

Constructing some spectra

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We start from the following

Problem 1. Given integers $n, a > 0$, does there exist spectrum $E = E_p$

for each prime p and a constant c (independent of p) satisfying the followings ?

$$\# \text{ of generators of } \sum_{*} \sum_{a \leq p^n} \pi_*(E) \otimes \mathbb{Z}_p < c$$

$$\text{and } \# \text{ of generators of } \sum_{*} \sum_{a \leq p^n} H_*(E) \otimes \mathbb{Z}_p < c.$$

As is easily seen the spectra $E = S^0$, $H_p = K(\mathbb{Z}_p)$, BP do not give any

answer of the above problem. However, for the case $n = 1$, an easy answer is

given by taking $E = V(1)$ or the following $H_p(1)$. We define

$$H_p(1) = \text{fibre of } p^1 : H_p \longrightarrow \Sigma^q H_p = R_1^0 \cdot H_p$$

$$\text{and } BP(1) = \text{fibre of } r_1 : BP \longrightarrow \Sigma^q BP = R_1^0 \cdot BP,$$

where $q = 2(p-1)$, R_1^j is a symbol having bidegree $(-1, 2p^j(p^i-1))$, r_A is the

Landweber-Novikov operation and p^A is a cohomology operation dual to that of

Milnor's (having a similar action to homology to r_A). Then $\pi_*(H_p(1))$ has

only two generators 1 and $h = \{R_1^0\}$ of degree 0 and $q-1$, and

$$H_*(BP(1); Z_p) \cong Z_p[\xi_1^p, \xi_2, \xi_3, \dots] \otimes \Lambda(\psi_1^0),$$

$$H_*(H_p(1); Z_p) \cong H_*(BP(1); Z_p) \otimes \Lambda(\tau_0, \tau_1', \tau_2', \dots)$$

where $H_*(H_p; Z_p) = Z_p[\xi_1, \xi_2, \dots] \otimes \Lambda(\tau_0, \tau_1, \dots)$ by Milnor, $\tau_i' = \tau_i - \tau_0 \xi_i$,

$\psi_1^0 = [R_1^0 \cdot \xi_1^{p-1}]$ of degree $pq-1$ and $H_*(BP; Z_p) = Z_p[\xi_1, \xi_2, \dots]$ for the mod p

reduction ξ_i of $m_i \in H_*(BP) = Z_{(p)}[m_1, m_2, \dots]$, $\deg \xi_i = \deg m_i = 2(p^i-1)$,

$\deg \tau_i = 2p^i-1$. The element ψ_1^0 is detected by the secondary operation ψ_1^0

associated with the relation $p^{p-1}p^1 = 0$. Taking p sufficiently large ($pq-2 \geq$

ap) we see that $H_p(1)$ gives an answer to the problem for $n = 1$. Also we may

regard that the spectrum $V(1)$ is the $(pq-2)$ -skeleton of $H_p(1)$ for $p \geq 3$.

Now our problem is to construct spectra of such a sort $H_p(n)$, $BP(n)$ for $n =$

$2, 3, \dots$. We consider

Problem 2. Given positive integer n , can we construct a chain complex

$C(n)$ satisfying the following conditions ? For $X = H_p$ or BP

$$C(n) = \Lambda(R_i^j; i+j \leq n) \otimes X^*(X),$$

for monomials x, y in $\Lambda(R_j^i)$, $\partial(x \otimes 1) = \sum y \otimes f_{x,y}$

$$f_{x,y} \equiv \begin{cases} -1 & \text{if } x = zR_i^j \text{ and } y = zR_k^j R_{i-k}^{j+k} \\ p_i^j \text{ or } r_i^j & \text{if } y = xR_i^j; \quad p_i^j = p^A, r_i^j = r_A \text{ for } A = p^j \Delta_i, \\ 0 & \text{otherwise} \end{cases}$$

modulo p and higher terms in the subalgebra generated by r_i^j 's.

The chain complex is represented by a sequence

$$C(n) : X_0 = X \xrightarrow{\partial_0} X_1 \xrightarrow{\partial_1} X_2 \rightarrow \dots \rightarrow X_{\binom{n+1}{2}} = X$$

where X_r is the product (wedge) of $x.X$ for monomials x in $\Lambda(R_1^j)$ of the length r .

We denote by $H_p(n)$ resp. $BP(n)$ a fibre (tower, realization or desuspension of iterated cones) of the above sequence $C(n)$ if it exists.

Lemma (i) Assume the existence of $C(n)$. Then there exists $BP(n)$ for $p \geq \frac{1}{4}(n^2 + n + 2)$ and $H_p(n)$ for $p \geq \frac{1}{4}(n^2 + 3n + 4)$ or $p = 3, n = 2$.

They are unique if the inequalities hold.

(ii) Assume the existence of $BP(n)$. Then there exists a spectral sequence : $E_2 = H_*(C(n); Y_*(BP)) \implies Y_*(BP(n))$, which collapses if $p > \frac{1}{4}(n^2 + n + 2)$ and if $Y = S, H, H_p, V(m)$ or $= BP$. Similar spectral sequence exists for $H_p(n)$.

In general, a fibre of $C(n)$ exists if $[\Sigma^{k-2} X_r, X_{r+k}] = 0$ for $k \geq 3$. Then the lemma is proved by the fact $BP^*(BP) = 0$ for $*$ $\not\equiv 0 \pmod{q}$ and also counting the number of Bocksteins in the monomials of $H_p(H_p) = H_*(H_p; Z_p)$ of

appropriate degrees.

Corollary. $\pi_*(H_p(n)) = H_*(C(n); Z_p)$, so the number of the generators of $\pi_*(H_p(n))$ is not greater than $2^{\binom{n+1}{2}}$.

In $H_*(BP; Z_p)$ r_i^j acts same as p_i^j . Consider the subalgebra P^* of the mod p Steenrod algebra $H_p^*(H_p)$ spanned by P^A , then the associated graded algebra $E^0(P^*)$ is the envelopping algebra over a Lie algebra mod p spanned by p_i^j with the relation $[p_k^j, p_{i-k}^{j+k}] = p_i^j$. So, modulo p and higher terms, $C(n)$ changes to May's resolution of (non-restricted) Lie algebra $\{p_i^j\}$, and we have

Lemma. There exist spectral sequences : $'E_2 = Z_p[\xi_1^{p^n}, \dots, \xi_n^p, \xi_{n+1}, \dots] \otimes \Lambda(\psi_i^j; i+j \leq n) \implies H_*(C(n); H_*(BP; Z_p))$ and $'E_2 = 'E_2 \otimes \Lambda(\tau_0, \tau_1', \tau_2', \dots) \implies H_*(C(n); H_*(H_p; Z_p))$, where $\deg \psi_i^j = 2p^{j+1}(p^i-1)-1$.

Corollary. If $C(n)$ exists then Problem 1 is affirmative by $H_p(n)$.

Note 1. $C(n)$ may be regarded as a sort of (unusual) resolution of $H_p(n)$ or $BP(n)$.

Note 2. Let p be sufficiently large w. r. t. n . If $H_p(n)$ exists and the above associated spectral sequences collapse, then we define a spectrum $VB(n)$ as the $(p^n q - 2)$ -skeleton of $H_p(n)$:

$$H_*(VB(n); \mathbb{Z}_p) \cong \Lambda(\tau_0, \tau_1', \tau_2', \dots, \tau_n') \otimes \Lambda(\psi_1^j; i+j < n).$$

Similarly, for the $(p^n q - 2)$ -skeleton $B(\binom{n}{2})$ of $BP(n)$,

$$H_*(B(\binom{n}{2}); \mathbb{Z}_p) \cong \Lambda(\psi_1^j; i+j < n).$$

Note 3. If $V(n)$ and $B(\binom{n}{2})$ exist then $V(n) \wedge B(\binom{n}{2})$ may be regarded as $VB(n)$.

Now we can prove the following

Theorem. For $p \geq 3$, $BP(2)$, $H_p(2)$, $VB(2)$ and $B(3)$ exist. For $p \geq 5$,

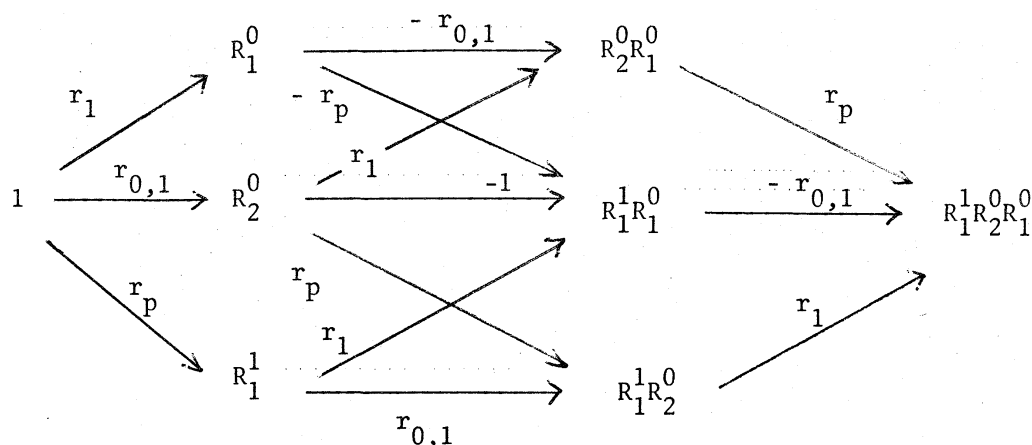
$BP(3)$, $H_p(3)$ and $VB(3)$ exist.

The main part of the proof is the construction of $C(2)$ and $C(3)$. If $C(n)$ is constructed for $X = BP$, it is also constructed for $X = H_p$ just by changing r_i^j by p_i^j . So we construct $C(2)$ and $C(3)$ for $X = BP$ only.

First consider the case $n = 2$. Since $r_1 = r_1^0$, $r_p = r_1^1$ and $r_{0,1} = r_2^0$ enjoy the relations

$$[r_1, r_p] = r_{0,1} \quad \text{and} \quad [r_1, r_{0,1}] = [r_p, r_{0,1}] = 0,$$

$C(2)$ is defined by the formulas in Problem 2 without taking modulus, that is, it is represented by the following diagram (replacing $x.BP$ by x) :



Note 4. $BP(1)$ is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_p e^{pq} \cup_{\alpha_1} e^{(p+1)q} \cup e^{2pq-1} \dots$$

where $p^p(S^0) = e^{pq}$ and $p^1(e^{pq}) = e^{(p+1)q}$ in mod p cohomology.

$BP(2)$ is mod p equivalent to

$$S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1 \beta_1^{p-1}} e^{p^2q-1} \cup_p e^{p^2q} \cup e^{(p^2+p)q-2} \cup e^{(p^2+p)q-1} \dots$$

where $p^{p^2}(S^0) = e^{p^2q}$.

Note 5. There exists a chain map $g : C(2) \longrightarrow \Sigma^{pq} C(2)$ of degree 1 such as

$$\begin{array}{ccccccc}
 & & 1 & & R_1^0 & R_2^0 & R_1^1 & \dots \\
 & & \downarrow & & \swarrow & \downarrow & \swarrow & \\
 g \downarrow & & 0 & & r_{p-1} & v_1 & -p & \\
 & & (\emptyset) & & & 1 & &
 \end{array}$$

Moreover $gg = 0$. This induces maps $g : BP(2) \longrightarrow \Sigma^{pq-1} BP(2)$ and

$$g : H_p(2) \longrightarrow \Sigma^{pq-1} H_p(2) \text{ such that } g_*(\psi_1^0) = 1.$$

Note 6. By considering a fibre of the sequence $H_p(2) \xrightarrow{g} \Sigma^{pq-1} H_p(2) \xrightarrow{g} \Sigma^{2(pq-1)} H_p(2) \longrightarrow \dots$, we can obtain $V(2)$ for $p > 3$, since g kills ψ_1^0 . But this breaks for $p = 3$ and we obtain $V(1\frac{1}{2})$ only.

Note 7. By a chain equivalence R_2^0 and $R_1^1 R_1^0$ are cancelled, and we obtain an equivalent chain complex

$$C(2)' : \begin{array}{ccccc} & & h & \xrightarrow{-r_{0,1} - r_1 r_p} & g \\ & \nearrow r_1 & \downarrow r_p r_p & \searrow & \downarrow r_p \\ 1 & & & & hk. \\ & \searrow r_p & \uparrow r_1 r_1 & \nearrow & \uparrow r_1 \\ & & h_1 & \xrightarrow{r_{0,1} - r_p r_1} & k \end{array}$$

Moreover, combining with the reduced form of g , as in Note 6, we obtain a BP-resolution of S^0 up to degree $p^2 q - 2$ which is essentially same as "BP-resolution" of Thomas-Zahler used in proving the non-triviality of some γ_i .

Finally we consider the case $n = 3$. The operations $\{r_i^j; i+j \leq 3\}$ are no more closed under $[\ , \]$:

$$\begin{aligned} [r_1, r_{p^2}] &= r_{p^2-p} r_{0,1}, & [r_p, r_{p^2}] &= r_{0,p} + f r_{0,1}, & [r_{0,1}, r_{p^2}] &= r_{0,0,1}, \\ [r_1, r_{0,p}] &= r_{0,0,1}, & [r_p, r_{0,p}] &= r_{p-1} r_{0,0,1}, & [r_{0,1}, r_{0,p}] &= 0, \\ [r_{p^2}, r_{0,p}] &= (r_{p^2-1} - r_{0,p-1}) r_{0,0,1} & \text{and} & [r_i^j, r_{0,0,1}] &= 0 \text{ for } i+j \leq 3, \end{aligned}$$

where f is uniquely determined by the second equality (explicitly : $f = -r_{p^2-1} + \sum_{i=0}^{p-2} \frac{(-1)^i}{i+1} r_{p-i-1} r_{p^2-pi-p} r_{0,i}$).

However we can construct $C(3)$ by taking $f_{x,y} = -1$ or r_i^j for the first two cases in Problem 2 and modifying $f_{x,y}$ for the third case as follows. If gr_m^n is a term of above $[r_i^j, r_k^\ell]$ then

$$f_{x,y} = -g \quad \text{for } x = zR_m^n, y = zR_k^\ell R_i^j \quad (R_i^j, R_k^\ell, R_m^n \notin z).$$

In order to complete the definition of f in $C(3)$, we must add two more extra cases :

$$f_{x,y} = \begin{cases} -r_{p^2-p-1} & \text{for } x = zR_3^0 R_2^0, y = zR_1^2 R_2^1 R_1^0 \quad (x = 1, R_1^1) \\ -f' & \text{for } x = zR_3^0 R_2^0, y = zR_1^2 R_2^1 R_1^1 \quad (x = 1, R_1^0), \end{cases}$$

where f' is determined by

$$f' r_{0,1} = [r_{p-1}, r_{p^2}] - [r_p, r_{p^2-1}].$$

To check the condition $[f, f] = 0$, we need various relations in $[,]$. For example, between $R_3^0 R_2^0$ and $R_1^2 R_2^1 R_1^0$ there are 10 monomials connected by

non-trivial maps. They are cancelled by $[1, f] = [1, r_{p^2-1}] = 0$ and

$$[f', r_1] = [r_{p-1}, r_{p^2-p}] + [r_p, r_{p^2-p-1}].$$

Consequently we can construct $C(3)$, and then $H_p(3)$ and $BP(3)$ for $p > 3$

by applying the first lemma. For $p = 3$ and $X = BP$, $C(3)$ is realized except the last term, then $B(3)$ is obtained as a skeleton.

Note 8. Let $B(2)$ be the $((p^2+p)q - 2)$ -skeleton of $B(3)$ then there are cofiberings

$$S^0 \longrightarrow B(1) \longrightarrow S^{pq-1},$$

$$B(1) \longrightarrow B(2) \longrightarrow \Sigma^{p^2q-1} B(1),$$

$$B(2) \longrightarrow B(3) \longrightarrow \Sigma^{(p^2+p)q-1} B(2)$$

$$\text{and } B(3) \equiv_p S^0 \cup_{\beta_1} e^{pq-1} \cup_{\alpha_1 \beta_1^{p-1}} e^{p^2q-1} \cup_{\beta_1} e^{(p^2+p)q-2} \cup_p e^{(p^2+p)q-1} \cup \dots$$

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