

EXISTENCE AXIOM OF PRE-INDEPENDENCE RELATIONS IN NSOP₁ THEORIES

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1. INTRODUCTION

This note is based on the author's talk in RIMS Symposia, "Model theoretic aspects of the notion of independence and dimension", held on December 4–6, 2023.

Kaplan and Ramsey introduced Kim-independence in [8], which partially inherits what non-forking independence satisfies in simple theories, assuming the theory is NSOP₁, and it successfully gives a characterization of NSOP₁ theories. However, it is still difficult to say that Kim-independence is a complete generalization of non-forking independence since it is still not known whether Kim-independence is equivalent to non-forking independence in simple theories over an arbitrary set. The goal of our work is finding a pre-independence relation over sets which generalizes feature of non-forking independence in simple theories over sets and Kim-independence in NSOP₁ theories over models. In other words, we want to find a pre-independence relation $\downarrow$ such that:

- (i) $\downarrow = \downarrow^f$ in simple theories over an arbitrary set,
- (ii) $\downarrow = \downarrow^K$ in NSOP₁ theories over models,

and satisfies properties what $\downarrow^K$ satisfies in NSOP₁ theories over models (such as existence, symmetry, independence theorem, etc.), in NSOP₁ theories over an arbitrary set.

As a partial achievement of this goal, we prove that $\downarrow^{Kf}$ satisfies (i), (ii) above, and existence in NSOP₁ theories over sets. We leave a sketch of proof of the main theorem (Theorem 3.12) below.

2. PRELIMINARY AND NOTATION

We quote the following notions of pre-independence relations from [1], [2], and [3].

Definition 2.1. [1][2][3, Definition 2.4] A *pre-independence relation* is an invariant ternary relation $\downarrow$ on sets. If a triple of sets (a, b, c) is in the pre-independence relation $\downarrow$, then we write it $a \downarrow_c b$ and say "*a is $\downarrow$ -independent from b over c*". Throughout this paper we will consider the following properties for a pre-independence relation. (If it is clear in the context, then we omit the words in the parenthesis.)

- (i) Monotonicity (over d): If $aa' \downarrow_d bb'$, then $a \downarrow_d b$.
- (ii) Base monotonicity (over d): If $a \downarrow_d bb'$, then $a \downarrow_{db} b'$.
- (iii) Transitivity (over d): If $a \downarrow_{db} c$ and $b \downarrow_d c$, then $ab \downarrow_d c$.

- (iv) Right extension (over d): If $a \perp_d b$, then for all c , there exists $c' \equiv_{db} c$ such that $a \perp_d bc'$.
- (v) Existence (over d): $a \perp_d d$ for all a . We say a set d is an *extension base* for $\perp$ if $\perp$ satisfies existence over d .
- (vi) Finite character (over d): If $a \not\perp_d b$, then there exist finite $a' \subseteq a$ and $b' \subseteq b$ such that $a' \not\perp_d b'$.
- (vii) Strong finite character (over d): If $a \not\perp_d b$, then there exist finite subtuple $b' \subseteq b$, finite tuples x', y' of variables with $|x'| \leq |a|$, $|y'| = |b'|$, and a formula $\varphi(x', y') \in \mathcal{L}(d)$ such that $\varphi(x', b') \in \text{tp}(a/db)$ and $a' \not\perp_d b'$ for all $a' \models \varphi(x', b')$.

Definition 2.2. We say a formula with parameters $\varphi(x, a)$ divides over a set B , if there exists an indiscernible sequence $(a_i)_{i < \omega}$ over B with $a_0 = a$ such that $\{\varphi(x, a_i)\}_{i < \omega}$ is inconsistent. We say $\varphi(x, a)$ forks over B if there are dividing formulas $\psi_0(x, b_0), \dots, \psi_{n-1}(x, b_{n-1})$ over B such that $\varphi(x, a) \vdash \bigvee_{i < n} \psi_i(x, b_i)$.

We write $a \perp_C^f b$ if $\text{tp}(a/bC)$ has no forking formula over C . We write $a \perp_C^i b$ if there exists a global invariant type over C containing $\text{tp}(a/bC)$. We write $a \perp_C^u b$ if $\text{tp}(a/bC)$ is finitely satisfiable in C .

The definitions of Kim-independence was introduced by Kaplan and Ramsey in [8]. We import a generalized version of it introduced by Mutchnik in [17].

Definition 2.3. Let $\perp$ be a pre-independence relation, κ an infinite cardinal, A a set of parameters. We call a sequence $(b_i)_{i < \kappa}$ an $\perp$ -Morley sequence over A if it is indiscernible over A and $b_i \perp_A b_{<i}$ for all $i < \kappa$. Let $\varphi(x, y)$ be a formula. We say $\varphi(x, b) \perp$ -Kim-divides over A if there exists an $\perp$ -Morley sequence $(b_i)_{i < \kappa}$ over A with $b_0 = b$ such that $\{\varphi(x, b_i)\}_{i < \kappa}$ is inconsistent. We say $\varphi(x, b) \perp$ -Kim-forks over A if there exist $\psi_0(x, b_0), \dots, \psi_n(x, b_n)$ such that $\varphi(x, b) \vdash \bigvee_{i \leq n} \psi_i(x, b_i)$ and each $\psi_i(x, b_i) \perp$ -divides over A . We say a partial type $\Sigma(x) \perp$ -Kim-forks over A if it implies a formula that $\perp$ -Kim-forks over A .

For a given pre-independence relation $\perp$, we say a is $\perp$ -Kim-independent from b over C and write $a \perp_C^{K\perp} b$ if $\text{tp}(a/bC)$ does not $\perp$ -forks over C . We will write $\perp^{Kf}, \perp^{Ki}$ for $\perp^{K\perp f}, \perp^{K\perp i}$ respectively, to simplify notation.

By $a \perp_C^{Kd^i} b$, we mean $\text{tp}(a/bC)$ has no $\perp^i$ -Kim-dividing formula.

We will frequently use a notion of ill-founded tree $\mathcal{T}_\alpha^\delta$ which is originally introduced in [8] by Kaplan and Ramsey. In this note we import a generalized version which appears in [13], with some additional notations.

Definition 2.4. [8][13] Suppose α and δ are ordinals. We define $\mathcal{T}_\alpha^\delta$ to be the set of functions η so that

- (i) $\text{dom}(\eta)$ is an end-segment of α of the form $[\beta, \alpha)$ for β equal to 0 or a successor ordinal. If α is a successor or 0, we allow $\beta = \alpha$, i.e. $\text{dom}(\eta) = \emptyset$. Note that $\mathcal{T}_0^\delta = \{\emptyset\}$.
- (ii) $\text{ran}(\eta) \subseteq \delta$.
- (iii) Finite support: the set $\{\gamma \in \text{dom}(\eta) : \eta(\gamma) \neq 0\}$ is finite.

If $\delta = \omega$, then we just write $\mathcal{T}_\alpha$.

Let $\mathcal{L}_{s,\alpha} = \{\triangleleft, <_{lex}, \wedge, \{P_\beta\}_{\beta < \alpha}\}$ for each ordinal α , where $\triangleleft, <_{lex}$ are binary relation symbols, $\wedge$ is a binary function symbol, and P_β is a unary relation symbol

for each β . If it is clear in the context, we will omit ' α ' in $\mathcal{L}_{s,\alpha}$ and just write it $\mathcal{L}_s$. We interpret $\mathcal{T}_\alpha^\delta$ as an $\mathcal{L}_s$ -structure by defining each symbol as below.

- (iv) $\eta \triangleleft \nu$ if and only if $\eta \subseteq \nu$. Write $\eta \perp \nu$ if $\neg(\eta \triangleleft \nu)$ and $\neg(\nu \triangleleft \eta)$.
- (v) $\eta \wedge \nu = \eta|_{[\beta,\alpha]} = \nu|_{[\beta,\alpha]}$ where $\beta = \min\{\gamma : \eta|_{[\gamma,\alpha]} = \nu|_{[\gamma,\alpha]}\}$, if non-empty (note that β will not be a limit, by finite support). Define $\eta \wedge \nu$ to be the empty function if this set is empty (note that this cannot occur if α is a limit).
- (vi) $\eta <_{lex} \nu$ if and only if $\eta \triangleleft \nu$ or, $\eta \perp \nu$ with $\text{dom}(\eta \wedge \nu) = [\gamma + 1, \alpha]$ and $\eta(\gamma) < \nu(\gamma)$.
- (vii) For each ordinal $\beta < \alpha$, let $P_\beta^{\alpha,\delta} = \{\eta \in \mathcal{T}_\alpha^\delta : \text{dom}(\eta) = [\beta, \alpha]\}$ (the β -th floor in $\mathcal{T}_\alpha^\delta$). If it is clear in the context, we omit α and δ , just write P_β . Note that if β is limit then P_β is empty.
- (viii) Canonical inclusion: For $\alpha < \alpha'$, $\mathcal{T}_\alpha^2$ can be embedded in $\mathcal{T}_{\alpha'}^2$ with respect to $\mathcal{L}_{s,\alpha'}$ by a map $f_{\alpha,\alpha'} : \mathcal{T}_\alpha^2 \rightarrow \mathcal{T}_{\alpha'}^2 : \eta \mapsto \eta \cup \{(\beta, 0) : \beta \in \alpha' \setminus \alpha\}$. Unless otherwise stated, we regard $\mathcal{T}_\alpha^2$ as $f_{\alpha,\alpha'}(\mathcal{T}_\alpha^2)$ in $\mathcal{T}_{\alpha'}^2$. Note that by finite support, $\mathcal{T}_\alpha^2$ can be regarded as $\bigcup_{\beta < \alpha} \mathcal{T}_\beta^2$ with respect to canonical inclusions, for each limit ordinal α .
- (ix) $\eta \perp_{lex} \nu$ if and only if $\eta <_{lex} \nu$ and $\eta \not\triangleleft \nu$. For an indexed set $\{a_\eta\}_{\eta \in \mathcal{T}_\alpha^2}$ and $\eta \in \mathcal{T}_\alpha^2$, by $a_{\perp_{lex} \eta}$ we mean the set $\{a_\nu : \nu \perp_{lex} \eta\}$.
- (x) For each $\eta \in \mathcal{T}_\alpha^\delta$, let $t(\eta)$ be an ordinal such that $\text{dom}(\eta) = [t(\eta), \alpha]$. If α is not limit, then $t(\emptyset) = \alpha$.
- (xi) For each $\eta \in \mathcal{T}_\alpha^\delta$, $i < \delta$, let $(i)^\frown \eta = \eta \cup \{(\alpha, i)\} \in \mathcal{T}_{\alpha+1}^\delta$.
- (xii) $(i)^\frown \mathcal{T}_\alpha^\delta = \{(i)^\frown \eta : \eta \in \mathcal{T}_\alpha^\delta\} \subseteq_{\alpha+1}^\delta$ for each α , δ and $i < \delta$.
- (xiii) We let $\eta^\frown (i) := \eta \cup \{(t(\eta) - 1, i)\}$, for each $i < \delta$ and $\eta \in \mathcal{T}_\alpha^\delta$ with $\text{dom}(\eta) \neq [0, \alpha]$.
- (xiv) $\text{dom}^*(\eta)$ is $\{i \in \text{dom}(\eta) : i \text{ is not a limit ordinal}\}$ for each $\eta \in \mathcal{T}_\alpha^\delta$.
- (xv) For $\beta < \alpha$, $\zeta_\beta \in \mathcal{T}_\alpha^\delta$ is a function from $[\beta, \alpha]$ to δ such that $\zeta(i) = 0$ for all $i \in [\beta, \alpha]$.
- (xvi) For all $v \subseteq \alpha$, $\mathcal{T}_{\alpha|v}^\delta$ is the set of all tuples $\eta \in \mathcal{T}_\alpha^\delta$ such that $t(\eta) \in v$ and $\eta(i) = 0$ for all $i \in \text{dom}(\eta) \setminus v$.

Now we recall the notions of indiscernibility of tree and modeling property (cf. [11], [12], and [18]). Let M be a structure in a language $\mathcal{L}$. For a tuple $\bar{a}$ of elements in M and a subset A of M , by $\text{qftp}_{\mathcal{L}}(\bar{a}/A)$ and $\text{tp}_{\mathcal{L}}(\bar{a}/A)$, we mean the set of quantifier-free $\mathcal{L}_A$ -formulas and the set of $\mathcal{L}_A$ -formulas realized by $\bar{a}$ in M respectively. If there is no confusion, we may omit the subscript $\mathcal{L}$.

Let $\mathcal{I}$ be a structure in a language $\mathcal{L}_{\mathcal{I}}$. For a set $\{b_i : i \in \mathcal{I}\}$ and a finite tuple $\bar{i} = (i_0, \dots, i_n)$ in $\mathcal{I}$, we write $\bar{b}_{\bar{i}}$ for the tuple $(b_{i_0}, \dots, b_{i_n})$. Let $\mathbb{M}$ be a monster model of a complete theory T in a language $\mathcal{L}$. By $\bar{a}_{\bar{\eta}} \equiv_{\Delta, A} \bar{b}_{\bar{\nu}}$ (or $\text{tp}_{\Delta}(\bar{a}_{\bar{\eta}}/A) = \text{tp}_{\Delta}(\bar{b}_{\bar{\nu}}/A)$), we mean that for any $\mathcal{L}_A$ -formula $\varphi(\bar{x}) \in \Delta$ where $\bar{x} = x_0 \cdots x_n$, $\bar{a}_{\bar{\eta}} \models \varphi(\bar{x})$ if and only if $\bar{b}_{\bar{\nu}} \models \varphi(\bar{x})$.

Definition 2.5. Let $\mathbb{M}$ be a monster model in a language $\mathcal{L}$ and $\mathcal{I}$ be an index structure in a language $\mathcal{L}_{\mathcal{I}}$.

- (i) For $(b_i)_{i \in \mathcal{I}}$ of tuples of elements in $\mathbb{M}$ and a subset A of $\mathbb{M}$, we say that $(b_i)_{i \in \mathcal{I}}$ is $\mathcal{I}$ -indiscernible over A if for any finite tuples $\bar{i}$ and $\bar{j}$ in $\mathcal{I}$,

$$\text{qftp}_{\mathcal{L}_{\mathcal{I}}}(\bar{i}) = \text{qftp}_{\mathcal{L}_{\mathcal{I}}}(\bar{j}) \text{ implies } \text{tp}_{\mathcal{L}}(\bar{b}_{\bar{i}}/A) = \text{tp}_{\mathcal{L}}(\bar{b}_{\bar{j}}/A).$$

- (ii) We say $(b_i)_{i \in \mathcal{I}}$ is $\mathcal{I}$ -locally based on $(a_i)_{i \in \mathcal{I}}$ over A if for all $\bar{i}$ and a finite set of $\mathcal{L}_A$ -formulas Δ , there is $\bar{j}$ such that $\text{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{i}) = \text{qftp}_{\mathcal{L}_\mathcal{I}}(\bar{j})$ and $\bar{b}_i \equiv_{\Delta, A} \bar{a}_{\bar{j}}$.

We say that $\mathcal{I}$ -indexed sets have the *modeling property* if for any $\mathcal{I}$ -indexed set $(a_i)_{i \in \mathcal{I}}$, there is an $\mathcal{I}$ -indiscernible set $(b_i)_{i \in \mathcal{I}}$, which is $\mathcal{I}$ -locally based on $(a_i)_{i \in \mathcal{I}}$.

Definition 2.6. For ordinals α and δ , let $\mathcal{T}_\alpha^\delta$ be a tree in the language $\mathcal{L}_s$ with interpretations in Definition 2.4. We refer to a $\mathcal{T}_\alpha^\delta$ -indexed indiscernible set as a *s-indiscernible tree*. We say that $(b_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ is *s-locally based* on $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ over A if $(b_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ is $\mathcal{T}_\alpha^\delta$ -locally based on $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ over A .

Notation 2.7. For a tree $\mathcal{T}_\alpha^\delta$ in the language $\mathcal{L}_s$, by $\bar{\eta} \sim_s \bar{\nu}$, we mean $\text{qftp}_{\mathcal{L}_s}(\bar{\eta}) = \text{qftp}_{\mathcal{L}_s}(\bar{\nu})$ and say they are *s-isomorphic*.

Fact 2.8. If δ is infinite, then for any ordinal α , a set A , and a tree of parameters $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$, there exists an s-indiscernible $(b_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ over A which is s-locally based on $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ over A .

The proof of the above fact can be found in [11], [12], and [18]. We call Fact 2.8 the *s-modeling property*.

The following is easy to show but important when we want to construct a sequence of s-indiscernible trees whose members are pairwise distinct. It will be implicitly used when we apply Lemma 3.9 and Lemma 3.10 in the proof of the main theorem.

Remark 2.9. Let δ and δ' be infinite cardinal with $\delta < \delta'$. If $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\delta}$ is s-indiscernible over A , then there exists an s-indiscernible tree $(b_\eta)_{\eta \in \mathcal{T}_\alpha^{\delta'}}$ over A such that $a_\eta = b_\eta$ for all $\eta \in \mathcal{T}_\alpha^\delta$ by compactness.

The following two techniques are useful when we want to construct an indiscernible sequence or tree whose information is based on a priorly given sequence or tree. Note that Fact 2.11 can be proved by using the same argument in [8, Lemma 5.10].

Fact 2.10. [10, Lemma 1.5] Let A be a set, δ a cardinal, and $\kappa = \beth_{\lambda^+}(\lambda)$ where $\lambda = 2^{|A|+|T|+\delta}$. For any sequence of parameters $(a_i)_{i < \kappa}$ with $|a_i| = |a_j| \leq \delta$ for all $i, j < \kappa$, there exists $(b_i)_{i < \omega}$ such that

- (i) for all $n < \omega$, there exists $i_0 < \dots < i_{n-1} < \kappa$ such that $b_0 \dots b_{n-1} \equiv_A a_{i_0} \dots a_{i_{n-1}}$,
- (ii) $(b_i)_{i < \omega}$ is indiscernible over A .

Fact 2.11. [8, Lemma 5.10] Let A be a set, δ a cardinal, δ' an infinite cardinal, and $\kappa = \beth_{\lambda^+}(\lambda)$ where $\lambda = 2^{|A|+|T|+\delta+\delta'}$. For any tree of parameters $(a_\eta)_{\eta \in \mathcal{T}_\kappa^{\delta'}}$ with $|a_\eta| = |a_\nu| \leq \delta$ for all $\eta, \nu \in \mathcal{T}_\kappa^{\delta'}$, there exists $(b_\eta)_{\eta \in \mathcal{T}_\omega^{\delta'}}$ such that

- (i) for all $n < \omega$, there exists $u \subseteq \kappa$ with $|u| = n$ such that $(b_\eta)_{\eta \in \mathcal{T}_n^{\delta'}} \equiv_A (a_\eta)_{\eta \in \mathcal{T}_{\kappa|u}^{\delta'}}$,
- (ii) for all finite $u, v \subseteq \omega$ with $|u| = |v|$, $(b_\eta)_{\eta \in \mathcal{T}_{\omega|u}^{\delta'}} \equiv_A (b_\eta)_{\eta \in \mathcal{T}_{\omega|v}^{\delta'}}$.

3. EXISTENCE OF $\perp$ -KIM-INDEPENDENCE

Lemma 3.1. Let T be any complete theory and $\mathbb{M}$ its monster model. Let $A, B \subseteq \mathbb{M}$ be sets and κ an infinite cardinal. Suppose that we have a sequence $(b_i)_{i < \kappa} \in \mathbb{M}$ such

that $b_i \downarrow_A^f b_{<i} B$ for all $i < \kappa$. Then for any set $C \subseteq \mathbb{M}$, there exists $(b'_i)_{i < \kappa} \equiv_{AB} (b_i)_{i < \kappa}$ such that $b'_i \downarrow_A^f b'_{<i} BC$ for all $i < \kappa$.

Lemma 3.2. Let T be any complete theory and $\mathbb{M}$ its monster model. Let A, B be small sets in the monster model and κ a sufficiently large infinite cardinal. Suppose that we have an indiscernible sequence $(b_i)_{i < \kappa}$ over AB such that $b_i \downarrow_A^f b_{<i} B$ for all $i < \kappa$. Then for any small set C , there exists $(b'_i)_{i < \kappa} \equiv_{AB} (b_i)_{i < \kappa}$ such that

- (i) $b'_i \downarrow_A^f b'_{<i} BC$ for all $i < \kappa$,
- (ii) $(b'_i)_{i < \kappa}$ is indiscernible over ABC .

If $\downarrow^f$ satisfies base monotonicity additionally, then $(b'_i)_{i < \kappa}$ is a $\downarrow^f$ -Morley sequence over D for any $A \subseteq D \subseteq ABC$.

Lemma 3.3. Let T be any complete theory, α an ordinal, λ an infinite cardinal, B a set, and $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ an s -indiscernible tree over B . If $(a'_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is s -locally based on $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ over B , then $(a'_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda} \equiv_B (a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$.

Lemma 3.4. Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, and B a set. If $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is an s -indiscernible tree over B , then it is s -indiscernible over $Ba_\emptyset$.

Lemma 3.5. Let T be any complete theory, α an ordinal, κ an infinite cardinal, and A, B, C sets. Let $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\kappa}$ be an s -indiscernible tree over BC such that $A \downarrow_C^f B(a_\eta)_{\eta \in \mathcal{T}_\alpha^\kappa}$. If $(a'_\eta)_{\eta \in \mathcal{T}_\alpha^\kappa}$ is s -locally based on $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\kappa}$ over ABC , then $A \downarrow_C^f B(a'_\eta)_{\eta \in \mathcal{T}_\alpha^\kappa}$.

Lemma 3.6. Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, and κ a cardinal. Let B be a set, $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ an s -indiscernible tree over B , and $(b_i)_{i < \kappa}$ a $\downarrow^f$ -Morley sequence over B with $b_0 := a_\emptyset$. Then there exists $(b'_i)_{i < \kappa} \equiv_{Bb_0} (b_i)_{i < \kappa}$ such that

- (i) $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is s -indiscernible over $Bb'_{<\kappa}$,
- (ii) $b'_i \downarrow_B^f (a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda} b'_{<i}$ for each $0 < i < \kappa$.

Definition 3.7. [4] A sequence of trees $(A_i)_{i < \beta}$ is said to be *mutually s -indiscernible* over a set D if A_i is s -indiscernible over $DA_{\neq i}$ for each $i < \beta$.

Remark 3.8. Let α be a successor ordinal. If a sequence of trees $((a_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i < \beta}$ is mutually s -indiscernible over D , then it is mutually s -indiscernible over $Da_\emptyset^{<\beta}$ by Lemma 3.4.

Lemma 3.9. Let T be any complete theory, α an ordinal, λ an infinite cardinal, and D a set. Assume that a mutually s -indiscernible sequence $((a_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i < \beta}$ over D is given. If there exists $((\hat{a}_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i \leq \beta}$ such that

- (i) $(\hat{a}_\eta^\beta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is s -indiscernible over $D(a_\eta^{<\beta})_{\eta \in \mathcal{T}_\alpha^\lambda}$,
- (ii) $(\hat{a}_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is s -locally based on $(a_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda}$ over $D(\hat{a}_\eta^{<i})_{\eta \in \mathcal{T}_\alpha^\lambda} (a_\eta^{>i})_{\eta \in \mathcal{T}_\alpha^\lambda} (\hat{a}_\eta^\beta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ for each $i < \beta$,
- (iii) $(\hat{a}_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda}$ is s -indiscernible over $D(\hat{a}_\eta^{<i})_{\eta \in \mathcal{T}_\alpha^\lambda} (a_\eta^{>i})_{\eta \in \mathcal{T}_\alpha^\lambda} (\hat{a}_\eta^\beta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ for each $i < \beta$,

then $((\hat{a}_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i < \beta} \equiv_D ((a_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i < \beta}$ and $((\hat{a}_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i \leq \beta}$ is mutually s -indiscernible over D .

Lemma 3.10. *Let T be any complete theory, α a successor ordinal, λ an infinite cardinal, κ a sufficiently large cardinal with $\kappa > \lambda$ and $\text{cf}(\kappa) = \kappa$, D a set, $(b_i)_{i < \kappa}$ a $\downarrow^f$ -Morley sequence over D , and $(a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda}$ an s -indiscernible tree over $Db_{<\kappa}$ with $a_\emptyset = b_\emptyset$. If $b_i \downarrow_D^f (a_\eta)_{\eta \in \mathcal{T}_\alpha^\lambda} (b_k)_{0 < k < i}$ for each $0 < i < \kappa$, then there exist $(A_i := (a_\eta^i)_{\eta \in \mathcal{T}_\alpha^\lambda})_{i < \kappa}$ and $(b'_i)_{i < \kappa} \equiv_{Db_0} (b_i)_{i < \kappa}$ such that $a_\eta^0 := a_\eta$ for each $\eta \in \mathcal{T}_\alpha^\lambda$, $b'_i = a_\emptyset^i$ for each $i < \kappa$, and satisfy*

- (i) $(A_i)_{i < \kappa}$ is indiscernible over D ,
- (ii) $(A_i)_{i < \kappa}$ is mutually s -indiscernible over D ,
- (iii) $b'_j \downarrow_D^f A_i (b'_k)_{i < k < j}$ for all $i < j < \kappa$.

Lemma 3.11. *Let T be any complete theory, α an ordinal, λ an infinite cardinal, $n < \omega$, $\phi(x), \varphi_0(x, y_0), \dots, \varphi_{n-1}(x, y_{n-1}) \in \mathcal{L}$, and $b_0, \dots, b_{n-1}$ tuples of parameters. Suppose that*

$$(i) \vdash \forall x(\phi(x) \leftrightarrow \varphi_0(x, b_0) \vee \dots \vee \varphi_{n-1}(x, b_{n-1})),$$

and for a sufficient large cardinal κ with $\kappa > \lambda$, there exist $(A_k := (a_\eta^k)_{\eta \in \mathcal{T}_\alpha^\lambda})_{k < \kappa}$, $f : \alpha^ \rightarrow n$, and $D := (b'_0, \dots, b'_{n-1}) \equiv (b_0, \dots, b_{n-1})$ satisfying*

- (ii) $\{\varphi_{f(t(\nu))}(x, a_\nu^k) \mid \nu \trianglelefteq \eta\}$ is consistent for each $k < \kappa$ and $\eta \in \mathcal{T}_\alpha^\lambda$,
- (iii) $a_\eta^k \equiv b_{f(t(\eta))}$ for all $k < \kappa$ and $\eta \in \mathcal{T}_\alpha^\lambda$,
- (iv) $(A_k)_{k < \kappa}$ is indiscernible over D ,
- (v) A_0 is s -indiscernible over D ,

where $\alpha^ := \alpha + 1$ if α is 0 or a successor, $\alpha^* := \alpha$ if α is limit. Then there exists $j < n$ and such that*

$$\{\varphi_{f(t(\nu))}(x, a_\nu^k) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b'_j)\}$$

is consistent for each $\eta \in \mathcal{T}_\alpha^\lambda$ and $k < \kappa$.

Now we are ready to prove the main theorem.

Theorem 3.12. *Suppose T is NSOP₁. Then $\downarrow^{K^f}$ satisfies existence over an arbitrary set.*

Proof. It is enough to show that $a \downarrow_E^{K^f} \emptyset$ for any tuple of parameters a and a set E . Without loss of generality, we may assume $E = \emptyset$. So we show $a \downarrow^{K^f} \emptyset$. Suppose not. Then there exists $\phi(x) \in \text{tp}(a)$ such that $\phi(x) \not\downarrow^f\text{-Kim-forks}$ over $\emptyset$. So there exist $\varphi_0(x, b_0), \dots, \varphi_{n-1}(x, b_{n-1})$ such that $\phi(x) \models \varphi_0(x, b_0) \vee \dots \vee \varphi_{n-1}(x, b_{n-1})$ and $\varphi_i(x, b_i) \not\downarrow^f\text{-Kim-divides}$ over $\emptyset$ for all $i < n$.

Let $\mu' < \lambda < \mu$ be sufficiently large infinite cardinals with self-cofinality such that $\mu' > \beth_{(2^{|T|+\omega})^+}(2^{|T|+\omega})$, $\lambda > \beth_{(2^{\mu'})^+}(2^{\mu'})$, and $\mu > \beth_{(2^\lambda)^+}(2^\lambda)$. Since $\varphi_i(x, b_i) \not\downarrow^f\text{-Kim-divides}$ over $\emptyset$, there exists a $\downarrow^f$ -Morley sequence $(b'_i)_{i < \lambda}$ over $\emptyset$ with $b'_i = b_i$ such that $\{\varphi_i(x, b'_i)\}_{i < \lambda}$ is inconsistent, for each $i < n$. By replacing $\varphi_i(x, b_i)$ with $\phi(x) \wedge \varphi_i(x, b_i)$, we may assume that $\vdash \forall x(\phi(x) \leftrightarrow \varphi_0(x, b_0) \vee \dots \vee \varphi_{n-1}(x, b_{n-1}))$.

For each ordinal α , let $\alpha^* := \alpha + 1$ if α is 0 or a successor, and $\alpha^* := \alpha$ if α is limit.

Claim 3.12.1. There exist a sequence of s -indiscernible trees $((a_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda})_{\alpha < \mu}$ over $\emptyset$ and a sequence of functions $(f_\alpha : \alpha^* \rightarrow n)_{\alpha < \mu}$ such that

- (i) $a_\eta^\alpha \equiv b_{f_\alpha(t(\eta))}$ for each $\eta \in \mathcal{T}_\alpha^\lambda$,
- (ii) $(a_{\eta \smallfrown (l)}^\alpha)_{l < \lambda} \equiv (b_{f_\alpha(t(\eta)-1)}^l)_{l < \lambda}$ for each $\eta \in \mathcal{T}_\alpha^\lambda$ with $t(\eta) \neq 0$,

- (iii) $\{\varphi_{f_\alpha(t(\nu))}(x, a_\nu^\alpha) \mid \nu \trianglelefteq \eta\}$ is consistent for each $\eta \in \mathcal{T}_\alpha^\lambda$,
- (iv) $a_{\eta \smallfrown (l)}^\alpha \downarrow^f (a_\nu^\alpha)_{\nu \trianglelefteq \eta \smallfrown (0)} (a_{\eta \smallfrown (l')}^\alpha)_{l' < l}$ for each $0 < l < \lambda$ and $\eta \in \mathcal{T}_\alpha^\lambda$ with $t(\eta) \neq 0$,

for each $\alpha < \mu$, and

- (v) $(a_\eta^{\alpha_0})_{\eta \in \mathcal{T}_{\alpha_0}^\lambda} \subseteq (a_\eta^{\alpha_1})_{\eta \in \mathcal{T}_{\alpha_1}^\lambda}$ by canonical inclusion and $f_{\alpha_0} \subseteq f_{\alpha_1}$ for each $\alpha_0 < \alpha_1 < \mu$.

Proof of Claim 3.12.1. For $(a_\eta^0)_{\eta \in \mathcal{T}_0^\lambda}$ and $f_0 : 1 \rightarrow n$, choose any $j < n$ and let $a_\emptyset^0 := b_j$ and $f_0(0) = j$. Then they satisfy all conditions we want.

Suppose that we have constructed $((a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda})_{\beta < \alpha}$ and $(f_\beta : \beta^* \rightarrow n)_{\beta < \alpha}$ satisfying all conditions, for some $\alpha < \mu$. First we assume that α is a successor. Then there exists $\beta < \mu$ such that $\alpha = \beta + 1$. Suppose β is also a successor. Choose any sufficiently large cardinal κ with $\kappa > \lambda$ and $\text{cf}(\kappa) = \kappa$. Then there exists a $\downarrow^f$ -Morley sequence $(b^k)_{k < \kappa}$ over $\emptyset$ such that $(b^l)_{l < \lambda} \equiv (b_{f_\beta(\beta)}^l)_{l < \lambda}$ and $b^0 = a_\emptyset^\beta$.

By Lemma 3.2, there exists $D := (b_0^*, \dots, b_{n-1}^*) \equiv (b_0, \dots, b_{n-1})$ such that $(b^k)_{k < \kappa}$ is a $\downarrow^f$ -Morley sequence over D .

By applying s-modeling property, Lemma 3.3, Lemma 3.4, and automorphism, we may assume that $(a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$ is s-indiscernible over D .

By Lemma 3.6, there exists $(b'_k)_{k < \kappa} \equiv_{Db^0} (b^k)_{k < \kappa}$ such that $(a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$ is s-indiscernible over $Db'_{<\kappa}$ and $b'_k \downarrow_D^f (a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda} b'_{<k}$ for all $0 < k < \kappa$.

By Lemma 3.10, there exist $(A_k := (a_\eta^{\beta,k})_{\eta \in \mathcal{T}_\beta^\lambda})_{k < \kappa}$ and $(b''_k)_{k < \kappa} \equiv_{Db'_0} (b'_k)_{k < \kappa}$ such that $a_{\eta}^{\beta,0} := a_\eta^\beta$ for each $\eta \in \mathcal{T}_\beta^\lambda$, $b''_k = a_{\emptyset}^{\beta,k}$ for each $k < \kappa$, and satisfy

- (I) $(A_k)_{k < \kappa}$ is indiscernible over D ,
- (II) $(A_k)_{k < \kappa}$ is mutually s-indiscernible over D ,
- (III) $b''_j \downarrow_D^f A_i(b''_k)_{i < k < j}$ for all $0 < j < \kappa$.

By Lemma 3.11, there exists $j < n$ such that

$$\{\varphi_{f_\beta(t(\nu))}(x, a_\nu^{\beta,k}) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b_j^*)\}$$

is consistent for each $\eta \in \mathcal{T}_\beta^\lambda$ and $k < \kappa$. For each $\eta \in \mathcal{T}_\alpha^\lambda$, let

$$\hat{a}_\eta^\alpha = \begin{cases} a_\nu^{\beta,l} & \text{if } \eta = (l) \smallfrown \nu \\ b_j^* & \text{if } \eta = \emptyset, \end{cases}$$

and $f_\alpha = f_\beta \cup \{(\alpha, j)\}$. Let $(\hat{a}_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda}$ be an s-indiscernible tree over $\emptyset$ which is locally based on $(\hat{a}_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda}$ over $\emptyset$. Then by Lemma 3.3, $(\hat{a}_\eta^\alpha)_{\eta \in \mathcal{T}_\beta^\lambda} \equiv (a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$. By automorphism, we can find an s-indiscernible tree $(a_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda}$ containing $(a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$ with respect to canonical inclusion, satisfying (i), (ii), (iii), (iv), and (v) with f_α .

Now we suppose that β is limit. Again, we choose a sufficiently large cardinal κ with $\kappa > \lambda$ and $\text{cf}(\kappa) = \kappa$. By Lemma 3.3 and s-modeling property, we can find $D := (b_0^*, \dots, b_{n-1}^*) \equiv (b_0, \dots, b_{n-1})$ such that $(a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$ is s-indiscernible over D . Since κ is sufficiently large, we can find $(A_k := (a_\eta^{\beta,k})_{\eta \in \mathcal{T}_\beta^\lambda})_{k < \kappa}$ such that $(A_k)_{k < \kappa}$ is indiscernible and mutually s-indiscernible over D , by applying Lemma 3.9 and s-modeling property repeatedly. By Lemma 3.11, there exists $j < n$ such that

$$\{\varphi_{f_\beta(t(\nu))}(x, a_\nu^{\beta,k}) \mid \nu \trianglelefteq \eta\} \cup \{\varphi_j(x, b_j^*)\}$$

is consistent for each $\eta \in \mathcal{T}_\beta^\lambda$ and $k < \kappa$. By the same construction above, we can find $(a_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda}$ and $f_\alpha : \alpha^* \rightarrow n$ satisfying (i), (ii), (iii), (iv), and (v).

If α is limit, then we just take $(a_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda} := \bigcup_{\beta < \alpha} (a_\eta^\beta)_{\eta \in \mathcal{T}_\beta^\lambda}$ and $f_\alpha := \bigcup_{\beta < \alpha} f_\beta$. By finite support, they satisfy all conditions we want. This completes proof of Claim 3.12.1. $\square$

Claim 3.12.2. There exists an s-indiscernible tree $(a_\eta)_{\eta \in \mathcal{T}_\mu^\lambda}$ over $\emptyset$ and $\varphi(x, y) \in \mathcal{L}$, and a $\downarrow^f$ -Morley sequence $(b^l)_{l < \lambda}$ such that

- (†) $\{\varphi(x, b^l)\}_{l < \lambda}$ is inconsistent,
- (‡) $(a_{\eta \smallfrown (l)})_{l < \lambda} \equiv (b^l)_{l < \lambda}$ for each $\eta \in \mathcal{T}_\mu^\lambda$ with $t(\eta) \neq 0$,
- (††) $\{\varphi(x, a_\nu) \mid \nu \sqsubseteq \eta\}$ is consistent for each $\eta \in \mathcal{T}_\mu^\lambda$,
- (†††) $a_{\eta \smallfrown (l)} \downarrow^f (a_\nu)_{\nu \sqsupseteq \eta \smallfrown (0)} (a_{\eta \smallfrown (l')})_{l' < l}$ for each $0 < l < \lambda$ and $\eta \in \mathcal{T}_\mu^\lambda$ with $t(\eta) \neq 0$.

Proof of Claim 3.12.2. Let $((a_\eta^\alpha)_{\eta \in \mathcal{T}_\alpha^\lambda})_{\alpha < \mu}$ and $(f_\alpha : \alpha^* \rightarrow n)_{\alpha < \mu}$ be sequences given by Claim 3.12.1. Take unions $(a'_\eta)_{\eta \in \mathcal{T}_\mu^\lambda} := \bigcup_{\alpha < \mu} ((a_\eta^\alpha)_{\alpha < \mu})$ and $f := \bigcup_{\alpha < \mu} f_\alpha$. By pigeonhole principle, we may assume that f is monochromatic. $\square$

Recall the choice of μ' in the beginning of the proof.

Claim 3.12.3. There exists a mutually indiscernible $(I_m = (c_l^m)_{l < \lambda})_{m < \mu'}$ such that

- (*) $c_l^m \downarrow^f I_{< m} c_{< l}^m$ for all $m < \mu'$, $l < \lambda$,
- (*) $I_{m_0} \dots I_{m_{n-1}} \equiv I_0 \dots I_{n-1}$ for all $m_0 < \dots < m_{n-1} < \mu'$,
- (**) $\{\varphi(x, c_{g(m)}^m) \mid m < \mu'\}$ is consistent for each $g : \mu' \rightarrow \lambda$,
- (**) $\{\varphi(x, c_l^m) \mid l < \lambda\}$ is inconsistent for each $m < \mu'$,

Proof of Claim 3.12.3. Let $(a_\eta)_{\eta \in \mathcal{T}_\mu^\lambda}$, $\varphi(x, y)$, and $(b^l)_{l < \lambda}$ be given by Claim 3.12.2. By Fact 2.11, there exists $(a'_\eta)_{\eta \in \mathcal{T}_\omega^\lambda}$ such that

- (‡) for all $n < \omega$, there exists $u \subseteq \kappa$ with $|u| = n$ such that $(a'_\eta)_{\eta \in \mathcal{T}_n^\lambda} \equiv (a_\eta)_{\eta \in \mathcal{T}_{\kappa \restriction u}^\lambda}$,
- (‡‡) for all finite $u, v \subseteq \omega$ with $|u| = |v|$, $(a'_\eta)_{\eta \in \mathcal{T}_{\omega \restriction u}^\lambda} \equiv (a'_\eta)_{\eta \in \mathcal{T}_{\omega \restriction v}^\lambda}$.

For each $m < \omega$, $l < \lambda$, let $c_l^m = a_{\zeta_{m+1} \smallfrown (l+1)}$ and $I_m := (c_l^m)_{l < \lambda}$. Then $(I_m)_{m < \omega}$ is mutually indiscernible and satisfies (*), (*), and (**). (**) is by the assumption that T is NSOP₁. By using EM-type of $(I_m)_{m < \omega}$ over $\emptyset$, we can extend $(I_m)_{m < \omega}$ to $(I_m)_{m < \mu'}$. $\square$

From now we work on $(I_m)_{m < \mu'}$ obtained in Claim 3.12.3. Choose any small model M .

Claim 3.12.4. There exists $(J_m := (d_i^m)_{i < \omega})_{m < \mu'}$ such that

- (***) $\varphi(x, d_i^m) \downarrow^f$ -Kim-divides over M for each $m < \mu'$ and $i < \omega$.
- (***) J_m is a subsequence of I_m for each $m < \mu'$,
- (****) $d_0^m \downarrow_M^{K^i} J_{< m}$ for each $m < \mu'$.

Proof of Claim 3.12.4. I_0 is $\downarrow^f$ -Morley sequence over $\emptyset$ and M is small, there exists $I'_0 \equiv I_0$ such that I'_0 is a $\downarrow^f$ -Morley sequence over M by Lemma 3.2. By replacing automorphic image, we may assume $I'_0 = I_0$. Let $d_i^0 := c_i^0$ for each $i < \omega$ and put $J_0 := (d_i^0)_{i < \omega}$. Note that $e \downarrow_M^{K^i} \emptyset$ for all e since M is a model. Thus $d_0^0 \downarrow_M^{K^i} \emptyset$. (***) is clear.

Now suppose that we have constructed $(J_m)_{m < \mu_0}$ satisfying (**), (***), and (****) for some $\mu_0 < \mu'$. Note that I_{μ_0} is indiscernible over $J_{<\mu_0}$ and $c_l^{\mu_0} \downarrow^f J_{<\mu_0} c_l^{\mu_0}$ for all $l < \lambda$ by (*). By the choice of μ' and λ , we can apply Lemma 3.2 to obtain $(I'_{\mu_0}) \equiv_{J_{<\mu_0}} I_{\mu_0}$ which is a $\downarrow^f$ -Morley sequence over M . By replacing automorphic image again, we may assume $I'_{\mu_0} = I_{\mu_0}$. Let $d_i^{\mu_0} := c_i^{\mu_0}$ for each $i < \omega$ and put $J_{\mu_0} := (d_i^{\mu_0})_{i < \omega}$.

Since $\downarrow^f$ is stronger than $\downarrow^{K^i}$ over models, J_{μ_0} is a $\downarrow^{K^i}$ -Morley sequence over M . Thus we have $J_{<\mu_0} \downarrow_M^{K^{d^i}} d_0^{\mu_0}$ by Kim's lemma over models in NSOP₁. Since $\downarrow^i$ -Kim-dividing and $\downarrow^i$ -Kim-forking are equivalent over models in NSOP₁, we have $J_{<\mu_0} \downarrow_M^{K^i} d_0^{\mu_0}$. By symmetry of $\downarrow^{K^i}$ over models, we have $d_0^{\mu_0} \downarrow_M^{K^i} J_{<\mu_0}$. Thus J_{μ_0} satisfies (****) and (****). (****) is clear.

By continuing this we can find $(J_m)_{m < \mu'}$ satisfying all conditions we want. $\square$

In particular, $d_0^m \downarrow_M^{K^i} d_0^m$ for all $m < \mu'$. By Fact 2.10, we may assume $(d_0^m)_{m < \mu'}$ is indiscernible over M . Thus $(d_0^m)_{m < \mu'}$ is a $\downarrow^{K^i}$ -Morley sequence over M . But $\{\varphi(x, d_0^m)\}_{m < \mu'}$ is consistent, thus $\varphi(x, d_0^0)$ does not $\downarrow^{K^i}$ -Kim-divides over M by Kim's lemma. It yields a contradiction with (**). $\square$

Corollary 3.13. *Suppose T is NSOP₁. Then for all C and $p \in S(C)$, there exists a $\downarrow^{K^f}$ -Morley sequence in p over C .*

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