

On limit behavior of eigenvalues spacing for 1-D random Schrödinger operators

By

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Abstract

A limit distribution for spacing of random eigenvalues is obtained. The eigenvalues arise from 1-D Schrödinger operators with decaying random potentials.

§ 1. Background

Let

$$H_l = -\frac{d^2}{dt^2} + q \quad \text{on } (0, l) \text{ with Dirichlet boundary condition.}$$

where q is a random potential. Let $\{X_t\}_{t \geq 0}$ be a Brownian motion on a compact Riemannian manifold M and F be a smooth non-constant function on M . For a positive decreasing function $a(t)$, a typical random decaying potential q is given by

$$q(t) = a(t)F(X_t).$$

Assume here for simplicity

$$a(t) \sim ct^{-\alpha} \text{ as } t \rightarrow \infty \text{ with some constant } c > 0.$$

Case 1. If $\alpha = 0$, Goldseid-Molchanov-Pastur([1]) proved that the operator H_∞ has only point spectrum distributing densely on $[0, \infty)$.

Case 2. Kotani-Ushiroya([2]) obtained the followings:

(i) $0 < \alpha < \frac{1}{2} \implies H_\infty$ has only point spectrum distributing densely on $[0, \infty)$.

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(ii) $\alpha = \frac{1}{2} \implies H_\infty$ has only point spectrum distributing densely on $[0, E_0]$ and purely singular continuous spectrum on $[E_0, \infty)$, where E_0 can be determined exactly from F and $a(t)$.

(iii) $\alpha > \frac{1}{2}$ and $\int_M F(x)dx = 0 \implies H_\infty$ has purely absolutely continuous spectrum on $[0, \infty)$.

Motivated by condensed matter physics, Molchanov investigated finer structure of the spectrum in Case 1. Let $\{E_1(l) < E_2(l) \leq \dots\}$ be the eigenvalues of H_l and for an interval I set

$$N_l(I) = \#\{k; E_k(l) \in I\}.$$

Molchanov ([3]): For each $k = 0, 1, 2, \dots$ and $E > 0, a > 0$

$$(1.1) \quad P\left(N_l\left(\left(E - \frac{a}{2l}, E + \frac{a}{2l}\right)\right) = k\right) \xrightarrow{l \rightarrow \infty} e^{-an(E)} \frac{(an(E))^k}{k!}$$

holds, where

$$n(E) = \frac{dN(E)}{dE}.$$

To investigate the spacing between two neighboring eigenvalues let

$$\Delta_n = n(E_{n+1}(nt) - E_n(nt)), \quad t = N(E)^{-1}.$$

Molchanov's result (1.1) implies in particular that, in Case 1

$$\lim_{n \rightarrow \infty} P(\Delta_n \leq x) = 1 - e^{-n(E)x/N(E)}$$

holds. A similar problem can be considered in the case of random matrices $\{X(\omega)\}$, where $X(\omega)$ is a sample of real symmetric $n \times n$ matrix whose elements $\{x_{ij}(\omega)\}_{1 \leq i \leq j \leq n}$ are independent and distributed by an identical Gaussian law. Then it is known that we have a different limit distribution.

For random Schrödinger operators in higher dimensions we can ask the same question. In discrete case, the result corresponding to Molchanov's one was established by Minami ([4]) in point spectrum region. In continuous case, Combes-Germinet-Klein ([5]) obtained recently a similar result.

The purpose of this note is to investigate the limit behavior of Δ_n in Case 2. In the third section we try to obtain a kind of central limit theorem, however the proof has not been completed yet. We would like to postpone making the argument rigorous in future.

§ 2. First limit theorem (Law of large numbers)

For $E > 0$, setting $\kappa = \sqrt{E}$, we have the equivalence:

$$(2.1) \quad \begin{aligned} & -x_t'' + a(t)F(X_t(\omega))x_t = Ex_t \\ & \iff \\ & \begin{pmatrix} x_t \\ y_t \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ \kappa^{-1}a(t)F(X_t(\omega)) - \kappa & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}. \end{aligned}$$

Since we are considering the Dirichlet boundary condition, we assume

$$x_0 = 0, \quad y_0 = 1.$$

We transform the variables $\{x_t, y_t\}$ to

$$\begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} = \begin{pmatrix} \cos \kappa t & -\sin \kappa t \\ \sin \kappa t & \cos \kappa t \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix},$$

and set

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = r_t \begin{pmatrix} \sin \theta_t \\ \cos \theta_t \end{pmatrix}, \quad \begin{pmatrix} \tilde{x}_t \\ \tilde{y}_t \end{pmatrix} = \tilde{r}_t \begin{pmatrix} \sin \tilde{\theta}_t \\ \cos \tilde{\theta}_t \end{pmatrix}, \quad \theta_t = \kappa t + \tilde{\theta}_t.$$

Then the equation (2.1) is equivalent to

$$(2.2) \quad \frac{d}{dt} \begin{pmatrix} \log \tilde{r}_t^2 \\ \tilde{\theta}_t \end{pmatrix} = \kappa^{-1} a(t) F(X_t) \begin{pmatrix} \sin 2(\kappa t + \tilde{\theta}_t) \\ -\sin^2(\kappa t + \tilde{\theta}_t) \end{pmatrix},$$

and it holds that

$$(2.3) \quad \begin{aligned} \tilde{\theta}_t &= -\frac{1}{2\kappa} \int_0^t a(s) F(X_s) ds + \frac{1}{2\kappa} \operatorname{Re} \int_0^t a(s) F(X_s) e^{2i(\kappa s + \tilde{\theta}_s)} ds, \\ \tilde{r}_t^2 &= \exp \left(\frac{1}{\kappa} \operatorname{Im} \int_0^t a(s) F(X_s) e^{2i(\kappa s + \tilde{\theta}_s)} ds \right), \\ \frac{\partial \theta_t}{\partial \kappa} &= \int_0^t \frac{\tilde{r}_s^2}{\tilde{r}_t^2} ds + \frac{1}{2\kappa^2} \int_0^t \frac{\tilde{r}_s^2}{\tilde{r}_t^2} a(s) F(X_s) \left(1 - \operatorname{Re} e^{2i(\kappa s + \tilde{\theta}_s)} \right) ds. \end{aligned}$$

We remark here that Sturm oscillation theorem implies for $n = 1, 2, \dots$

$$(2.4) \quad \theta_t(\kappa) = n\pi \iff \kappa = \sqrt{E_n(t)}.$$

Our method to investigate the problem is to use the identities (2.3) and the relation (2.4).

First we study a non-random estimate of $E_n(nt)$ depending only on F and $a(s)$. Set

$$\kappa_+ = \max_{t \geq 0, x \in M} (F(x)a(t)), \quad \kappa_- = \min_{t \geq 0, x \in M} (F(x)a(t)).$$

Then the mini-max principle for eigenvalues shows

$$\left(\frac{n\pi}{l}\right)^2 + \kappa_- \leq E_n(l) \leq \left(\frac{n\pi}{l}\right)^2 + \kappa_+.$$

Hence, if $\left(\frac{\pi}{t}\right)^2 + \kappa_- \geq 0$

$$(2.5) \quad \sqrt{\left(\frac{\pi}{t}\right)^2 + \kappa_-} \leq \sqrt{E_n(nt)} \leq \sqrt{\left(\frac{\pi}{t}\right)^2 + \kappa_+}$$

Set

$$\kappa_0 = \frac{\pi}{\sqrt{(-\kappa_-) \vee 0}}, \quad \text{where } \frac{1}{0} = \infty$$

Then, if $t < \kappa_0$ is valid, then it holds necessarily that $E_n^{(nt)} > 0$. On the other hand, from (2.3) it follows that

$$(2.6) \quad |\tilde{\theta}_t| \leq \frac{\|F\|_\infty}{\kappa} \int_0^t a(s) ds,$$

hence

$$\left| \tilde{\theta}_{nt} \left(\sqrt{E_n(nt)} \right) \right| \leq \frac{\|F\|_\infty}{\sqrt{E_n(nt)}} \int_0^{nt} a(s) ds \leq \frac{t}{\pi} \|F\|_\infty \int_0^{nt} a(s) ds$$

holds. Then the identity

$$n\pi = \theta_{nt} \left(\sqrt{E_n(nt)} \right) = nt \sqrt{E_n(nt)} + \tilde{\theta}_{nt} \left(\sqrt{E_n(nt)} \right)$$

shows

$$\left| \sqrt{E_n(nt)} - \frac{\pi}{t} \right| = \frac{\left| \tilde{\theta}_{nt} \left(\sqrt{E_n(nt)} \right) \right|}{nt} \leq \frac{\|F\|_\infty}{n\pi} \int_0^{nt} a(s) ds \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

and gives

Proposition 2.1. *As long as t remains on any compact set of $(0, \kappa_0)$, $\sqrt{E_n(nt)}$ converges to π/t as $n \rightarrow \infty$ uniformly with respect to t .*

In order to investigate the behavior of the difference

$$\Delta_n = n(E_{n+1}(nt) - E_n(nt)),$$

we have to know the asymptotic behavior of

$$Y_t(\kappa, \beta) = \int_0^t a(s) F(X_s) e^{\beta i(\kappa s + \tilde{\theta}_s)} ds \quad \text{for } \beta \in \mathbb{R}.$$

We impose a condition on $a(t)$:

$$(2.7) \quad \int_0^\infty a(s)^2 s^\delta ds < \infty \quad \text{for some } \delta > 0.$$

Lemma 2.2. *Assume $a(t)$ satisfies (2.7) and F fulfils*

$$\int_M F(x) dx = 0.$$

Let K be a compact set of $(0, \infty)$. Then, for any $\beta \in \mathbb{R}$

$$\lim_{t \rightarrow \infty} Y_t(\kappa, \beta) = Y_\infty(\kappa, \beta) \quad \text{uniformly on } K \quad a.s..$$

Moreover, for any $\epsilon < \delta/2$ and $\beta \in \mathbb{R}$

$$\sup_{t \geq 0, \kappa, \kappa_1 \in K} \frac{|Y_t(\kappa, \beta) - Y_t(\kappa_1, \beta)|}{|\kappa - \kappa_1|^\epsilon} < \infty \quad a.s.$$

is valid.

Proof. The first statement is (2) of Lemma 2.2 of ([2]). The necessary estimates for the second statement are given in the sublemma of ([2]). They proved the estimates

$$E \left(\sup_{t \geq 0} |Y_t(\kappa, \beta) - Y_t(\kappa_1, \beta)|^{2p} \right) \leq \text{const.} |\kappa - \kappa_1|^{p\eta} \quad \text{if } \kappa, \kappa_1 \in K.$$

Then Kolmogorov theorem implies the conclusion of Lemma. □

Remark. In the proof of the sublemma, they estimated

$$E \left(\sup_{s \in (t_0, t)} |J(s)|^{2p} \right)$$

by decomposing it into the 5 terms in page 254. However the third term should be

$$\text{const.} \left\{ |\lambda - \lambda'|^\eta \int_{t_0}^t (1 + |s|^\eta) a(s)^2 ds \right\}^p.$$

Therefore the statement of the sublemma must be

$$E \left(\sup_{t \in [0, \infty)} |I(t, \omega, \lambda, \lambda')|^{2p} \right) \leq \text{const.} |\lambda - \lambda'|^{p\eta}$$

instead of

$$E \left(\sup_{t \in [0, \infty)} |I(t, \omega, \lambda, \lambda')|^{2p} \right) \leq \text{const.} |\lambda - \lambda'|^{2p\eta}.$$

Then we can improve Proposition 2.1:

Theorem 2.3. *Assume the condition (2.7) and F has mean 0. Then, we have*

$$n \left(\sqrt{E_n(nt)} - \frac{\pi}{t} \right) \xrightarrow[n \rightarrow \infty]{} -\frac{1}{t} \tilde{\theta}_\infty \left(\frac{\pi}{t} \right) = \pi \int_0^\infty a(s) F(X_s) \sin^2 \left(\frac{\pi s}{t} + \tilde{\theta}_s \left(\frac{\pi}{t} \right) \right) ds \quad a.s..$$

The convergence is compact uniform with respect to $t \in (0, \kappa_0)$.

Remark. We denote $\tilde{\theta}_t$ by $\tilde{\theta}_t(\kappa)$ if it is necessary to make the κ -dependence explicit.

Setting

$$\Lambda_n(t) = n \left(\sqrt{E_{n+1}(nt)} - \sqrt{E_n(nt)} \right),$$

which is an equivalent quantity with Δ_n , we can conclude

Theorem 2.4. *Suppose $a(t)$ satisfies the condition (2.7) and F has mean 0. Then, if $t < \kappa_0$*

$$\Lambda_n(t) = \frac{\pi}{t} + O(n^{-\epsilon}) \quad a.s. \quad \text{for any } \epsilon < \delta/2.$$

This holds compact uniformly in $t \in (0, \kappa_0)$.

Proof. From (2.3) we have

$$\theta_t(\kappa) = \kappa t - \frac{1}{2\kappa} \int_0^t a(s) F(X_s) ds + \frac{1}{2\kappa} \operatorname{Re} \int_0^t a(s) F(X_s) e^{2i(\kappa s + \tilde{\theta}_s)} ds.$$

Hence

$$\begin{aligned} \theta_t(\kappa) - \theta_t(\kappa_1) &= (\kappa - \kappa_1)t - \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \int_0^t a(s) F(X_s) ds \\ (2.8) \quad &+ \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \operatorname{Re} Y_t(\kappa, 2) + \frac{1}{2\kappa_1} \operatorname{Re} (Y_t(\kappa, 2) - Y_t(\kappa_1, 2)) \end{aligned}$$

is valid. Now we assume t moves in a compact set K of $(0, \kappa_0)$ and set

$$l_- = \min_{t \in \tilde{K}} \sqrt{\left(\frac{\pi}{t} \right)^2 + \kappa_-}, \quad l_+ = \max_{t \in \tilde{K}} \sqrt{\left(\frac{\pi}{t} \right)^2 + \kappa_+},$$

where $\tilde{K}$ is a compact set such that $K \subset (\tilde{K})^\circ \subset \tilde{K} \subset (0, \kappa_0)$. Then the inequality (2.5) assures that $\sqrt{E_n(nt)}$ and $\sqrt{E_{n+1}(nt)}$ move in $[l_-, l_+]$. Therefore, setting $\kappa = \sqrt{E_{n+1}(nt)}$, $\kappa_1 = \sqrt{E_n(nt)}$ and $t \rightarrow nt$, we have

$$\begin{aligned} \pi &= t\Lambda_n(t) - \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \int_0^{nt} a(s) F(X_s) ds + \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \operatorname{Re} Y_{nt}(\kappa, 2) \\ (2.9) \quad &+ \frac{1}{2\kappa_1} \operatorname{Re} (Y_{nt}(\kappa, 2) - Y_{nt}(\kappa_1, 2)). \end{aligned}$$

Then Lemma 2.2 implies

$$(2.10) \quad \begin{aligned} \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \int_0^{nt} a(s) F(X_s) ds &= O \left(\frac{\Lambda_n(t)}{n} \right), \\ \frac{1}{2} \left(\frac{1}{\kappa} - \frac{1}{\kappa_1} \right) \operatorname{Re} Y_{nt}(\kappa, 2) &= O \left(\frac{\Lambda_n(t)}{n} \right), \\ \frac{1}{2\kappa_1} \operatorname{Re} (Y_{nt}(\kappa, 2) - Y_{nt}(\kappa_1, 2)) &= O \left(\frac{\Lambda_n(t)}{n} \right)^\epsilon. \end{aligned}$$

A priori we know $\Lambda_n(t)/n$ is bounded, hence from (2.9) the boundedness of $\Lambda_n(t)$ follows. Substituting this consequence to the estimates (2.10), we see

$$\Lambda_n(t) - \frac{\pi}{t} = O(n^{-\epsilon}).$$

□

§ 3. Second limit theorem

In this section we assume

$$a(t) = t^{-\alpha} \text{ for } t \geq 1 \text{ with } \alpha \in \left(\frac{1}{2}, 1 \right) \text{ and } \int_M F(x) dx = 0.$$

To investigate the asymptotic behavior of the error term we consider a sequence of stochastic processes with parameters c_1 and c_2

$$\Theta_t^{(n)}(c_1, c_2) = n^{\alpha - \frac{1}{2}} \left(\tilde{\theta}_{nt} \left(\kappa + \frac{c_1}{n} \right) - \tilde{\theta}_{nt} \left(\kappa + \frac{c_2}{n} \right) \right).$$

Define

$$g(x) = (L + 2i\kappa)^{-1} F(x), \text{ where } L = \frac{1}{2}\Delta.$$

Then, Ito's formula implies

$$\int_0^t e^{2i\kappa s} F(X_s) ds = e^{2i\kappa t} g(X_t) + M_t, \text{ with a martingale } M_t = M_t(\kappa).$$

Then we have

$$\begin{aligned} \int_0^t a(s) F(X_s) e^{2i(\kappa s + \tilde{\theta}_s)} ds &= \int_0^t a(s) e^{2i\tilde{\theta}_s} d(e^{2i\kappa s} g(X_s)) + \int_0^t a(s) e^{2i\tilde{\theta}_s} dM_s \\ &= a(0)g(X_0) - a(t)e^{2i(\tilde{\theta}_t + \kappa t)}g(X_t) - \int_0^t a'(s)e^{2i(\kappa s + \tilde{\theta}_s)}g(X_s) ds \\ &\quad - 2i \int_0^t a(s)e^{2i(\kappa s + \tilde{\theta}_s)}\tilde{\theta}'_s g(X_s) ds + \int_0^t a(s)e^{2i\tilde{\theta}_s} dM_s. \end{aligned}$$

Therefore, substituting (2.3) into the above identity leads us to

$$(3.1) \quad \int_0^t a(s) F(X_s) e^{2i(\kappa s + \tilde{\theta}_s)} ds = -a(0)g(X_0) + a(t)e^{2i(\tilde{\theta}_t + \kappa t)}g(X_t) \\ - \frac{i}{2\kappa} \int_0^t a(s)^2 F(X_s) g(X_s) ds + I_t(\kappa) + J_t(\kappa) + N_t(\kappa),$$

where

$$(3.2) \quad I_t(\kappa) = - \int_0^t a'(s) e^{2i(\kappa s + \tilde{\theta}_s)} g(X_s) ds \\ J_t(\kappa) = - \frac{i}{2\kappa} \int_0^t a(s)^2 e^{4i(\kappa s + \tilde{\theta}_s)} F(X_s) g(X_s) ds + \frac{i}{\kappa} \int_0^t a(s)^2 e^{2i(\kappa s + \tilde{\theta}_s)} F(X_s) g(X_s) ds \\ N_t(\kappa) = \int_0^t a(s) e^{2i\tilde{\theta}_s} dM_s.$$

Setting $\kappa_1 = c_1/n + \kappa$, $\kappa_2 = c_2/n + \kappa$ we have

$$\Theta_t^{(n)}(c_1, c_2) = -n^{\alpha-\frac{1}{2}} \left(\frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \int_0^{nt} a(s) F(X_s) ds \\ + n^{\alpha-\frac{1}{2}} \operatorname{Re} \int_0^{nt} a(s) F(X_s) \left(\frac{e^{2i(\kappa_1 s + \tilde{\theta}_s(\kappa_1))}}{2\kappa_1} - \frac{e^{2i(\kappa_2 s + \tilde{\theta}_s(\kappa_2))}}{2\kappa_2} \right) ds \\ = -n^{\alpha-\frac{1}{2}} \left(\frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \int_0^{nt} a(s) F(X_s) ds \\ + n^{\alpha-\frac{1}{2}} \left(\frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \operatorname{Re} \int_0^{nt} a(s) F(X_s) e^{2i(\kappa_1 s + \tilde{\theta}_s(\kappa_1))} ds \\ + \frac{n^{\alpha-\frac{1}{2}}}{2\kappa_2} \operatorname{Re} \left(\int_0^{nt} a(s) F(X_s) \left(e^{2i(\kappa_1 s + \tilde{\theta}_s(\kappa_1))} - e^{2i(\kappa_2 s + \tilde{\theta}_s(\kappa_2))} \right) ds \right).$$

Since, $\kappa_1 - \kappa_2 = (c_1 - c_2)/n$

$$\left| n^{\alpha-\frac{1}{2}} \left(\frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \int_0^{nt} a(s) F(X_s) ds \right| \leq C n^{\alpha-\frac{1}{2}-1+1-\alpha} = C n^{-\frac{1}{2}}, \\ \left| n^{\alpha-\frac{1}{2}} \left(\frac{1}{2\kappa_1} - \frac{1}{2\kappa_2} \right) \operatorname{Re} \int_0^{nt} a(s) F(X_s) e^{2i(\kappa_1 s + \tilde{\theta}_s(\kappa_1))} ds \right| \leq C n^{\alpha-\frac{1}{2}-1+1-\alpha} = C n^{-\frac{1}{2}},$$

hold with some constant C . Hence

$$\Theta_t^{(n)}(c_1, c_2) = \frac{n^{\alpha-\frac{1}{2}}}{2\kappa} \operatorname{Re} \left(\int_0^{nt} a(s) F(X_s) \left(e^{2i(\kappa_1 s + \tilde{\theta}_s(\kappa_1))} - e^{2i(\kappa_2 s + \tilde{\theta}_s(\kappa_2))} \right) ds \right) + \eta_t^{(n)} \\ \left| \eta_t^{(n)} \right| \leq \operatorname{const.} n^{-\frac{1}{2}}$$

is valid. Denote g corresponding to κ by g_κ . Then there exists a constant C' such that

$$|g_{\kappa_1}(x) - g_{\kappa_2}(x)| \leq C' |\kappa_1 - \kappa_2| = \text{const.} n^{-1}.$$

We use the inequality

$$|e^{2i\kappa_1 s} - e^{2i\kappa_2 s}| \leq 2 |\kappa_1 - \kappa_2| s = \text{const.} n^{-1}.$$

Then we have

$$\begin{aligned} n^{\alpha-\frac{1}{2}} \int_1^{nt} |a'(s)| s n^{-1} ds &\leq \text{const.} n^{-\frac{1}{2}}, & n^{\alpha-\frac{1}{2}} \int_1^{nt} a(s)^2 s n^{-1} ds &\leq \text{const.} n^{\frac{1}{2}-\alpha}, \\ n^{\alpha-\frac{1}{2}} \int_1^{nt} |a'(s)| ds &\leq \text{const.} n^{-\frac{1}{2}}, & n^{\alpha-\frac{1}{2}} \int_1^{nt} a(s)^2 ds &\leq \text{const.} n^{\frac{1}{2}-\alpha} \end{aligned}$$

hence

$$\begin{aligned} (3.3) \quad \Theta_t^{(n)}(c_1, c_2) &= n^{\alpha-\frac{1}{2}} \operatorname{Re} \left(S_t^{(n)}(\kappa_1) - S_t^{(n)}(\kappa_2) \right) + \xi_t^{(n)} \\ \left| \xi_t^{(n)} \right| &\leq \text{const.} n^{\frac{1}{2}-\alpha}, \end{aligned}$$

where

$$S_t^{(n)}(\kappa) = \int_0^{nt} a(s) e^{2i\tilde{\theta}_s(\kappa)} d \frac{M_s(\kappa)}{2\kappa}.$$

Set

$$T_t^{(n)}(c_1, c_2) = n^{\alpha-\frac{1}{2}} \operatorname{Re} \left(S_t^{(n)}(\kappa_1) - S_t^{(n)}(\kappa_2) \right).$$

Now we compute $\langle T_t^{(n)}(c_1, c_2) \rangle_t$ and obtain the limit of $T_t^{(n)}(c_1, c_2)$ when $n \rightarrow \infty$. Introduce the notation:

$$\begin{aligned} [g_1, g_2](x) &= L(g_1 g_2)(x) - g_1(x) (L g_2)(x) - g_2(x) (L g_1)(x) \\ &= (\nabla g_1, \nabla g_2)(x). \end{aligned}$$

To simplify the notations we set

$$g_\kappa(s, x) = \frac{1}{2\kappa} e^{2i\tilde{\theta}_s(\kappa)} e^{2i\kappa s} g_\kappa(x).$$

Then

$$\begin{aligned} \left\langle S^{(n)}(\kappa), S^{(n)}(\kappa') \right\rangle_t &= \int_0^{nt} a(s)^2 [g_\kappa(s), g_{\kappa'}(s)](X_s) ds, \\ \left\langle S^{(n)}(\kappa), \overline{S^{(n)}(\kappa')} \right\rangle_t &= \int_0^{nt} a(s)^2 \left[g_\kappa(s), \overline{g_{\kappa'}(s)} \right](X_s) ds. \end{aligned}$$

and

$$\begin{aligned}
& n^{1-2\alpha} \left\langle T^{(n)}(c_1, c_2), T^{(n)}(c'_1, c'_2) \right\rangle_t \\
&= \left\langle \operatorname{Re} \left(S^{(n)}(\kappa_1) - S^{(n)}(\kappa_2) \right), \operatorname{Re} \left(S^{(n)}(\kappa'_1) - S^{(n)}(\kappa'_2) \right) \right\rangle_t \\
&= \int_0^{nt} a(s)^2 \left[\operatorname{Re} (g_{\kappa_1}(s, \cdot) - g_{\kappa_2}(s, \cdot)), \operatorname{Re} (g_{\kappa'_1}(s, \cdot) - g_{\kappa'_2}(s, \cdot)) \right] (X_s) ds
\end{aligned}$$

hold, where $\kappa'_1 = c'_1/n + \kappa$, $\kappa'_2 = c'_2/n + \kappa$. Hence

$$\begin{aligned}
& \left\langle T^{(n)}(c_1, c_2), T^{(n)}(c'_1, c'_2) \right\rangle_t \\
&= n^{2\alpha-1} \int_0^{nt} a(s)^2 \left[\operatorname{Re} (g_{\kappa_1}(s, \cdot) - g_{\kappa_2}(s, \cdot)), \operatorname{Re} (g_{\kappa'_1}(s, \cdot) - g_{\kappa'_2}(s, \cdot)) \right] (X_s) ds
\end{aligned}$$

is valid. Now applying Lemma 2.2 for $\beta = 0, 2$ and (2.3) imply

$$\sup_{s \geq 0} \frac{\left| \tilde{\theta}_s(\kappa_1) - \tilde{\theta}_s(\kappa) \right|}{|\kappa_1 - \kappa|^\epsilon} < \infty \quad \text{a.s.},$$

which shows

$$\sup_{s \geq 0} \frac{\left| e^{2i\tilde{\theta}_s(\kappa_1)} - e^{2i\tilde{\theta}_s(\kappa)} \right|}{|\kappa_1 - \kappa|^\epsilon} < \infty \quad \text{a.s.}$$

for any $\epsilon < \alpha - \frac{1}{2}$. Hence we see

$$\begin{aligned}
& \left\langle T^{(n)}(c_1, c_2), T^{(n)}(c'_1, c'_2) \right\rangle_t \\
&= n^{2\alpha-1} \int_0^{nt} a(s)^2 \left[\operatorname{Re} \left(e^{2ic_1s/n} - e^{2ic_2s/n} \right) g_\kappa(s, \cdot), \operatorname{Re} \left(e^{2ic'_1s/n} - e^{2ic'_2s/n} \right) g_\kappa(s, \cdot) \right] (X_s) ds \\
&+ O(n^{-\epsilon}) \quad \text{a.s.}
\end{aligned}$$

Because of the strong averaging property of $\{X_t\}$, the main term is

$$\frac{n^{2\alpha-1}}{8\kappa^2} \langle [g_\kappa, \overline{g_\kappa}] \rangle \int_0^{nt} a(s)^2 \operatorname{Re} \left(e^{2ic_1s/n} - e^{2ic_2s/n} \right) \overline{(e^{2ic'_1s/n} - e^{2ic'_2s/n})} ds = l_t((c_1, c_2), (c'_1, c'_2)),$$

where

$$l_t((c_1, c_2), (c'_1, c'_2)) = \frac{\langle [g_\kappa, \overline{g_\kappa}] \rangle}{8\kappa^2} \int_0^t s^{-2\alpha} \operatorname{Re} (e^{2ic_1s} - e^{2ic_2s}) \overline{(e^{2ic'_1s} - e^{2ic'_2s})} ds,$$

and $\langle g \rangle$ denote the integral of g by the Riemannian volume element dx . To give a complete proof we have to use the martingale representation if the term $e^{2i\tilde{\theta}_s(\kappa)} e^{2i\kappa s}$ appears. The detail of the proof will be given elsewhere. We remark

$$\langle [g_\kappa, \overline{g_\kappa}] \rangle = (-\Delta g_\kappa, g_\kappa) = -2 \operatorname{Re} \left((L + 2i\kappa)^{-1} F, F \right).$$

Let $\{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \geq 0}$ be a Gaussian system with covariance

$$l_{t \wedge t'}((c_1, c_2), (c'_1, c'_2)).$$

Since $T_t^{(n)}(c_1, c_2)$ is a sequence of martingales, the martingale central limit theorem implies

Lemma 3.1. *For fixed $\kappa > 0$, as $n \rightarrow \infty$ $\left\{\Theta_t^{(n)}(c_1, c_2)\right\}_{t \geq 0, c_1, c_2 \geq 0}$ converges weakly to*

$$\{Z(t, c_1, c_2)\}_{t \geq 0, c_1, c_2 \geq 0}.$$

From now on we make a heuristic argument to guess a result. Set

$$\kappa + \frac{c_2}{n} = \sqrt{E_n(nt)}, \quad \kappa + \frac{c_1}{n} = \sqrt{E_{n+1}(nt)} \quad \text{with } \kappa = \frac{\pi}{t}.$$

Then we can describe the difference by $\Theta_t^{(n)}$ as follows:

$$\begin{aligned} & n^{\alpha - \frac{1}{2}} \left(\Lambda_n(t) - \frac{\pi}{t} \right) \\ &= n^{\alpha - \frac{1}{2}} \left(\Lambda_n(t) - \frac{1}{t} \left(\theta_{nt} \left(\sqrt{E_{n+1}(nt)} \right) - \theta_{nt} \left(\sqrt{E_n(nt)} \right) \right) \right) \\ &= -\frac{n^{\alpha - \frac{1}{2}}}{t} \left(\tilde{\theta}_{nt} \left(\kappa + \frac{c_1}{n} \right) - \tilde{\theta}_{nt} \left(\kappa + \frac{c_2}{n} \right) \right) \\ &= -\frac{1}{t} \Theta_t^{(n)}(c_1, c_2). \end{aligned}$$

From Theorems 2.3, 2.4 it follows that

$$\begin{aligned} c_2 &\xrightarrow{n \rightarrow \infty} -\frac{1}{t} \tilde{\theta}_\infty \left(\frac{\pi}{t} \right) = \pi \int_0^\infty a(s) F(X_s) \sin^2 \left(\frac{\pi s}{t} + \tilde{\theta}_s \left(\frac{\pi}{t} \right) \right) ds, \\ c_1 - c_2 &\xrightarrow{n \rightarrow \infty} \frac{\pi}{t} \quad \text{a.s..} \end{aligned}$$

Hence, to obtain the conclusion we have to investigate the limiting behavior of joint distributions:

$$\tilde{\theta}_{nt}(\kappa), \quad \Theta_t^{(n)}(c_1, c_2).$$

The complete proof will be given elsewhere.

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