S^{spin}(s,\tau)$ is written in the form (7), and $\omega_\tau = \omega_\sigma \chi$. From (7), the character $\xi := \omega_\pi(\omega_\sigma\chi)^{-1}$ satisfies $\xi^2 = 1$. In the case of $\xi = 1$ (i.e., $\omega_\pi = \omega_\tau$), $\pi_v$ is equivalent to $\tau_v$ at almost all $v$ since they have identical Satake parameters. Hence $\pi$ is a CAP representation asscoiated to Klingen parabolic subgroup. In the case that $\xi \neq 1$, calculating $L_S(s,\pi,\wedge^2)$, we see

$$\begin{aligned} & L_S(s,\omega_\pi)L_S^{st}(s,\pi,\omega_\pi) \\ = \; & L_S(s,\omega_\pi\xi^{-1})L_S(s-2,\chi\omega_\pi\xi^{-1})L_S(s+2,\chi^{-1}\omega_\pi\xi^{-1})L_S^{st}(s,\sigma,\omega_\pi\xi^{-1}). \end{aligned}$$

Twisting both sides by $\omega_\pi^{-1}\xi$,

$$\begin{aligned} L_S(s,\xi)L_S^{st}(s,\pi,\xi) &= \zeta_S(s)L_S(s-2,\chi)L_S(s+2,\chi^{-1})L_S^{st}(s,\sigma) \\ &= \zeta_S(s)L_S(s+t,\chi_1)L_S(s-t,\chi_1^{-1})L_S^{st}(s,\sigma). \end{aligned} \tag{9}$$

Here $\chi_1$ is the unitalization of $\chi$ and we write $\chi_\infty = |\cdot|^{2+t}(\mathrm{sign})^{\frac{1\pm1}{2}}$. Applying lemma 1 to (9), we find that $L_S^{st}(s,\pi,\xi\chi_1)$ (resp. $L_S^{st}(s,\pi,\xi\chi_1^{-1})$) has a simple pole at $s=1+t$ (resp. $s=1-t$), if $\Re(t)>0$ (resp. $\Re(t)<0$). However, $\pi$ is cuspidal, so $t$ is allowed to be $\pm1$ and $\pi$ is a CAP representation along Klingen parabolic subgroup. If $\Re(t)=0$, we can also coclude $t=0$ by considering the possiblity of the location of the poles. In this case, if $\chi_1^2 \neq 1$, then $(\xi\chi_1)^2 \neq 1$ and we find that $L_S^{st}(s,\pi,\xi\chi_1)$ has a simple pole at $s=1$, twisting (9) by $\chi_1$. This conflicts to [2]. If $\xi^2=1$ but $\chi_1\xi \neq 1$, we find that $L_S^{st}(s,\pi,\xi\chi_1,st)$ has at least double pole at $s=1$, which conflicts to [2], too. Thus, the remained possibility of $\chi_1$ is only $\chi_1 = \xi$, i.e., some quadratic character. This is just the Base Change lift type. □

**REMARK 2** *Conversely, for given spinor L-function $L(s,\sigma)L(s,\sigma,\chi_{K/\mathbb{Q}})$ of base change type, [5] gives generic non-cuspform and cuspform which is fixed by paramodular groups, if $\sigma$ is holomorphic.*

The next lemma used in the proof of previous theorem follows from the results of Jacquet, Shalika [1] and Shahidi [7].

**LEMMA 1** *Let $\pi \in \Pi(GL_2(\mathbb{A}))$ be cuspidal. Then,*

*i) $L_S(s,\pi,\eta,st) \neq 0$ for every unitary $\eta \in \widehat{\mathbb{Q}^\times \backslash \mathbb{A}^\times}$ at $\Re s \geq 1$,*

*ii) if $\pi$ comes from a größencharacter of a quadratic extension $K$ over $\mathbb{Q}$, then*

$$\begin{cases} ord_{s=1}L_S(s,\pi,\eta,st) = -1 & \text{if } \eta = \chi_{K/\mathbb{Q}}, \\ ord_{s=1}L_S(s,\pi,\eta,st) = 0 & \text{otherwise.} \end{cases}$$

*iii) if $\pi$ does not come from größencharacters, $ord_{s=1}L_S(s,\pi,\eta,st) = 0$ for every unitary $\eta \in \widehat{\mathbb{Q}^\times \backslash \mathbb{A}^\times}$.*

Complementing Kudla-Rallis [2] by Proposition 2, we can give the following characterization of cuspidality of $\Pi(Sp_2(\mathbb{A}))$ by standard $L$-functions:

**THEOREM 2** *Non CAP $\pi \in \Pi(Sp_2(\mathbb{A}))$ is cuspidal, iff all the i) $\sim$ iii) are satisfied: For unitary $\eta \in \widehat{\mathbb{Q}^\times \backslash \mathbb{A}^\times}$,*

*i)* $L_S(s,\pi,\eta,st)$ *is entire at* $\Re s > 1$;

*ii)* *if* $\eta^2 = 1$, $ord_{s=1} L_S(s,\pi,\eta,st) \geq -1$;

*iii)* *if* $\eta^2 \neq 1$, $ord_{s=1} L_S(s,\pi,\eta,st) \geq 0$.

**PROOF.** By corollary 7.2.3, Theorem 7.2.5 of [2], and Soudry [8], cuspidal $\pi$ satisfies $i), ii)$. (If the standard $L$-function has a simple pole at $s = 2$, then $\pi$ is liftable to $O(2)$, and is a CAP representation.) Hence, our task is to show that both of $i), ii)$ are not satisfied by non-cuspidal $\pi'$ which is induced from by cuspidal $\sigma \in \Pi(GL_2(\mathbb{Q}_\mathbb{A}))$. Put $\chi_1 = \chi/|\chi|$ and let $\chi_\infty = |\cdot|^{t+si} sign^a$ with $t, s \in \mathbb{R}$ and $a = 0$ or $1$. In the case of $|\chi_\infty| \neq |\cdot|, |\cdot|^3$, it holds

$$\begin{cases} L_S(s,\pi',\chi_1,st) \text{ has a double pole at a point in the reagion } \Re s \geq 1 & \text{if } \chi_1^2 = 1 \\ L_S(s,\pi',\chi_1,st) \text{ is not eintire in } \Re s \geq 1 & \text{otherwise.} \end{cases}$$

Indeed, from (6), $L_S(s,\pi',\chi_1,st) = \zeta_S(s-t)\zeta_S(s+t)L_S(s,\sigma,\chi_1,st)$, if $\chi_1^2 = 1$. Obviously, this $L$-function has a simple pole at $1+|t|$, if $t \neq 0$. In the case of $|\chi_\infty| = |\cdot|$ or $|\cdot|^3$, we can say

$$\begin{cases} L_S(s,\pi,\chi_1,st) \text{ has a simple pole at } s = 2 & \text{if } \chi_1^2 = 1 \\ L_S(s,\pi,\chi_1,st) \text{ is not eintire in } \Re s \geq 1 & \text{otherwise.} \end{cases}$$

We are going to see that $L_S(s,\pi,\chi_1,st)$ has a double pole at a point in the region $\Re s \geq 1$ if $\chi_1^2 = 1$, and that $L_S(s,\pi,\chi_1,st)$ is not eintire in $\Re s \geq 1$ otherwise. If $\chi_1^2 = 1$, then

$$L_S(s,\pi,\chi_1,st) = \zeta_S(s)^2 L(s,\sigma,\chi_1,st),$$

which has a double (at least) pole at $s = 1$. If $\chi_1^2 \neq 1$, then

$$L_S(s,\pi,\chi_1,st) = \zeta_S(s) L(s,\chi_1^2) L(s,\sigma,\chi_1,st),$$

which has a simple (at least) pole at $s = 1$. This completes the proof. □

**REMARK 3** *If $\pi$ satisfies the generalized Ramanujan conjecture, we only need to see ii),iii) for the cuspidality of $\pi$.*

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