

ON STARLIKE, CONVEX AND CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

SHIGEYOSHI OWA (近畿大・理工 尾和重義)

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of complex order b ($b \neq 0$, complex) if and only if $f(z)/z \neq 0$ ($z \in \mathbb{U}$) and

$$(1.2) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}_0^*(b)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of complex order b . A function $f(z)$ in $\mathcal{A}$ is said to be convex of complex order b ($b \neq 0$, complex) if and only if $f'(z) \neq 0$ ($z \in \mathbb{U}$) and

$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

Also we denote by $\mathcal{K}_0(b)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of complex order b . Note that $f(z) \in \mathcal{K}_0(b)$ if and only if $zf'(z) \in \mathcal{S}_0^*(b)$.

REMARK 1. The class $\mathcal{S}_0^*(b)$ was introduced by Nasr and Aouf [3], and the class $\mathcal{K}_0(b)$ was introduced by Nasr and Aouf [4].

REMARK 2. Letting $b = 1 - \alpha$, we observe that $\mathcal{S}_0^*(1-\alpha) = \mathcal{S}^*(\alpha)$ and $\mathcal{K}_0(1-\alpha) = \mathcal{K}(\alpha)$, where $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike functions and convex functions of order α , respectively.

A function $f(z)$ belonging to the class Λ is said to be close-to-convex of complex order b ($b \neq 0$, complex) if and only if there exists a function $g(z) \in S_0^*(1)$ satisfying the condition

$$(1.4) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} \left[\frac{f'(z)}{g'(z)} - 1 \right] \right\} > 0 \quad (z \in U).$$

We denote by $C_0(b)$ the subclass of Λ consisting of functions which are close-to-convex of complex order b . We note that $C_0(1-\alpha) = C(\alpha)$, where $C(\alpha)$ is the class of close-to-convex functions of order α .

2. FORMER RESULTS

THEOREM A (Nasr and Aouf [3]), A function $f(z)$ is in the class $S_0^*(b)$ if and only if there exists a probability measure $\mu(t)$ ($0 \leq t < 2\pi$) such that

$$(2.1) \quad f(z) = z \exp \left\{ \int_0^{2\pi} -2b \log(1 - ze^{-it}) d\mu(t) \right\}.$$

THEOREM B (Nasr and Aouf [3]), If a function $f(z)$ is in the class $S_0^*(b)$ with $\operatorname{Re}(b) > 0$, then

$$(2.2) \quad |a_n| \leq \frac{1}{(n-1)!} \prod_{m=0}^{n-2} |2b + m| \quad (n \geq 2).$$

The equality in (2.2) is attained for the function

$$(2.3) \quad f(z) = \frac{z}{(1-z)^b} = z + \sum_{n=2}^{\infty} \prod_{m=0}^{n-2} \left(\frac{2b+m}{m+1} \right) z^n.$$

THEOREM C (Nasr and Aouf [3]), If a function $f(z)$ is in the class $S_0^*(b)$, then

$$(2.4) \quad |a_3 - \mu a_2^2| \leq |b| \max(1, |4b\mu - 2b - 1|),$$

where μ is a complex number.

THEOREM D (Nasr and Aouf [3]), If a function $f(z)$ is in the class

$S_0^*(b)$, then

$$(2.5) \quad \left| \frac{zf'(z)}{f(z)} \right| \leq 1 + \frac{2|b|r}{1-r} \quad (|z| = r < 1).$$

The equality in (2.5) is attained for the function

$$(2.6) \quad f(z) = \frac{z}{(1-z)^{2b}}.$$

3. SOME RESULTS

In order to derive our results, we have to recall here the following lemma.

LEMMA I (Miller and Mocanu [2]). Let $q(z)$ be univalent in the unit disk $\mathbb{U}$, $\theta(w)$ and $\phi(w)$ be analytic in the domain $\mathbb{D}$ containing $q(\mathbb{U})$, and $\phi(w) \neq 0$ for $w \in q(\mathbb{U})$. Set

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z).$$

Suppose that

(i) $Q(z)$ is starlike (univalent) in $\mathbb{U}$ with $Q(0) = 0$, $Q'(0) \neq 0$;

and

$$(ii) \quad \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$(3.1) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of the subordination (3.1).

REMARK 3. The univalent function $q(z)$ is said to be a dominant of the differential subordination

$$(3.2) \quad \Psi(p(z), zp'(z)) \prec h(z)$$

if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3.2). In particular, if $\tilde{q}(z)$ is a dominant of (3.2) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (3.2), then $\tilde{q}(z)$ is said to be the best dominant of (3.2).

THEOREM I (Obradović, Aouf and Owa [5]), If a function $f(z)$ is in the class $S_0^*(b)$, then

$$(3.3) \quad \left\{ \frac{f(z)}{z} \right\}^a \prec \frac{1}{(1-z)^{2ab}},$$

where a is a complex number, $a \neq 0$, and either $|2ab + 1| \leq 1$ or $|2ab - 1| \leq 1$. The function $1/(1-z)^{2ab}$ is the best dominant of the differential subordination (3.3).

PROOF. Letting $q(z) = 1/(1-z)^{2ab}$, $\theta(w) = 1$ and $\phi(w) = 1/abw$ in Lemma 1, we see that $Q(z) = 2z/(1-z)$ and $h(z) = (1+z)/(1-z)$. Therefore, $Q(z)$ is starlike (univalent) in the unit disk $\mathbb{U}$, $Q(0) = 0$, $Q'(0) = 2 \neq 0$, and

$$\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{1}{1-z} \right\} > 0 \quad (z \in \mathbb{U}).$$

Thus, the conditions (i) and (ii) in Lemma 1 are satisfied.

Also, we see that $q(z)$ is univalent in $\mathbb{U}$ by Royster [6]. We define the function $p(z)$ by $p(z) = (f(z)/z)^a$ for $f(z) \in S_0^*(b)$. Then $p(z)$ is analytic in $\mathbb{U}$, $p(z) = 1 + p_1z + p_2z^2 + \dots$, and $p(z) \neq 0$ for $0 < |z| < 1$. Since

$$(3.4) \quad \begin{aligned} \theta(p(z)) + zp'(z)\phi(p(z)) &= 1 + \frac{1}{ab} \frac{zp'(z)}{p(z)} \\ &= 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right), \end{aligned}$$

$f(z) \in S_0^*(b)$ implies that

$$(3.5) \quad 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z} = h(z).$$

Consequently, with the help of Lemma 1, we observe that

$$(3.6) \quad 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1+z}{1-z} \\ \Rightarrow \left(\frac{f(z)}{z} \right)^a \prec \frac{1}{(1-z)^{2ab}}.$$

This completes the assertion of Theorem 1.

Taking $a = -1/2b$ in Theorem 1, we have

COROLLARY 1. If a function $f(z)$ is in the class $S_0^*(b)$, then

$$(3.7) \quad \left(\frac{z}{f(z)} \right)^{1/2b} \prec 1 - z,$$

and

$$(3.8) \quad \left| \left(\frac{z}{f(z)} \right)^{1/2b} - 1 \right| < |z| \quad (z \in \mathbb{U}).$$

REMARK 4. If $b = 1 - \alpha$ ($0 \leq \alpha < 1$), Corollary 1 becomes that

$$f(z) \in S^*(\alpha) \Rightarrow \left| \left(\frac{z}{f(z)} \right)^{1/2(1-\alpha)} - 1 \right| < |z| \quad (z \in \mathbb{U}).$$

This is the former result given by Todorov [8].

COROLLARY 2. If a function $f(z)$ is in the class $K_0(b)$, then

$$(3.9) \quad (f'(z))^a \prec \frac{1}{(1-z)^{2ab}},$$

where a is a complex number, $a \neq 0$, and either $|2ab + 1| \leq 1$ or $|2ab - 1| \leq 1$.

Next, we show

THEOREM 2 (Obradović, Aouf and Owa [5]). Let a function $f(z)$ be

in the class $S_0^*(b)$, and a be a complex number, $a \neq 0$, $0 < 2ab \leq 1$. Then

$$(3.10) \quad \operatorname{Re} \left\{ \frac{f(z)}{z} \right\}^a > \frac{1}{2^{2ab}} \quad (z \in \mathbb{U})$$

and

$$(3.11) \quad \left| \left(\frac{f(z)}{z} \right)^{-a} - 2^{2ab-1} \right| < 2^{2ab-1} \quad (z \in \mathbb{U}).$$

The estimates are best possible.

PROOF. Define the function $g(\theta) = \cos(\mu\theta) - \cos^\mu \theta$ for $0 < \mu \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$. Then we see that $g(\theta)$ is an even function of θ and

$$g'(\theta) = \mu \{ \cos^{\mu-1} \theta \sin \theta - \sin(\mu\theta) \}$$

$$\geq \mu \{ \sin \theta - \sin(\mu\theta) \}$$

$$\geq 0$$

for $0 < \mu \leq 1$ and $0 \leq \theta \leq \pi/2$. This proves that $g(\theta) \geq g(0) = 0$, that is, that

$$\cos(\mu\theta) - \cos^\mu \theta \geq 0$$

for $0 < \mu \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$.

Letting $\mu = 2ab$ and $\theta = \phi/2 - \pi/2$ ($0 \leq \phi \leq 2\pi$), we obtain that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{1}{(1 - e^{i\phi})^{2ab}} \right\} &= \left(2 \sin \frac{\phi}{2} \right)^{-\mu} \cos \{ \mu(\phi/2 - \pi/2) \} \\ &= (2 \cos \theta)^{-\mu} \cos(\mu\theta). \end{aligned}$$

Therefore, using Theorem 1, we have

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\}^a > \operatorname{Re} \left\{ \frac{1}{(1 - e^{i\phi})^{2ab}} \right\} \geq \frac{1}{2^{2ab}} \quad (0 \leq \phi \leq 2\pi).$$

This completes the first half of the theorem.

The second half of the theorem follows from the above and the

result given by Wilken and Feng [10].

Further, taking the function $f(z) = z/(1 - z)^{2b}$, we see that the results of the theorem are sharp.

In order to derive our next results, we need the following lemma.

LEMMA 2 (Jack [1]). Let $w(z)$ be regular in the unit disk U and such that $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0 , then we have

$$z_0 w'(z_0) = k w(z_0),$$

where k is real and $k \geq 1$.

With the aid of the above lemma, we prove

THEOREM 3. If a function $f(z)$ belonging to A satisfies

$$(3.12) \quad \left| \frac{f'(z)}{g'(z)} - 1 \right|^\alpha \left| \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2} \right|^\beta < |b|^{\alpha+\beta} \quad (z \in U)$$

for some $\alpha \geq 0$, $\beta \geq 0$, and $g(z) \in S_0^*(1)$, then $f(z) \in C_0(b)$.

PROOF. Defining the function $w(z)$ by

$$(3.13) \quad w(z) = \frac{1}{b} \left(\frac{f'(z)}{g'(z)} - 1 \right)$$

for $f(z)$ belonging to A and $g(z)$ belonging to $S_0^*(1)$, we see that $w(z)$ is regular in the unit disk U and $w(0) = 0$. Noting that

$$(3.14) \quad b z w'(z) = \frac{zf''(z)}{g'(z)} - \frac{zf'(z)g''(z)}{g'(z)^2},$$

we know that (3.12) can be written

$$(3.15) \quad |b w(z)|^\alpha |b z w'(z)|^\beta < |b|^{\alpha+\beta}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$(3.16) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then, Lemma 2 leads to

$$(3.17) \quad |bw(z_0)|^\alpha |bz_0 w'(z_0)|^\beta = |b|^{\alpha+\beta} k^\beta \quad (k \geq 1) \\ \geq |b|^{\alpha+\beta}$$

which contradicts our condition (3.12). Therefore, we conclude that

$|w(z)| < 1$ for all $z \in U$, that is, that

$$\left| \frac{1}{b} \left(\frac{f'(z)}{g'(z)} - 1 \right) \right| < 1 \quad (z \in U).$$

This implies that

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{f'(z)}{g'(z)} - 1 \right) \right\} > 0 \quad (z \in U),$$

which proves that $f(z) \in C_0(b)$.

Letting $g(z) = z \in S_0^*(1)$ in Theorem 3, we have

COROLLARY 3. If a function $f(z)$ belonging to A satisfies

$$(3.18) \quad |f'(z) - 1|^\alpha |zf''(z)|^\beta < |b|^{\alpha+\beta} \quad (z \in U)$$

for some $\alpha \geq 0$, $\beta \geq 0$, then $f(z) \in C_0(b)$.

THEOREM 4. If a function $f(z)$ belonging to A satisfies

$$(3.19) \quad \left| a \left(\frac{zf'(z)}{f(z)} - 1 \right) + (1-a) \frac{z^2 f''(z)}{f(z)} \right| < |b| \quad (z \in U)$$

for $0 \leq a \leq 1$, $|b| \leq 1$, then $f(z) \in S_0^*(b)$.

PROOF. Let

$$(3.20) \quad w(z) = \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right)$$

for $f(z) \in \mathcal{A}$. Then the function $w(z)$ is regular in the unit disk $\mathbb{U}$ and $w(0) = 0$. Making use of the logarithmic differentiations in both sides of (3.20), we have

$$(3.21) \quad \frac{zf''(z)}{f'(z)} = bw(z) + \frac{bzw'(z)}{1 + bw(z)}.$$

It follows that

$$(3.22) \quad a \left(\frac{zf'(z)}{f(z)} - 1 \right) + (1 - a) \frac{z^2 f''(z)}{f(z)} \\ = bw(z) \left\{ 1 + (1 - a)bw(z) + (1 - a) \frac{zw'(z)}{w(z)} \right\},$$

or

$$(3.23) \quad \left| bw(z) \left\{ 1 + (1 - a) \left(bw(z) + \frac{zw'(z)}{w(z)} \right) \right\} \right| < |b|.$$

Assume that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then we can write $w(z_0) = e^{i\theta}$. Therefore, using Lemma 2, we see that

$$(3.24) \quad \left| bw(z_0) \left\{ 1 + (1 - a) \left(bw(z_0) + \frac{z_0 w'(z_0)}{w(z_0)} \right) \right\} \right| \\ = |b| |1 + (1 - a)(k + be^{i\theta})| \\ \geq |b| |1 + (1 - a)(1 - |b|)| \\ \geq |b|$$

which contradicts (3.23). Thus we have

$$(3.25) \quad |w(z)| = \left| \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < 1 \quad (z \in \mathbb{U}).$$

Noting that (3.25) implies

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \quad (z \in \mathbb{U}),$$

we complete the proof of Theorem 4.

Further, we derive

THEOREM 5. If a function $f(z)$ belonging to $\mathcal{A}$ satisfies

$$(3.26) \quad \left| \frac{f'(z)}{g'(z)} - 1 \right|^\alpha \left| \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} \right|^\beta < |b|^\alpha \left(\frac{|b|}{1 + |b|} \right)^\beta$$

for some $\alpha \geq 0$, $\beta \geq 0$, $g(z) \in \mathcal{S}_0^*(1)$, and for all $z \in \mathbb{U}$, then $f(z) \in \mathcal{C}_0(b)$.

PROOF. Defining the function $w(z)$ by (3.13), we have

$$(3.27) \quad \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} = \frac{bzw'(z)}{1 + bw(z)}.$$

Therefore, the condition (3.26) leads to

$$(3.28) \quad |bw(z)|^\alpha \left| \frac{bzw'(z)}{1 + bw(z)} \right|^\beta < |b|^\alpha \left(\frac{|b|}{1 + |b|} \right)^\beta.$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$(3.29) \quad \max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Then letting $w(z_0) = e^{i\theta}$, and applying Lemma 2, we obtain that

$$(3.30) \quad |bw(z_0)|^\alpha \left| \frac{bz_0 w'(z_0)}{1 + bw(z_0)} \right|^\beta = |b|^\alpha \left| \frac{bk w(z_0)}{1 + bw(z_0)} \right|^\beta \\ \geq |b|^\alpha \left(\frac{|b|}{1 + |b|} \right)^\beta$$

which contradicts our condition (3.26). Thus we see that $f(z) \in C_0(b)$.

Taking $g(z) = z \in S_0^*(1)$ in Theorem 5, we have

COROLLARY 4. If a function $f(z)$ belonging to A satisfies

$$(3.31) \quad |f'(z) - 1|^\alpha \left| \frac{zf''(z)}{f'(z)} \right|^\beta < |b|^\alpha \left(\frac{|b|}{1 + |b|} \right)^\beta \quad (z \in U)$$

for some $\alpha \geq 0$, $\beta \geq 0$, then $f(z) \in C_0(b)$.

4. COEFFICIENT INEQUALITIES

In this section, we consider the coefficient inequalities of functions to be in the classes $S_0^*(b)$, $K_0(b)$ and $C_0(b)$.

THEOREM 6. If a function $f(z)$ belonging to A satisfies

$$(4.1) \quad \sum_{n=2}^{\infty} (n + |b| - 1) |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then $f(z) \in S_0^*(b)$.

PROOF. It is easy to see that the condition (4.1) implies that

$$(4.2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} n a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right|$$

$$< \frac{\sum_{n=2}^{\infty} n |a_n|}{1 - \sum_{n=2}^{\infty} |a_n|}$$

$$\leq |b|.$$

Thus we have $f(z) \in S_0^*(b)$.

Similarly, we have

THEOREM 7. If a function $f(z)$ belonging to A satisfies

$$(4.3) \quad \sum_{n=2}^{\infty} n(n + |b| - 1) |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then $f(z) \in K_0(b)$.

REMARK 5. Letting $b = 1 - \alpha$ ($0 \leq \alpha < 1$) in Theorem 6 and Theorem 7, we have the corresponding results by Silverman [7].

THEOREM 8. If a function $f(z)$ belonging to A satisfies

$$(4.4) \quad \sum_{n=2}^{\infty} n |a_n| \leq |b| \quad (b \text{ is complex, } b \neq 0),$$

then $f(z) \in C_0(b)$.

Now, we introduce the subclasses $S_1^*(b)$, $K_1(b)$ and $C_1(b)$ of A consisting of functions $f(z)$ which satisfy the coefficient inequalities (4.1), (4.3) and (4.4), respectively.

REMARK 6. It follows from the definitions of $S_1^*(b)$, $K_1(b)$ and $C_1(b)$ that

$$\begin{aligned} \text{(i)} \quad f(z) \in S_1^*(b) &\implies |a_n| \leq \frac{|b|}{n + |b| - 1} & (n \geq 2), \\ \text{(ii)} \quad f(z) \in K_1(b) &\implies |a_n| \leq \frac{|b|}{n(n + |b| - 1)} & (n \geq 2), \\ \text{(iii)} \quad f(z) \in C_1(b) &\implies |a_n| \leq \frac{|b|}{n} & (n \geq 2). \end{aligned}$$

Next, we prove

THEOREM 9. If a function $f(z)$ is in the class $S_1^*(b)$, then

$$(4.5) \quad |z| - \frac{|b|}{1 + |b|} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{1 + |b|} |z|^2$$

for $z \in U$. If $0 < |b| \leq 1$, then

$$(4.6) \quad 1 - \frac{2|b|}{1 + |b|} |z| \leq |f'(z)| \leq 1 + \frac{2|b|}{1 + |b|} |z|$$

for $z \in U$. The estimates in (4.5) and (4.6) are sharp.

PROOF. Noting that

$$(4.7) \quad \sum_{n=2}^{\infty} |a_n| \leq \frac{|b|}{1 + |b|}$$

for $f(z) \in S_1^*(b)$, we have

$$(4.8) \quad |f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \geq |z| - \frac{|b|}{1 + |b|} |z|^2$$

and

$$(4.9) \quad |f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |z| + \frac{|b|}{1 + |b|} |z|^2.$$

Further, if $0 < |b| \leq 1$, then we have

$$(4.10) \quad \frac{1 + |b|}{2} \sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} (n + |b| - 1) |a_n| \leq |b|,$$

or

$$(4.11) \quad \sum_{n=2}^{\infty} n |a_n| \leq \frac{2|b|}{1 + |b|}.$$

Thus we obtain

$$(4.12) \quad |f'(z)| \geq 1 - |z| \sum_{n=2}^{\infty} n |a_n| \geq 1 - \frac{2|b|}{1 + |b|} |z|$$

and

$$(4.13) \quad |f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n|a_n| \leq 1 + \frac{2|b|}{1+|b|} |z|.$$

Finally, taking the function

$$(4.14) \quad f(z) = z - \frac{|b|}{1+|b|} z^2,$$

we see that the estimates in (4.5) and (4.6) are sharp.

Using the same manner as in the proof of Theorem 9, we have

THEOREM IO. If a function $f(z)$ is in the class $K_1(b)$, then

$$(4.15) \quad |z| - \frac{|b|}{2(1+|b|)} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{2(1+|b|)} |z|^2$$

and

$$(4.16) \quad 1 - \frac{|b|}{1+|b|} |z| \leq |f'(z)| \leq 1 + \frac{|b|}{1+|b|} |z|$$

for $z \in \mathbb{U}$. The estimates in (4.15) and (4.16) are sharp for the function

$$(4.17) \quad f(z) = z - \frac{|b|}{2(1+|b|)} z^2.$$

THEOREM II. If a function $f(z)$ is in the class $C_1(b)$, then

$$(4.18) \quad |z| - \frac{|b|}{2} |z|^2 \leq |f(z)| \leq |z| + \frac{|b|}{2} |z|^2$$

and

$$(4.19) \quad 1 - |b||z| \leq |f'(z)| \leq 1 + |b||z|$$

for $z \in \mathbb{U}$. The estimates in (4.8) and (4.9) are sharp for the function

$$(4.20) \quad f(z) = z - \frac{|b|}{2} z^2.$$

Finally, we give

CONJECTURE. Let a function $f(z) \in A$ be defined by

$$(4.21) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

Then

(i) $f(z) \in S_0^*(b)$ if and only if

$$\sum_{n=2}^{\infty} (n + |b| - 1) a_n \leq |b|.$$

(ii) $f(z) \in K_0(b)$ if and only if

$$\sum_{n=2}^{\infty} n(n + |b| - 1) a_n \leq |b|.$$

(iii) $f(z) \in C_0(b)$ with $g(z) = z \in S_0^*(1)$ if and only if

$$\sum_{n=2}^{\infty} n a_n \leq |b|.$$

REMARK 7. The above conjectures (i), (ii) and (iii) are true for $b = 1 - \alpha$ ($0 \leq \alpha < 1$).

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Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan