

Wave front propagation and the discriminant of a tame polynomial

田邊晋 (Susumu Tanabé)

熊本大学自然科学研究科数理科学講座

Department of Mathematics, Kumamoto University

ABSTRACT. *In this note we present a description of a wave front starting from an algebraic hypersurface surface as a pull-back of the discriminantal loci of a tame polynomial by a polynomial mapping. As an application we give examples of wave fronts which define free/almost free divisors near the focal point.*

1 Preliminaries on the wave fronts

In this section we prepare fundamental notations and lemmata to develop our studies in further sections. Let us denote by $Y := \{(z, u) \in \mathbb{C}^{n+1}; F(z) + u = 0\}$ the complexified initial wave front set defined by a polynomial $F(z) \in \mathbb{R}[z_1, \dots, z_n]$, $z = (z_1, \dots, z_n)$. Of course the real initial wave front set is $Y \cap \mathbb{R}^{n+1}$.

Let us consider the traveling of the ray starting from a point $(z, u) \in Y$ along unit vectors perpendicular to the hypersurface tangent to Y at (z, u) . It will reach at the point $(x_1, \dots, x_{n+1})$

$$\begin{aligned} x_j &= \pm t \frac{1}{|(d_z F(z), 1)|} \frac{\partial F(z)}{\partial z_j} + z_j, 1 \leq j \leq n, \\ x_{n+1} &= \pm t \frac{1}{|(d_z F(z), 1)|} + u \text{ with } (z, u) \in Y, \end{aligned} \quad (1.1)$$

at the moment t . Further on, we denote by $x' = (x_1, \dots, x_n)$, $x = (x', x_{n+1})$. We see that (x, t) and (z, u) satisfying the relation (1.1) are located on the zero loci of two phase functions

$$\psi_{\pm}(x, t, z, u) = (\langle x' - z, d_z F(z) \rangle + (x_{n+1} - u)) \pm t |(d_z F(z), 1)|, \quad (1.2)$$

each of which corresponds to the backward $\psi_+(x, t, z, u)$ (resp. the forward $\psi_-(x, t, z, u)$) wave propagation. To simplify the argument, we will not distinguish forward and backward wave propagations in future. This leads us to introduce an unified phase function

$$\begin{aligned} \psi(x, t, z, u) &:= \psi_+(x, t, z, u) \cdot \psi_-(x, t, z, u) \\ &= (\langle x' - z, d_z F(z) \rangle + (x_{n+1} + u))^2 - t^2 |(d_z F(z), 1)|^2, \end{aligned} \quad (1.3)$$

Let us denote by W_t the wave front at time t with the initial wave front Y i.e. $Y = W_0$.

Lemma 1.1. *For $x \in W_t$, the point (x, t) belongs to the critical value set of the projection,*

$$\begin{aligned} \{(z, u) \in Y : \psi(x, t, z, u) = 0\} &\rightarrow \mathbb{C}^{n+2} \\ (x, t, z, u) &\mapsto (x, t). \end{aligned}$$

We can understand this fact in several ways. Instead of purely geometrical interpretation, in our previous publication [9] we adopted investigation of the singular loci of the integral of type,

$$I(x, t) = \int_{\gamma} H(z, u) \left(\frac{1}{\psi_+(x, t, z, u)} + \frac{1}{\psi_-(x, t, z, u)} \right) dz \wedge du$$

for $\gamma \in H_n(Y)$ and $H(z, u) \in \mathcal{O}_{\mathbb{C}^{n+1}}$. The above integral ramifies around its singular loci W_t and by the general theory of the Gel'fand-Leray integrals (cf. [11]), W_t is contained in the critical value set mentioned in the Lemma 1.1.

According to the Lemma 1.1, The set $LW := \cup_{t \in \mathbb{C}} W_t \subset \mathbb{C}^{n+1}$ (the real part of it is the large wave front after Arnol'd [1] I, 22.1) can be interpreted as a subset of the discriminant of the function (called the phase function)

$$\Psi(x, t, z) := (\langle x' - z, d_z F(z) \rangle + x_{n+1} + F(z))^2 - t^2(|d_z F(z)|^2 + 1) \quad (1.4)$$

for $x' = (x_1, \dots, x_n)$. This is a set of (x, t) for which the algebraic variety

$$X_{x,t} := \{z \in \mathbb{C}^n : \Psi(x, t, z) = 0\}$$

has singular points.

Remark 1.1. Masaru Hasegawa [7] and Toshizumi Fukui (Saitama University) study the wave front W_t as a discriminantal loci of the function,

$$\Phi(x, t, z) = -\frac{1}{2}(|(x' - z, x_{n+1} + F(z))|^2 - t^2),$$

that measures the tangency of the sphere $\{(z, z_{n+1}) \in \mathbb{R}^{n+1} : |(z - x', z_{n+1} - x_{n+1})|^2 = t^2\}$ with the hypersurface $Y \cap \mathbb{R}^{n+1}$. In some cases, this approach allows us to get less complicated expression of the defining equation of LW in comparison with ours in Theorem 2.5.

We assume that the variety $X_{x,t}$ has at most isolated singular points for a point (x, t) of the space-time. Among those points, we choose a focal point $(x_0, t_0) \in \mathbb{C}^{n+2}$ i.e. the point where the maximum of the sum of all local Milnor numbers is attained. If we denote by $z^{(1)}, \dots, z^{(k)}$ the singular points located on X_{x_0, t_0} and Milnor numbers corresponding to these points by $\mu(z^{(i)})$, $i = 1, \dots, k$, the following inequality holds for the focal point

$$\text{sum of Milnor numbers of singular points on } X_{x,t} \leq \sum_{i=1}^k \mu(z^{(i)}),$$

for every $(x, t) \in \mathbb{C}^{n+2}$.

Assume that the quotient ring

$$\frac{\mathbb{C}[z]}{(d_z \Psi(x_0, t_0, z))\mathbb{C}[z]} \quad (1.5)$$

is a μ dimensional $\mathbb{C}$ vector space such that it admits a basis $\{e_1(z), \dots, e_\mu(z)\}$ that contains a set of basis elements as follows,

$$e_1(z) = 1, e_{j+1}(z) = (z_j - z_j^{(i)}), 1 \leq j \leq n, \quad (1.6)$$

for a fixed $i \in [1, k]$. Here we remark that $\sum_{i=1}^k \mu(z^{(i)}) \leq \mu$. The denominator $(d_z \Psi(x_0, t_0, z))\mathbb{C}[z]$ of the expression (1.5) means the Jacobian ideal of the polynomial $\Psi(x_0, t_0, z)$.

Now we decompose the difference

$$\Psi(x, t, z) - \Psi(x_0, t_0, z) = \sum_{j=1}^m s_j(x, t) e_j(z)$$

by means a set of polynomials in z , $\{e_1(z), \dots, e_\mu(z), e_{\mu+1}(z), \dots, e_m(z)\}$ and a set of polynomials in (x, t) ,

$$\begin{aligned} \iota : \mathbb{C}^{n+2} &\rightarrow \mathbb{C}^m \\ (x, t) &\mapsto \iota(x, t) := (s_1(x, t), \dots, s_m(x, t)) \end{aligned} \quad (1.7)$$

thus defined. In this way we introduce a set of polynomials $\{e_{\mu+1}(z), \dots, e_m(z)\}$ in addition to the basis of (1.5). We consider a polynomial $\varphi(z, s) \in \mathbb{C}[z, s]$ for $s = (s_1, \dots, s_m)$ defined by

$$\varphi(z, s) = \Psi(x_0, t_0, z) + \sum_{j=1}^m s_j e_j(z). \quad (1.8)$$

Locally this is a versal (but not miniversal) deformation of the holomorphic function germ $\Psi(x_0, t_0, z)$ at $z = z^{(i)}$.

2 Discriminant of a tame polynomial

Definition 2.1. *The polynomial $f(z) \in \mathbb{C}[z]$ is called tame if there is a compact set U of the critical points of $f(z)$ such that $\|d_z f(z)\| = \sqrt{(d_z f(z), \overline{d_z f(z)})}$ is away from 0 for all $z \notin U$.*

In the sequel we use the notation $s' = (s_2, \dots, s_m)$ and $s = (s_1, s')$.

Further on we impose the following conditions on $\varphi(z, s)$ introduced in (1.8). Assume that there exists an open set $0 \in V \subset \mathbb{C}^{m-1}$ such that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z \varphi(z, s))\mathbb{C}[z]} < \infty, \quad (2.1)$$

for every $s' \in V$ and $s_1 \in \mathbb{C}$. In addition to this, we assume that for every $s = (s_1, \dots, s_{n+1}, 0, \dots, 0) \in V$, the equality

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z (\Psi(x_0, t_0, z) + \sum_{j=2}^{n+1} s_j e_j(z)))\mathbb{C}[z]} = \mu, \quad (2.1)'$$

holds.

Lemma 2.1. *Under the conditions (1.5), (2.1), (2.1)' there exists a constructible subset $\tilde{U} \subset V$, such that $\varphi(z, s)$ is a tame polynomial for every $s \in \mathbb{C} \times \tilde{U}$ and*

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z \varphi(z, s))\mathbb{C}[z]} = \mu,$$

for every $s \in \mathbb{C} \times U$.

Proof

By [3], Proposition 3.1, (2.1)' yields the tameness of $\varphi(z, 0)$. After Proposition 3.2 of the same article, the set of s such that $\varphi(z, s)$ be tame is a constructible subset (i.e. locally closed set with respect to the Zariski topology) of the form $\mathbb{C} \times W$ for $W \subset V$. According to [3], Proposition 2.3, the set

$$T_n = \{s \in \mathbb{C} \times W : \dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{(d_z \varphi(z, s))\mathbb{C}[z]} \leq n\},$$

is Zariski closed for every n . We can take $\mathbb{C} \times \tilde{U} = T_{\mu} \setminus T_{\mu-1}$. **Q.E.D.**

Assumption I

(i) By shrinking $\tilde{U}$ if necessary, we assume that a constructible set $U \subset \tilde{U}$ can be given locally by holomorphic functions $(s_{\nu+1}, \dots, s_m)$ on the coordinate space with variables $(s_2, \dots, s_{\nu})$, $\nu \geq \mu$.

(ii) The image of the mapping ι of a neighbourhood of (x_0, t_0) is contained in $\mathbb{C} \times U$. In other words,

$$\iota(\mathbb{C}^{n+2}, (x_0, t_0)) \subset (\mathbb{C} \times U, \iota(x_0, t_0)).$$

For a fixed $\tilde{s}' = (\tilde{s}_2, \dots, \tilde{s}_m) \in U$ and the constructible subset $U \subset V$ of the Assumption I, (i) we see that $\varphi(z, s_1, \tilde{s}')$ is a tame polynomial for all $s_1 \in \mathbb{C}$. For such $\varphi(z, s_1, \tilde{s}')$, we define the following modules,

$$\mathcal{P}_{\varphi}(\tilde{s}') := \frac{\Omega_{\mathbb{C}^n}^{n-1}}{d_z \varphi(z, s_1, \tilde{s}') \wedge \Omega_{\mathbb{C}^n}^{n-2} + d\Omega_{\mathbb{C}^n}^{n-2}}, \quad (2.2)$$

$$\mathcal{B}_{\varphi}(\tilde{s}') := \frac{\Omega_{\mathbb{C}^n}^n}{d_z \varphi(z, s_1, \tilde{s}') \wedge d\Omega_{\mathbb{C}^n}^{n-2}}. \quad (2.3)$$

the module $\mathcal{B}_{\varphi}(\tilde{s}')$ is called an algebraic Brieskorn lattice. In considering the holomorphic forms multiplied by $\varphi(z, s_1, \tilde{s}')$ be zero in (2.2), (2.3) we can treat two modules as $\mathbb{C}[s_1]$ modules.

These modules contain the essential informations on the topology of the variety

$$Z_{(s_1, \tilde{s}')} = \{z \in \mathbb{C}^n : \varphi(z, s_1, \tilde{s}') = 0\}. \quad (2.4)$$

Let us denote by $D_{\varphi} \subset \mathbb{C} \times U$ the discriminantal loci of the polynomial $\varphi(z, s)$ i.e.

$$D_{\varphi} := \{s \in \mathbb{C} \times U : \exists z \in Z_s, \text{ s.t. } d_z \varphi(z, s) = \vec{0}\}. \quad (2.5)$$

Theorem 2.2. *For a fixed $\tilde{s}' = (\tilde{s}_2, \dots, \tilde{s}_m) \in U$, both $\mathcal{P}_{\varphi}(\tilde{s}')$ and $\mathcal{B}_{\varphi}(\tilde{s}')$ are free $\mathbb{C}[s_1]$ modules of rank μ .*

Proof First we show the statement on $\mathcal{B}_\varphi(\tilde{s}')$. After [5], Theorem 0.5, the algebraic Brieskorn lattice $\mathcal{B}_\varphi(\tilde{s}')$ is isomorphic to a free $\mathbb{C}[s_1]$ module of finite rank (so called the Brieskorn-Deligne lattice). The absence of the vanishing cycles at infinity for $\varphi(z, s_1, \tilde{s}')$ ensures this isomorphism.

On the other hand, for $(\tilde{s}_1, \tilde{s}') \in \mathbb{C} \times U$, the Corollary 0.2 of the same article tells us the following equality.

$$\begin{aligned} & \dim \operatorname{Coker}(s_1 - \tilde{s}_1 | \mathcal{B}_\varphi(\tilde{s}')) \\ &= \dim H_{n-1}(Z_{(\tilde{s}_1, \tilde{s}')})) + \text{sum of Milnor numbers of singular points on } Z_{(\tilde{s}_1, \tilde{s}')} \end{aligned}$$

For $(\tilde{s}_1, \tilde{s}') \in \mathbb{C} \times U \setminus D_\varphi$, the right hand side of the above equality equals

$$\sum_{s_1: Z_{(s_1, \tilde{s}')} \text{ singular}} \text{sum of Milnor numbers of singular points on } Z_{(s_1, \tilde{s}')}$$

by [3], Theorem 1.2.

Now we show that $\mathcal{B}_\varphi(\tilde{s}')$ is isomorphic to $\mathcal{P}_\varphi(\tilde{s}')$.

We show the bijectivity of the mapping $d : \mathcal{P}_\varphi(\tilde{s}') \rightarrow \mathcal{B}_\varphi(\tilde{s}')$. To see the injectivity, we remark that the condition $d(\omega + d\alpha + \beta \wedge d\varphi(z, s_1, \tilde{s}')) = d\omega + d\beta \wedge d\varphi(z, s_1, \tilde{s}') = 0$, $\alpha, \beta \in \Omega^{n-1}$ in $\mathcal{B}_\varphi(\tilde{s}')$, entails the existence of $\alpha' \in \Omega^{n-1}$ such that $d\omega = d\alpha' \wedge d\varphi(z, s_1, \tilde{s}')$, this in turn together with the de Rham lemma entails $\omega = \alpha' \wedge d\varphi(z, s_1, \tilde{s}') + d\beta'$ for some $\beta' \in \Omega^{n-1}$.

To see the surjectivity, it is enough to check that for every $\gamma \in \Omega^n$ the equation $d\omega = \gamma$ is solvable. **Q.E.D.**

Let us introduce a module for $\tilde{s}' = (\tilde{s}_2, \dots, \tilde{s}_m) \in U$,

$$Q_\varphi(\tilde{s}') := \frac{\Omega_{\mathbb{C}^n}^n}{d_z \varphi(z, s_1, \tilde{s}') \wedge \Omega_{\mathbb{C}^n}^{n-1}} \cong \frac{\mathbb{C}[z]}{(d_z \varphi(z, s_1, \tilde{s}')) \mathbb{C}[z]}, \quad (2.6)$$

that is a free $\mathbb{C}[s_1]$ module of rank μ because it is isomorphic to

$$\bigoplus_{s_1: Z_{(s_1, \tilde{s}')} \text{ singular}} \bigoplus_{z: \text{singular points on } Z_{(s_1, \tilde{s}')}} \mathbb{C}^{\mu(z)},$$

with $\mu(z)$: the Milnor number of the singular point $z \in Z_{(s_1, \tilde{s}')}$. Let us denote its basis by

$$\{g_1 dz, \dots, g_\mu dz\}, \quad (2.7)$$

such that the polynomials $\{g_1(z), \dots, g_\mu(z)\}$ consist a basis of the RHS of (2.6) as a free $\mathbb{C}[s_1]$ module.

According to [3], p.218, lines 5-6, the following is a locally trivial fibration,

$$Z_{(s_1, s')} \rightarrow (s_1, s') \in \mathbb{C} \times U \setminus D_\varphi.$$

This yields the next statement.

Corollary 2.3. *We can choose a basis $\{\omega_1, \dots, \omega_\mu\}$ of $\mathcal{P}_\varphi(\tilde{s}')$ independent of $\tilde{s}' \in U$.*

Due to the construction of U , we can consider the ring $\mathcal{O}_U$ of holomorphic functions on U . By the analytic continuation with respect to the parameter $s' \in U$, we see the following.

Lemma 2.4. *The modules $\mathcal{B}_\varphi(s')$, $\mathcal{P}_\varphi(s')$, $\mathcal{Q}_\varphi(s')$ are free $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ modules of rank μ .*

As the deformation polynomials $e_1, \dots, e_\mu$ arise from the special form of $\Psi(x, t, z)$ we are obliged to impose the following assumption.

Assumption II We assume that we can adopt $e_i(z)$ of (1.5), (1.6) as $g_i(z)$ in (2.7) $i = 1, \dots, \mu$ and they serve as a basis of $\mathcal{Q}_\varphi(s')$ as a free $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ module.

For the sake of simplicity, let us denote by $\text{mod}(d_z(\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z)))$ the residue class modulo the ideal $(d_z(\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z)))\mathbb{C}[z, s_1] \otimes \mathcal{O}_U$ in $\mathbb{C}[z, s_1] \otimes \mathcal{O}_U$. By virtue of the freeness of $\mathcal{Q}_\varphi(s')$, this residue class is uniquely determined. Our assumption (1.5), (1.6) together with the Weierstrass preparation theorem gives us a decomposition as follows,

$$\begin{aligned} & (\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z)) \cdot \frac{\partial \varphi(z, s)}{\partial s_i} \\ & \equiv \sum_{\ell=1}^{\mu} \sigma_i^\ell(s') \frac{\partial \varphi(z, s)}{\partial s_\ell} \text{ mod}(d_z(\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z))), \quad 1 \leq i \leq \mu \end{aligned} \quad (2.8)$$

$$\frac{\partial \varphi(z, s)}{\partial s_i} \equiv \sum_{\ell=1}^{\mu} \sigma_i^\ell(s') \frac{\partial \varphi(z, s)}{\partial s_\ell} \text{ mod}(d_z(\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z))), \quad \mu + 1 \leq i \leq m, \quad (2.9)$$

with $\sigma_i^\ell(s') \in \mathcal{O}_U$. In fact, according to an argument used in [4], Theorem A4, [10], Proposition 2 (both treat liftable vector fields in local case but they are valid for our situation), the following vector fields are tangent to the discriminant D_φ ,

$$\vec{v}_i := (s_1 + \sigma_i^1(s')) \frac{\partial}{\partial s_i} + \sum_{\ell=1, \ell \neq i}^{\mu} \sigma_i^\ell(s') \frac{\partial \varphi(z, s)}{\partial s_\ell}, \quad 1 \leq i \leq \mu \quad (2.10)$$

Here we recall the Assumption I, (i) that allows us to adopt $(s_1, s_2, \dots, s_\nu)$, $\nu \geq \mu$ as the local coordinates of $\mathbb{C} \times U$.

$$\vec{v}_i := -\frac{\partial}{\partial s_i} + \sum_{\ell=1}^{\mu} \sigma_i^\ell(s') \frac{\partial}{\partial s_\ell}, \quad \mu + 1 \leq i \leq \nu, \quad (2.11)$$

Evidently they are linearly independent over $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ because of the presence of the term $s_1 \frac{\partial}{\partial s_i}$ for every $1 \leq i \leq \mu$ and $-\frac{\partial}{\partial s_i}$ for $\mu + 1 \leq i \leq \nu$. Therefore they form a $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ free module of rank ν . Let us introduce the following matrix of which the i -th

row corresponds to the vector $\vec{v}_i$.

$$\Sigma(s) := \begin{pmatrix} s_1 + \sigma_1^1(s') & \sigma_1^2(s') & \cdots & \sigma_1^\mu(s') & 0 & \cdots & 0 & 0 \\ \sigma_2^1(s') & s_1 + \sigma_2^2(s') & \cdots & \sigma_2^\mu(s') & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_\mu^1(s') & \sigma_\mu^2(s') & \cdots & s_1 + \sigma_\mu^\mu(s') & 0 & \cdots & 0 & 0 \\ \sigma_{\mu+1}^1(s') & \sigma_{\mu+1}^2(s') & \cdots & \sigma_{\mu+1}^\mu(s') & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{\nu-1}^1(s') & \sigma_{\nu-1}^2(s') & \cdots & \sigma_{\nu-1}^\mu(s') & 0 & \cdots & -1 & 0 \\ \sigma_\nu^1(s') & \sigma_\nu^2(s') & \cdots & \sigma_\nu^\mu(s') & 0 & \cdots & 0 & -1 \end{pmatrix}. \quad (2.12)$$

In fact the following $\mu \times \mu$ submatrix of $\Sigma(s)$ contains the essential geometrical informations on D_φ .

$$\tilde{\Sigma}(s) := \begin{pmatrix} s_1 + \sigma_1^1(s') & \sigma_1^2(s') & \cdots & \sigma_1^\mu(s') \\ \sigma_2^1(s') & s_1 + \sigma_2^2(s') & \cdots & \sigma_2^\mu(s') \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\mu^1(s') & \sigma_\mu^2(s') & \cdots & s_1 + \sigma_\mu^\mu(s') \end{pmatrix}. \quad (2.13)$$

Theorem 2.5. 1) The algebra $\text{Der}_{\mathbb{C} \times U}(\log D_\varphi)$ of tangent fields to D_φ as a free $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ is generated by the vectors v_i , $1 \leq i \leq \nu$ of (2.10), (2.11).

2) The discriminantal loci D_φ is given by the equation

$$D_\varphi = \{s \in \mathbb{C} \times U : \det \tilde{\Sigma}(s) = 0\}.$$

3) The preimage of D_φ by the mapping ι contains the wave front $LW = \cup_{t \in \mathbb{C}} W_t \subset \mathbb{C}^{n+1}$ i.e. $LW \subset \iota^{-1}(D_\varphi)$.

Proof The tangency of vector fields $\vec{v}_i$'s to D_φ and their independence over $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ have already been shown.

First we shall prove 2). By virtue of the tangency of $\vec{v}_i$'s to D_φ and the equality,

$$\vec{v}_1 \wedge \cdots \wedge \vec{v}_\nu = \det \Sigma(s) \partial_{s_1} \wedge \cdots \wedge \partial_{s_\nu},$$

the function $\det \Sigma(s)$ shall vanish on D_φ . The statement on $Q_\varphi(s')$ of the Lemma 2.4 tells us that

$$\sharp\{s \in \mathbb{C} \times U : s_1 = \text{const} \cap D_\varphi\} = \mu, \quad (2.14)$$

in taking the multiplicity into account.

From (2.12), (2.13) we see that

$$\pm \det \Sigma(s) = \det \tilde{\Sigma}(s) = s_1^\mu + d_1(s') s_1^{\mu-1} + \cdots + d_\mu(s'),$$

with $d_i(s') \in \mathcal{O}_U$, $1 \leq i \leq \mu$. Thus the Weierstrass polynomial in s_1 , $\det \tilde{\Sigma}(s)$ shall be divided by the defining equation of D_φ which turns out to be also a Weierstrass polynomial in s_1 of degree μ . This proves 2).

Now we shall show that every vector $\vec{v}$ tangent to D_φ admits a decomposition like

$$\vec{v} = \sum_{i=1}^{\nu} a_i(s) \vec{v}_i, \quad (2.15)$$

for some $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$. For every i the following expression shall vanish on D_φ , because of the tangency of all vectors taking part in it,

$$\vec{v}_1 \wedge \cdots \wedge \vec{v}_{i-1} \wedge \vec{v} \wedge \vec{v}_{i+1} \wedge \cdots \wedge \vec{v}_\nu.$$

Therefore there exists $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$ such that the above expression equals to $a_i(s) \det \Sigma(s) \partial_{s_1} \wedge \cdots \wedge \partial_{s_m}$. This means that the vector $\vec{v} - \sum_{i=1}^{\nu} a_i(s) \vec{v}_i$ defines a zero vector at every $s \notin D_\varphi$, as the vectors $\vec{v}_1, \dots, \vec{v}_\nu$ form a frame outside D_φ . By the continuity argument on holomorphic functions, we see that the decomposition holds everywhere on $\mathbb{C} \times U$.

The statement 3) follows from Lemma 1.1, (1.4) and the definition (1.7) of the mapping ι . **Q.E.D.**

3 Gauss-Manin system for a tame polynomial

In this section, we will show that the above matrix $\tilde{\Sigma}(s)$, (2.13) can be obtained as the coefficient of the Gauss-Manin system defined for a tame polynomial $\varphi(z, s)$.

According to Lemma 2.4, every $\omega \in \mathcal{P}_\varphi(s')$ admits a unique decomposition as follows,

$$\omega = \sum_{i=1}^{\mu} a_i(s) \omega_i, \quad s \in \mathbb{C} \times U. \quad (3.1)$$

A generalisation of theorem 0.2 of [6] tells us that the following equivalence holds for every holomorphic $n-1$ form ω ,

$$\forall s \in \mathbb{C} \times U, \omega|_{Z_s} = 0 \text{ in } H^{n-1}(Z_s) \Leftrightarrow \omega = 0 \text{ in } \mathcal{P}_\varphi(s'). \quad (3.2)$$

We can prove the above statement (3.2) for every $n \geq 2$ in following a slightly modified argument explained in §2 of [6].

This theorem yields a corollary that ensures us the following equality for every vanishing cycle $\delta(s) \in H_{n-1}(Z_s)$,

$$\int_{\delta(s)} \omega = \sum_{i=1}^{\mu} a_i(s) \int_{\delta(s)} \omega_i, \quad s \in \mathbb{C} \times U, \quad (3.3)$$

for some $a_i(s) \in \mathbb{C}[s_1] \otimes \mathcal{O}_U$, $1 \leq i \leq \mu$. To show this along with the argument by L.Gavrilov [6], we simply need to replace his Lemma 2.2 by [5], Corollary 0.7.

Here we remark that for the basis of $\{e_1(z)dz, \dots, e_\mu(z)dz\}$ of $Q_\varphi(\tilde{s}')$ we can choose the basis $\{\omega_1, \dots, \omega_\mu\}$ of $\mathcal{P}_\varphi(\tilde{s}')$ such that

$$d\omega_i = e_i(z)dz + d_z \varphi(z, s) \wedge \epsilon_i,$$

for some $\epsilon_i \in \Omega^{n-1}$. That is to say, for every $\omega \in \Omega^{n-1}$ we can find the following two types of decomposition

$$\begin{aligned}\omega &= \sum_{i=1}^{\mu} c_i(s') d\omega_i + d_z \varphi(z, s) \wedge d\xi, \\ &= \sum_{i=1}^{\mu} c_i(s') (e_i(z) dz + d_z \varphi(z, s) \wedge \epsilon_i) + d_z \varphi(z, s) \wedge \eta,\end{aligned}$$

for some $c_i(s') \in \mathcal{O}_U$, $\xi \in \Omega^{n-2} \otimes \mathcal{O}_U$, $\eta \in \Omega^{n-1} \otimes \mathcal{O}_U$. In other words, for every $\eta \in \Omega^{n-1} \otimes \mathcal{O}_U$ one can find $\tilde{\xi} \in \Omega^{n-2} \otimes \mathcal{O}_U$ and $c_i(s')$, ξ as above that satisfy

$$\eta = - \sum_{i=1}^{\mu} c_i(s') \epsilon_i + d\xi + d_z \varphi(z, s) \wedge d\tilde{\xi}.$$

If we take ϵ_i as some representatives of $\mathcal{P}_{\varphi}(\tilde{s}')$, the above statement is reduced to that on $\mathcal{P}_{\varphi}(\tilde{s}')$ of Lemma 2.4.

As E.Brieskorn [2] showed, the following equality holds if we understand it as the property of the holomorphic sections in the cohomology bundle $H^{n-1}(Z_s)$ defined as the Leray's residue $\omega/d_z \varphi(z, s)$ for $\omega \in \Omega^n$,

$$\left(\frac{\partial}{\partial s_1}\right)^{-1} d\eta = d_z \varphi(z, s) \wedge \eta.$$

This yields that

$$\left(\frac{\partial}{\partial s_1}\right)^{-1} \mathcal{B}_{\varphi}(\tilde{s}') = d_z \varphi(z, s) \wedge \Omega^{n-1} / d_z \varphi(z, s) \wedge d\Omega^{n-2},$$

$$Q_{\varphi}(\tilde{s}') = \mathcal{B}_{\varphi}(\tilde{s}') / \left(\frac{\partial}{\partial s_1}\right)^{-1} \mathcal{B}_{\varphi}(\tilde{s}'),$$

we see that $\{e_1(z)dz, \dots, e_{\mu}(z)dz\}$ is a basis of $\mathcal{B}_{\varphi}(\tilde{s}')$ as an $\mathcal{O}_U[(\frac{\partial}{\partial s_1})^{-1}]$ module.

For such ω_i 's we have a decomposition in $Q_{\varphi}(\tilde{s}')$ as follows,

$$(\varphi(z, s) - s_1) d\omega_i = \sum_{\ell=1}^{\mu} \sigma_i^{\ell}(s') d\omega_{\ell} + d_z \varphi(z, s) \wedge \eta_i, \quad 1 \leq i \leq \mu \quad (3.4)$$

$\eta_i \in \Omega^{n-1}$. We see that (3.4) is equivalent to (2.8). This relation immediately entails the following equality for every $\delta(s) \in H_{n-1}(Z_s)$,

$$s_1 \frac{\partial}{\partial s_1} \int_{\delta(s)} \omega_i + \sum_{\ell=1}^{\mu} \sigma_i^{\ell}(s') \frac{\partial}{\partial s_1} \int_{\delta(s)} \omega_{\ell} + \int_{\delta(s)} \eta_i = 0, \quad (3.5)$$

in view of the fact $\int_{\delta(s)} \varphi(z, s) \frac{d\omega_i}{d_z \varphi(z, s)} = 0$ and the Leray's residue theorem

$$\frac{\partial}{\partial s_1} \int_{\delta(s)} \omega_i = \int_{\delta(s)} \frac{d\omega_i}{d_z \varphi(z, s)}.$$

After (3.3), every $\int_{\delta(s)} \eta_i$ admits an unique decomposition

$$\int_{\delta(s)} \eta_i = \sum_{j=1}^{\mu} b_i^j(s) \int_{\delta(s)} \omega_j, s \in \mathbb{C} \times U, \quad (3.6)$$

for some $b_i^j(s) \in \mathbb{C}[s_1] \times \mathcal{O}_U$, $1 \leq i, j \leq \mu$.

Let us consider a vector of fibre integrals

$$\mathbb{I}_Q := {}^t \left(\int_{\delta(s)} \omega_1, \dots, \int_{\delta(s)} \omega_{\mu} \right). \quad (3.7)$$

In summary we get

Proposition 3.1. 1) For a vector $\mathbb{I}_Q$, (3.5) we have the following Gauss-Manin system

$$\tilde{\Sigma} \cdot \frac{\partial}{\partial s_1} \mathbb{I}_Q + B(s) \mathbb{I}_Q = 0, \quad (3.8)$$

where $B(s) = (b_i^j(s))_{1 \leq i, j \leq \mu}$ for functions determined in (3.6).

2) The discriminantal loci D_{φ} of the tame polynomial $\varphi(z, s)$, $s \in \mathbb{C} \times U$ has an expression,

$$D_{\varphi} = \{s \in \mathbb{C} \times U : \det \tilde{\Sigma}(s) = 0\},$$

that corresponds to the singular loci of the system (3.8).

Remark 3.1. To see that the two statements on D_{φ} do not mean a simple coincidence, one may consult [10] Theorem 2.3 where he finds a description of the Gauss-Manin system for Leray's residues by means of the tangent vector fields to the discriminant loci.

4 Free and almost free wave fronts

Now we recall that the freeness of $Der_{\mathbb{C} \times U}(\log D_{\varphi})$ as a $\mathbb{C}[s_1] \otimes \mathcal{O}_U$ module, proven in the Theorem 2.5, means that D_{φ} defines a free divisor (in the sense of K.Saito) in the neighbourhood of every point $s \in D_{\varphi}$. We define the logarithmic tangent space $T_s^{\log} D_{\varphi}$ to D_{φ} at s :

$$T_s^{\log} D_{\varphi} = \{\vec{v}(s) : \vec{v}(s) \in Der_{\mathbb{C} \times U}(\log D_{\varphi})_s\} \quad (4.1)$$

We follow the presentation by David Mond [8] on the free and almost free divisors though the latter has been first introduced by J.N.Damon. To discuss when the large wave front LW becomes a free divisor, we need to make use of the notion of algebraic transversality. We recall here the Assumption I, (ii) on the image of the mapping ι that entails the following inclusion relation,

$$d_{x,t} \iota(T_{(x,t)} \mathbb{C}^{n+2}) \subset T_{\iota(x,t)}(\mathbb{C} \times U),$$

for (x, t) in the neighbourhood of (x_0, t_0) .

Definition 4.1. The mapping ι is algebraically transverse to D_{φ} at $(x_0, t_0) \in \mathbb{C}^{n+2}$ if and only if

$$d_{x,t} \iota(T_{(x_0,t_0)} \mathbb{C}^{n+2}) + T_{\iota(x_0,t_0)}^{\log} D_{\varphi} = T_{\iota(x_0,t_0)}(\mathbb{C} \times U). \quad (4.2)$$

Lemma 4.1. (*[8] Jacobian criterion for freeness*) *The divisor $\iota^{-1}(D_\varphi)$ is free if and only if ι is algebraically transverse to D_φ .*

To state a criterion of the freeness of $\iota^{-1}(D_\varphi)$, we need the following $m \times (\nu + n + 2)$ matrix $T(x, t)$.

$$T(x, t) = \begin{pmatrix} s_1 + \sigma_1^1(s'(x, t)) & \cdots & \sigma_1^\mu(s'(x, t)) & 0 & \cdots & 0 & \cdots & 0 \\ \sigma_2^1(s'(x, t)) & \cdots & \sigma_2^\mu(s'(x, t)) & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sigma_\mu^1(s'(x, t)) & \cdots & s_1 + \sigma_\mu^\mu(s'(x, t)) & 0 & \cdots & 0 & \cdots & 0 \\ \sigma_{\mu+1}^1(s'(x, t)) & \cdots & \sigma_{\mu+1}^\mu(s'(x, t)) & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \sigma_\nu^1(s'(x, t)) & \cdots & \sigma_\nu^\mu(s'(x, t)) & 0 & \cdots & -1 & \cdots & 0 \\ s_1(x, t)_{x_1} & \cdots & s_\mu(x, t)_{x_1} & s_{\mu+1}(x, t)_{x_1} & \cdots & s_\nu(x, t)_{x_1} & \cdots & s_m(x, t)_{x_1} \\ s_1(x, t)_{x_2} & \cdots & s_\mu(x, t)_{x_2} & s_{\mu+1}(x, t)_{x_2} & \cdots & s_\nu(x, t)_{x_2} & \cdots & s_m(x, t)_{x_2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ s_1(x, t)_{x_{n+1}} & \cdots & s_\mu(x, t)_{x_{n+1}} & s_{\mu+1}(x, t)_{x_{n+1}} & \cdots & s_\nu(x, t)_{x_{n+1}} & \cdots & s_m(x, t)_{x_{n+1}} \\ s_1(x, t)_t & \cdots & s_\mu(x, t)_t & s_{\mu+1}(x, t)_t & \cdots & s_\nu(x, t)_t & \cdots & s_m(x, t)_t \end{pmatrix}. \quad (4.3)$$

The first ν rows of the $T(x, t)$ correspond to those of $\Sigma(\iota(x, t))$ while the $(\nu + i)$ -th row corresponds to $\frac{\partial}{\partial x_i} \iota(x, t)$, $1 \leq i \leq n + 1$ and the last row to $\frac{\partial}{\partial t} \iota(x, t)$ for $\iota(x, t)$ of (1.7).

The Lemma 4.1 yields immediately the following statement in view of the Theorem 2.5.

Proposition 4.2. *The divisor $\iota^{-1}(D_\varphi)$ is free in the neighbourhood of (x, t) if and only if $\text{rank } T(x, t) \geq \nu$.*

After Theorem 2.5, in the neighbourhood of each of its point s , the hypersurface D_φ defines a germ of free divisor.

Definition 4.2. *The germ of hypersurface $\iota^{-1}(D_\varphi)$ at $(x_0, t_0) \in \mathbb{C}^{n+2}$ is an almost free divisor based on the germ of free divisor D_φ at $\iota(x_0, t_0) \in \mathbb{C} \times U$ if there is a map $i_0 : \iota^{-1}(D_\varphi) \rightarrow D_\varphi$ which is algebraically transverse to D_φ except at (x_0, t_0) such that $\iota^{-1}(D_\varphi) = i_0^{-1}(D_\varphi)$.*

In view of this definition, we get a criterion so that $\iota^{-1}(D_\varphi)$ be an almost free divisor.

Proposition 4.3. *The germ of hypersurface $\iota^{-1}(D_\varphi)$ at $(x_0, t_0) \in \mathbb{C}^{n+2}$ is an almost free divisor based on the germ of free divisor D_φ at $\iota(x_0, t_0) \in \mathbb{C} \times U$ if the following inequality holds at an isolated point $(x_0, t_0) \in \iota^{-1}(D_\varphi)$,*

$$\text{rank } \Sigma(\iota(x_0, t_0)) + \text{rank } d_{x,t} \iota(x_0, t_0) < \nu, \quad (4.4)$$

while at other points $(x, t) \neq (x_0, t_0)$ in the neighbourhood of (x_0, t_0) , $\text{rank } T(x, t) \geq \nu$.

5 Examples

1. Wave propagation on the plane

Let us consider the following initial wave front on the plane $Y := \{(z, u) \in \mathbb{C}^2; az^2 + z^4 + u = 0\}$, $z = \text{i.e.}$ $F(z) = az^2 + z^4$ for some real non-zero constant a . In this case our phase function has the following expression

$$\begin{aligned}\Psi(x, t, z) &= (x_1 + az^2 + z^4 + (x_2 - z)(2az + 4z^3))^2 - t^2(1 + (2az + 4z^3)^2), \\ &= -t^2 + x_2^2 + 4ax_1x_2z + (-4a^2t^2 + 4a^2x_1^2 - 2ax_2)z^2 \\ &\quad (-4a^2x_1 + 8x_1x_2)z^3 + (a^2 - 16at^2 + 16ax_1^2 - 6x_2)z^4 \\ &\quad -20ax_1z^5 + (6a - 16t^2 + 16x_1^2)z^6 - 24x_1z^7 + 9z^8.\end{aligned}\quad (5.1)$$

It is easy to see that $(x_1, x_2, t) = (0, -1/2a, 1/2a)$ is a focal point with a singular point $(z, u) = (0, 0)$ and the Milnor number $\mu(0) = 3$ (A_3 singularity i.e. the swallow tail) if $a \neq 1$ and $\mu(0) = 5$ (A_5 singularity) if $a = 1$,

$$\Psi(0, -a/2, a/2, z) = (-(1/a) + a^2)z^4 + (-(4/a^2) + 6a)z^6 + 9z^8. \quad (5.2)$$

The quotient ring (1.5) for this $\Psi(0, -1/2a, 1/2a, z)$ has dimension $\mu = 7$.

Especially we can choose $e_i = z^{i-1}$, $i = 1, \dots, 7$ as the basis (2.7). Now, in view of (5.1) we introduce additional deformation polynomials $e_8 = z^7$, together with entries of the mapping ι (1.7),

$$\begin{aligned}s_1 &= -t^2 + x_2^2, s_2 = 4ax_1x_2, s_3 = -4a^2t^2 + 4a^2x_1^2 - 2ax_2, s_4 = -4a^2x_1 + 8x_1x_2, \\ s_5 &= a^2 - 16at^2 + 16ax_1^2 - 6x_2, s_6 = -20ax_1, s_7 = 6a - 16t^2 + 16x_1^2, s_8 = -24x_1.\end{aligned}\quad (5.3)$$

$$\varphi(z, s) = 9z^8 + \sum_{i=1}^8 s_i z^{i-1}.$$

In this case, the constructible set U of the Assumption I,(i) coincides with $\mathbb{C}^7$.

By the aid of the computer algebra system SINGULAR, we calculate the residue class $\text{mod}(d_z(\varphi(z, 0) + \sum_{j=2}^m s_j e_j(z)))$ of the following polynomials that illustrate (2.8).

$$\begin{aligned}\varphi(z, s) &\equiv (1/4 * s_7 - 7/576 * s_8^2) * z^6 + (3/8 * s_6 - 1/96 * s_7 * s_8) * z^5 + (1/2 * s_5 - 5/576 * \\ &s_6 * s_8) * z^4 + (5/8 * s_4 - 1/144 * s_5 * s_8) * z^3 + (3/4 * s_3 - 1/192 * s_4 * s_8) * z^2 + (7/8 * \\ &s_2 - 1/288 * s_3 * s_8) * z + (s_1 - 1/576 * s_2 * s_8)\end{aligned}$$

$$\begin{aligned}z * \varphi(z, s) &\equiv (3/8 * s_6 - 5/144 * s_7 * s_8 + 49/41472 * s_8^3) * z^6 + (1/2 * s_5 - 5/576 * s_6 * s_8 - \\ &1/48 * s_7^2 + 7/6912 * s_7 * s_8^2) * z^5 + (5/8 * s_4 - 1/144 * s_5 * s_8 - 5/288 * s_6 * s_7 + 35/41472 * \\ &s_6 * s_8^2) * z^4 + (3/4 * s_3 - 1/192 * s_4 * s_8 - 1/72 * s_5 * s_7 + 7/10368 * s_5 * s_8^2) * z^3 + (7/8 * \\ &s_2 - 1/288 * s_3 * s_8 - 1/96 * s_4 * s_7 + 7/13824 * s_4 * s_8^2) * z^2 + (s_1 - 1/576 * s_2 * s_8 - 1/144 * \\ &s_3 * s_7 + 7/20736 * s_3 * s_8^2) * z + (-1/288 * s_2 * s_7 + 7/41472 * s_2 * s_8^2)\end{aligned}$$

$$\begin{aligned}z^2 * \varphi(z, s) &\equiv (1/2 * s_5 - 13/288 * s_6 * s_8 - 1/48 * s_7^2 + 91/20736 * s_7 * s_8^2 - 343/2985984 * \\ &s_8^4) * z^6 + (5/8 * s_4 - 1/144 * s_5 * s_8 - 7/144 * s_6 * s_7 + 35/41472 * s_6 * s_8^2 + 5/1728 * s_7^2 * s_8 - \\ &49/497664 * s_7 * s_8^3) * z^5 + (3/4 * s_3 - 1/192 * s_4 * s_8 - 1/72 * s_5 * s_7 + 7/10368 * s_5 * s_8^2 - 5/192 * \\ &s_6^2 + 25/10368 * s_6 * s_7 * s_8 - 245/2985984 * s_6 * s_8^3) * z^4 + (7/8 * s_2 - 1/288 * s_3 * s_8 - 1/96 * \\ &s_4 * s_7 + 7/13824 * s_4 * s_8^2 - 1/48 * s_5 * s_6 + 5/2592 * s_5 * s_7 * s_8 - 49/746496 * s_5 * s_8^3) * z^3 +\end{aligned}$$

$$(s_1 - 1/576 * s_2 * s_8 - 1/144 * s_3 * s_7 + 7/20736 * s_3 * s_8^2 - 1/64 * s_4 * s_6 + 5/3456 * s_4 * s_7 * s_8 - 49/995328 * s_4 * s_8^3) * z^2 + (-1/288 * s_2 * s_7 + 7/41472 * s_2 * s_8^2 - 1/96 * s_3 * s_6 + 5/5184 * s_3 * s_7 * s_8 - 49/1492992 * s_3 * s_8^3) * z + (-1/192 * s_2 * s_6 + 5/10368 * s_2 * s_7 * s_8 - 49/2985984 * s_2 * s_8^3)$$

$$z^3 * \varphi(z, s) \equiv (5/8 * s_4 - 1/18 * s_5 * s_8 - 7/144 * s_6 * s_7 + 217/41472 * s_6 * s_8^2 + 17/3456 * s_7^2 * s_8 - 49/93312 * s_7 * s_8^3 + 2401/214990848 * s_8^5) * z^6 + (3/4 * s_3 - 1/192 * s_4 * s_8 - 1/18 * s_5 * s_7 + 7/10368 * s_5 * s_8^2 - 5/192 * s_6^2 + 1/162 * s_6 * s_7 * s_8 - 245/2985984 * s_6 * s_8^3 + 1/576 * s_7^3 - 91/248832 * s_7^2 * s_8^2 + 343/35831808 * s_7 * s_8^4) * z^5 + (7/8 * s_2 - 1/288 * s_3 * s_8 - 1/96 * s_4 * s_7 + 7/13824 * s_4 * s_8^2 - 1/18 * s_5 * s_6 + 5/2592 * s_5 * s_7 * s_8 - 49/746496 * s_5 * s_8^3 + 65/20736 * s_6^2 * s_8 + 5/3456 * s_6 * s_7^2 - 455/1492992 * s_6 * s_7 * s_8^2 + 1715/214990848 * s_6 * s_8^4) * z^4 + (s_1 - 1/576 * s_2 * s_8 - 1/144 * s_3 * s_7 + 7/20736 * s_3 * s_8^2 - 1/64 * s_4 * s_6 + 5/3456 * s_4 * s_7 * s_8 - 49/995328 * s_4 * s_8^3 - 1/36 * s_5^2 + 13/5184 * s_5 * s_6 * s_8 + 1/864 * s_5 * s_7^2 - 91/373248 * s_5 * s_7 * s_8^2 + 343/53747712 * s_5 * s_8^4) * z^3 + (-1/288 * s_2 * s_7 + 7/41472 * s_2 * s_8^2 - 1/96 * s_3 * s_6 + 5/5184 * s_3 * s_7 * s_8 - 49/1492992 * s_3 * s_8^3 - 1/48 * s_4 * s_5 + 13/6912 * s_4 * s_6 * s_8 + 1/1152 * s_4 * s_7^2 - 91/497664 * s_4 * s_7 * s_8^2 + 343/71663616 * s_4 * s_8^4) * z^2 + (-1/192 * s_2 * s_6 + 5/10368 * s_2 * s_7 * s_8 - 49/2985984 * s_2 * s_8^3 - 1/72 * s_3 * s_5 + 13/10368 * s_3 * s_6 * s_8 + 1/1728 * s_3 * s_7^2 - 91/746496 * s_3 * s_7 * s_8^2 + 343/107495424 * s_3 * s_8^4) * z + (-1/144 * s_2 * s_5 + 13/20736 * s_2 * s_6 * s_8 + 1/3456 * s_2 * s_7^2 - 91/1492992 * s_2 * s_7 * s_8^2 + 343/214990848 * s_2 * s_8^4)$$

$$z^4 * \varphi(z, s) \equiv (3/4 * s_3 - 19/288 * s_4 * s_8 - 1/18 * s_5 * s_7 + 7/1152 * s_5 * s_8^2 - 5/192 * s_6^2 + 113/10368 * s_6 * s_7 * s_8 - 49/82944 * s_6 * s_8^3 + 1/576 * s_7^3 - 35/41472 * s_7^2 * s_8^2 + 6517/107495424 * s_7 * s_8^4 - 16807/15479341056 * s_8^6) * z^6 + (7/8 * s_2 - 1/288 * s_3 * s_8 - 1/16 * s_4 * s_7 + 7/13824 * s_4 * s_8^2 - 1/18 * s_5 * s_6 + 17/2592 * s_5 * s_7 * s_8 - 49/746496 * s_5 * s_8^3 + 65/20736 * s_6^2 * s_8 + 19/3456 * s_6 * s_7^2 - 553/746496 * s_6 * s_7 * s_8^2 + 1715/214990848 * s_6 * s_8^4 - 17/41472 * s_7^3 * s_8 + 49/1119744 * s_7^2 * s_8^2 - 2401/2579890176 * s_7 * s_8^5) * z^5 + (s_1 - 1/576 * s_2 * s_8 - 1/144 * s_3 * s_7 + 7/20736 * s_3 * s_8^2 - 17/288 * s_4 * s_6 + 5/3456 * s_4 * s_7 * s_8 - 49/995328 * s_4 * s_8^3 - 1/36 * s_5^2 + 11/1728 * s_5 * s_6 * s_8 + 1/864 * s_5 * s_7^2 - 91/373248 * s_5 * s_7 * s_8^2 + 343/53747712 * s_5 * s_8^4 + 35/10368 * s_6^2 * s_7 - 1085/2985984 * s_6^2 * s_8^2 - 85/248832 * s_6 * s_7^2 * s_8 + 245/6718464 * s_6 * s_7 * s_8^3 - 12005/15479341056 * s_6 * s_8^5) * z^4 + (-1/288 * s_2 * s_7 + 7/41472 * s_2 * s_8^2 - 1/96 * s_3 * s_6 + 5/5184 * s_3 * s_7 * s_8 - 49/1492992 * s_3 * s_8^3 - 1/18 * s_4 * s_5 + 13/6912 * s_4 * s_6 * s_8 + 1/1152 * s_4 * s_7^2 - 91/497664 * s_4 * s_7 * s_8^2 + 343/71663616 * s_4 * s_8^4 + 1/324 * s_5^2 * s_8 + 7/2592 * s_5 * s_6 * s_7 - 217/746496 * s_5 * s_6 * s_8^2 - 17/62208 * s_5 * s_7^2 * s_8 + 49/1679616 * s_5 * s_7 * s_8^3 - 2401/3869835264 * s_5 * s_8^5) * z^3 + (-1/192 * s_2 * s_6 + 5/10368 * s_2 * s_7 * s_8 - 49/2985984 * s_2 * s_8^3 - 1/72 * s_3 * s_5 + 13/10368 * s_3 * s_6 * s_8 + 1/1728 * s_3 * s_7^2 - 91/746496 * s_3 * s_7 * s_8^2 + 343/107495424 * s_3 * s_8^4 - 5/192 * s_4^2 + 1/432 * s_4 * s_5 * s_8 + 7/3456 * s_4 * s_6 * s_7 - 217/995328 * s_4 * s_6 * s_8^2 - 17/82944 * s_4 * s_7^2 * s_8 + 49/2239488 * s_4 * s_7 * s_8^3 - 2401/5159780352 * s_4 * s_8^5) * z^2 + (-1/144 * s_2 * s_5 + 13/20736 * s_2 * s_6 * s_8 + 1/3456 * s_2 * s_7^2 - 91/1492992 * s_2 * s_7 * s_8^2 + 343/214990848 * s_2 * s_8^4 - 5/288 * s_3 * s_4 + 1/648 * s_3 * s_5 * s_8 + 7/5184 * s_3 * s_6 * s_7 - 217/1492992 * s_3 * s_6 * s_8^2 - 17/124416 * s_3 * s_7^2 * s_8 + 49/3359232 * s_3 * s_7 * s_8^3 - 2401/7739670528 * s_3 * s_8^5) * z + (-5/576 * s_2 * s_4 + 1/1296 * s_2 * s_5 * s_8 + 7/10368 * s_2 * s_6 * s_7 - 217/2985984 * s_2 * s_6 * s_8^2 - 17/248832 * s_2 * s_7^2 * s_8 + 49/6718464 * s_2 * s_7 * s_8^3 - 2401/15479341056 * s_2 * s_8^5)$$

We omit $z^5 * \varphi(z, s)$, $z^6 * \varphi(z, s)$. The vector (2.9) is given as follows

$$-72z^7 \equiv (s_1, 2s_2, 3s_3, 4s_4, 5s_5, 6s_6, 7s_7).$$

We list the rows of the matrix $\iota^*(\Sigma)(x, t)$ below. In this way we introduce 8 vector fields $w_i(x, t) \in \mathbb{C}^8$, $1 \leq i \leq 7$.

$$w_1(x, t) = (-t^2 + 1/6ax_1^2x_2 + x_2^2, 1/3ax_1(a(-t^2 + x_1^2) + 10x_2), a^2(-3t^2 + (5x_1^2)/2) - (3ax_2)/2 + x_1^2x_2, 1/3x_1(-7a^2 - 8at^2 + 8ax_1^2 + 12x_2), a^2/2 - 8at^2 + (23ax_1^2)/6 - 3x_2, -6ax_1 -$$

$$4t^2x_1 + 4x_1^3, (3a)/2 - 4t^2 - 3x_1^2, 0)$$

$$w_2(x, t) = (0, 0, 0, 1/36(-3a^3 + a^2(-52t^2 + 48x_1^2) - 48t^2x_2 - 4a(32t^4 - 8t^2x_1^2 - 24x_1^4 + 9x_2)), -(1/36)x_1(9a^2 + 296at^2 + 54ax_1^2 - 144x_2), -(a^2/4) - 4at^2 - (16t^4)/3 + (10ax_1^2)/3 + (4t^2x_1^2)/3 + 4x_1^4 - 3x_2, -(1/6)x_1(15a + 80t^2 + 18x_1^2), 0)$$

$$w_3(x, t) = (1/108ax_1^2(15a + 80t^2 + 18x_1^2)x_2, 1/54ax_1(-15a^2(t^2 - x_1^2) - 28t^2x_2 - 2a(40t^4 - 31t^2x_1^2 - 9x_1^4 + 6x_2)), 1/36(-16a^2t^4 - 21a^3x_1^2 + 30ax_1^2x_2 + 3a^2(-2x_1^4 + x_2) + 36x_2(x_1^4 + x_2) + 2t^2(-18 + 3a^3 - 38a^2x_1^2 - 4ax_2 + 80x_1^2x_2)), 1/54x_1(21a^3 - 2a^2(67t^2 - 60x_1^2) - 168t^2x_2 + a(-640t^4 + 496t^2x_1^2 + 36(4x_1^4 + 3x_2))), 1/108(-9a^3 - 3a^2(52t^2 + 77x_1^2) - 144t^2x_2 - 2a(192t^4 + 952t^2x_1^2 + 81x_1^4 + 54x_2)), 1/9x_1(9a^2 + a(-44t^2 + 30x_1^2) + 4(-40t^4 + 31t^2x_1^2 + 9(x_1^4 + x_2))), 1/36(-9a^2 - 18a(8t^2 + 5x_1^2) - 4(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), 0)$$

$$w_4(x, t) = (1/648ax_1x_2(9a^2 + 18a(8t^2 + 5x_1^2) + 4(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), -(1/648)a(18a^3(t^2 - x_1^2) + 9a^2(32t^4 - 12t^2x_1^2 - 20x_1^4 + x_2) + 4x_2(48t^4 + 148t^2x_1^2 + 27x_2) + 8a(48t^6 + 220t^4x_1^2 - 27x_1^2(x_1^4 + x_2) + t^2(-241x_1^4 + 45x_2))), 1/216x_1(-9a^4 - 6a^3(34t^2 + 5x_1^2) + 4a(44t^2 + 45x_1^2)x_2 - 2a^2(256t^4 + 412t^2x_1^2 + 18x_1^4 + 69x_2) + 8x_2(48t^4 + 268t^2x_1^2 + 27(x_1^4 + x_2))), 1/648(9a^4 + 36a^3(3t^2 - 4x_1^2) + 4a^2(-600t^4 + 142t^2x_1^2 + 360x_1^4 + 27x_2) - 24t^2(27 + 48t^2x_2 + 148x_1^2x_2) - 16a(192t^6 + 880t^4x_1^2 - 108x_1^2(x_1^4 + x_2) + t^2(-964x_1^4 + 171x_2))), -(1/648)x_1(-27a^3 + 6a^2(868t^2 + 135x_1^2) + 2016t^2x_2 + 4a(3120t^4 + 5212t^2x_1^2 + 243x_1^4 + 351x_2)), 1/216(9a^3 + 24a^2(2t^2 - 5x_1^2) - 4a(336t^4 + 40t^2x_1^2 - 9(20x_1^4 + 3x_2)) - 32(48t^6 + 220t^4x_1^2 - 27x_1^2(x_1^4 + x_2) + t^2(-241x_1^4 + 36x_2))), 1/108x_1(45a^2 - 6a(256t^2 + 45x_1^2) - 4(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2))), 0)$$

$$w_5(x, t) = ((ax_1^2x_2(-45a^2 + 6a(256t^2 + 45x_1^2) + 4(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2))))/1944, -(1/972)ax_1(-45a^3(t^2 - x_1^2) + 6a^2(256t^4 - 211t^2x_1^2 - 45x_1^4 - 6x_2) + 56t^2(24t^2 + 25x_1^2)x_2 + 4a(816t^6 + 688t^4x_1^2 - 81x_1^2(x_1^4 + x_2) + t^2(-1423x_1^4 + 219x_2))), 1/648(-9a^4(2t^2 - 7x_1^2) - 3a^3(96t^4 + 476t^2x_1^2 + 30x_1^4 + 3x_2) - 4ax_2(48t^4 - 620t^2x_1^2 + 27(-5x_1^4 + x_2)) + 8x_1^2x_2(816t^4 + 1504t^2x_1^2 + 81(x_1^4 + x_2)) - 2a^2(192t^6 + 2512t^4x_1^2 + 54x_1^6 + 99x_1^2x_2 + 4t^2(511x_1^4 + 45x_2))), 1/972x_1(-63a^4 + 30a^3(7t^2 - 12x_1^2) - 336t^2(24t^2 + 25x_1^2)x_2 - 4a^2(3240t^4 - 2357t^2x_1^2 - 540x_1^4 + 81x_2) - 8a(3264t^6 + 2752t^4x_1^2 - 324x_1^2(x_1^4 + x_2) + t^2(-5692x_1^4 + 801x_2))), (1/1944)(27a^4 + 9a^3(36t^2 + 77x_1^2) - 6a^2(1200t^4 + 6116t^2x_1^2 + 405x_1^4 - 54x_2) - 72t^2(27 + 48t^2x_2 + 148x_1^2x_2) - 4a(2304t^6 + 30960t^4x_1^2 + 729x_1^2(x_1^4 + x_2) + 4t^2(6508x_1^4 + 513x_2))), 1/162x_1(-27a^3 - 30a^2(2t^2 + 3x_1^2) - 4a(936t^4 - 458t^2x_1^2 + 27(-5x_1^4 + x_2)) - 8(816t^6 + 688t^4x_1^2 - 81x_1^2(x_1^4 + x_2) + t^2(-1423x_1^4 + 144x_2))), 1/648(27a^3 + 18a^2(8t^2 + 15x_1^2) - 12a(336t^4 + 1832t^2x_1^2 - 27(-5x_1^4 + x_2)) - 8(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2))), 0)$$

$$w_6(x, t) = ((ax_1x_2(-27a^3 - 18a^2(8t^2 + 15x_1^2) + 12a(336t^4 + 1832t^2x_1^2 - 27(-5x_1^4 + x_2)) + 8(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2))))/11664, (1/11664)a(54a^4(t^2 - x_1^2) + 9a^3(32t^4 + 28t^2x_1^2 - 60x_1^4 + 3x_2) - 32t^2x_2(144t^4 + 1476t^2x_1^2 + 781x_1^4 + 108x_2) - 24a^2(336t^6 + 1496t^4x_1^2 + 27x_1^2(-5x_1^4 + x_2) - t^2(1697x_1^4 + 33x_2)) - 4a(2304t^8 + 31104t^6x_1^2 + 16t^4(-179x_1^4 + 171x_2) + 4t^2(-7393x_1^6 + 609x_1^2x_2) - 81(12x_1^8 + 12x_1^4x_2 + x_2^2))), (1/3888)x_1(27a^5 + 18a^4(18t^2 + 5x_1^2) - 6a^3(1696t^4 + 2820t^2x_1^2 + 90x_1^4 - 69x_2) + 8ax_2(336t^4 + 4796t^2x_1^2 - 81(-5x_1^4 + x_2)) + 16x_2(576t^6 + 8352t^4x_1^2 + 243x_1^2(x_1^4 + x_2) + t^2(7636x_1^4 + 432x_2)) - 4a^2(4416t^6 + 19456t^4x_1^2 + 27x_1^2(6x_1^4 + 11x_2) + 4t^2(2395x_1^4 + 453x_2))), (1/11664)(-27a^5 - 36a^4(t^2 - 12x_1^2) - 192t^2x_2(144t^4 + 1476t^2x_1^2 + 781x_1^4 + 108x_2) + 12a^3(96t^4 - 142t^2x_1^2 - 9(40x_1^4 + 3x_2)) - 16a^2(4176t^6 + 19428t^4x_1^2 + 324x_1^2(-5x_1^4 + x_2) - t^2(19583x_1^4 + 189x_2)) - 32a(2304t^8 + 31104t^6x_1^2 - 972x_1^4(x_1^4 + x_2) + 16t^4(-179x_1^4 + 162x_2) + t^2(-29572x_1^6 + 1971x_1^2x_2))), -(1/11664)x_1(81a^4 - 90a^3(68t^2 + 27x_1^2) + 36a^2(7120t^4 + 12124t^2x_1^2 + 405x_1^4 - 117x_2) + 4032t^2(24t^2 + 25x_1^2)x_2 + 8a(53568t^6 +$$

$$241824t^4x_1^2+2187x_1^2(x_1^4+x_2)+4t^2(30649x_1^4+5103x_2))), -(a^4/144)-(256t^8)/27-128t^6x_1^2+1/54a^3(6t^2+5x_1^2)+t^4((2864x_1^4)/243-(80x_2)/9)+4x_1^4(x_1^4+x_2)-1/324a^2(96t^4+536t^2x_1^2+180x_1^4+27x_2)+t^2(-1+(29572x_1^6)/243-(64x_1^2x_2)/27)-2/243a(1152t^6+5964t^4x_1^2+81x_1^2(-5x_1^4+x_2)+2t^2(-2155x_1^4+54x_2)), -((x_1(135a^3-18a^2(16t^2+45x_1^2)+36a(2032t^4+3664t^2x_1^2-27(-5x_1^4+x_2))+8(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))))/1944), 0)$$

$$w_7(x, t) = ((ax_1^2x_2(135a^3-18a^2(16t^2+45x_1^2)+36a(2032t^4+3664t^2x_1^2-27(-5x_1^4+x_2))+8(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))))/34992, (1/17496)ax_1(-135a^4(t^2-x_1^2)+18a^3(16t^4+29t^2x_1^2-45x_1^4-6x_2)-16t^2x_2(3024t^4+10416t^2x_1^2+3367x_1^4+864x_2)-36a^2(2032t^6+1632t^4x_1^2+27x_1^2(-5x_1^4+x_2)-t^2(3529x_1^4+25x_2))-8a(13824t^8+52896t^6x_1^2-729x_1^4(x_1^4+x_2)+8t^4(-3793x_1^4+1071x_2)+t^2(-35647x_1^6+99x_1^2x_2))), (1/11664)(27a^5(2t^2-7x_1^2)+9a^4(32t^4+60t^2x_1^2+30x_1^4+3x_2)-6a^3(1344t^6+18176t^4x_1^2+270x_1^6-99x_1^2x_2-4t^2(-3799x_1^4+33x_2))-8ax_2(576t^6-12384t^4x_1^2+243x_1^2(-5x_1^4+x_2)+4t^2(-7463x_1^4+108x_2))+16x_1^2x_2(13824t^6+66720t^4x_1^2+729x_1^2(x_1^4+x_2)+8t^2(4547x_1^4+594x_2))-4a^2(2304t^8+58752t^6x_1^2+16t^4(8161x_1^4+171x_2)+4t^2(10795x_1^6+3021x_1^2x_2))-81(-6x_1^8-11x_1^4x_2+x_2^2))), (1/17496)x_1(189a^5+18a^4(13t^2+60x_1^2)-36a^3(192t^4+167t^2x_1^2+180x_1^4-27x_2)-96t^2x_2(3024t^4+10416t^2x_1^2+3367x_1^4+864x_2)-8a^2(76176t^6+69168t^4x_1^2+972x_1^2(-5x_1^4+x_2)+t^2(-123677x_1^4+621x_2))-16a(55296t^8+211584t^6x_1^2-2916x_1^4(x_1^4+x_2)+32t^4(-3793x_1^4+999x_2)-t^2(142588x_1^6+2151x_1^2x_2))), (1/34992)(-81a^5-27a^4(4t^2+77x_1^2)+18a^3(192t^4+116t^2x_1^2+405x_1^4-54x_2)-576t^2x_2(144t^4+1476t^2x_1^2+781x_1^4+108x_2)-12a^2(16704t^6+230112t^4x_1^2+t^2(196468x_1^4-756x_2)-729x_1^2(-5x_1^4+x_2))-8a(27648t^8+718848t^6x_1^2+6561x_1^4(x_1^4+x_2)+96t^4(17017x_1^4+324x_2)+4t^2(138634x_1^6+35613x_1^2x_2))), (1/2916)x_1(81a^4+18a^3(58t^2+15x_1^2)-36a^2(240t^4+254t^2x_1^2+45x_1^4-9x_2)-8a(21312t^6+25104t^4x_1^2+243x_1^2(-5x_1^4+x_2)+2t^2(-14197x_1^4+648x_2))-16(13824t^8+52896t^6x_1^2-729x_1^4(x_1^4+x_2)+8t^4(-3793x_1^4+783x_2)-t^2(35647x_1^6+2448x_1^2x_2))), 1/11664(-81a^4+162a^3(8t^2-5x_1^2)-36a^2(96t^4+424t^2x_1^2+27(-5x_1^4+x_2))-24a(4608t^6+66528t^4x_1^2-243x_1^2(-5x_1^4+x_2)+8t^2(7463x_1^4+54x_2))-16(6912t^8+190080t^6x_1^2+2187x_1^4(x_1^4+x_2)+48t^4(9551x_1^4+135x_2)+t^2(729+165916x_1^6+34992x_1^2x_2))), 0) \\ w_8(x, t) = (4ax_1x_2, 2(-4a^2t^2+4a^2x_1^2-2ax_2), 3(-4a^2x_1+8x_1x_2), 4(a^2-16at^2+16ax_1^2-6x_2), -100ax_1, 6(6a-16t^2+16x_1^2), -168x_1, 72)$$

At the focal point $(x, t) = (0, -1/2a, 1/2a)$ the matrix $\iota^*(\Sigma)(0, -1/2a, 1/2a)$ has the following form with rank 5 if $a \neq 1$ and rank 3 if $a = 1$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & (-1+a^3)/(2a) & 0 & -(1/a^2)+(3a)/2 & 0 \\ 0 & 0 & 0 & A_1 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_3 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 0 & A_4 & 0 \\ 0 & 0 & 0 & A_5 & 0 & A_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_5 & 0 & A_6 & 0 \\ 0 & 0 & 0 & 4(-1+a^3)/a & 0 & 6(-(4/a^2)+6a) & 0 & 72 \end{pmatrix} \quad (5.4)$$

where $A_1 = \frac{-(2-5a^3+3a^6)}{36a^3}$, $A_2 = \frac{-(4-6a^3+3a^6)}{12a^4}$, $A_3 = \frac{-4+10a^3-9a^6+3a^9}{216a^5}$, $A_4 = \frac{(-2+a^3)^2(-2+3a^3)}{72a^6}$, $A_5 = -\frac{(-2+a^3)^2(2-5a^3+3a^6)}{1296a^7}$, $A_6 = -\frac{16-56a^3+68a^6-30a^9+3a^{12}}{432a^8}$.

Thus together with the data

$$d_{x,t}\iota(0, -1/2a, 1/2a) \quad (5.5)$$

$$= \begin{pmatrix} 0 & -2 & 0 & -(4/a) - 4a^2 & 0 & -20a & 0 & -24 \\ 0 & -(1/a) & 0 & -2a & 0 & -6 & 0 & 0 \\ -(1/a) & 0 & -4a & 0 & -16 & 0 & -(16/a) & 0 \end{pmatrix}$$

we conclude that $\text{rank } T(0, -1/2a, 1/2a) = 8 = \nu$ if $a \neq 1$. Therefore after Proposition 4.2, the germ of the large wave front LW defines a free divisor in the neighbourhood of the focal point $(0, -1/2a, 1/2a)$ for $a \neq 1$.

In the case $a = 1$, $\text{rank } \iota^*(\Sigma)(0, -1/2, 1/2) = \text{rank } \iota^*(\tilde{\Sigma})(0, -1/2, 1/2) + 1 = 3$ and

$$\text{rank } T(0, -1/2, 1/2) = 6 < 8. \quad (5.6)$$

We see that the focal point $(0, -1/2, 1/2)$ is an isolated point after the following reasoning. The matrices above (5.4), (5.5) entail the following relationship

$$\text{span}_{\mathbb{C}}\{v_1(\iota(0, -1/2, 1/2)), \dots, v_8(\iota(0, -1/2, 1/2))\}$$

$$\cap \text{span}_{\mathbb{C}}\left\{\frac{\partial \iota}{\partial t}, \frac{\partial \iota}{\partial x_1}, \frac{\partial \iota}{\partial x_2}\right\}_{(0, -1/2, 1/2)} = \{0\}.$$

This means that the germ of the integral varieties of the vector fields $\{v_1(s), \dots, v_8(s)\}$ (i.e. the stratum of A_5 singularities of the discriminantal loci $D_{\varphi, \iota(0, -1/2, 1/2)}$) and the image $\iota(\mathbb{C}^3)$ intersect transversally at $\iota(0, -1/2, 1/2)$. In addition to that we can verify that the limit of tangent vectors to the stratum of A_4 singularities adjacent to A_5 stratum near $\iota(0, -1/2, 1/2)$ generated by the rows of the following matrix

$$\lim_{s_5 \rightarrow 0} \frac{\Sigma(\iota(0, -1/2, 1/2) + (0, 0, 0, 0, s_5, 0, 0)) - \iota^*(\Sigma)(0, -1/2, 1/2)}{s_5}$$

$$= \frac{\partial \Sigma(\iota(0, -1/2, 1/2) + (0, 0, 0, 0, s_5, 0, 0))}{\partial s_5} \Big|_{s_5=0}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3/8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(5/144) & 0 & 3/8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(7/72) & 0 & 0 \\ 0 & 0 & 0 & 0 & 5/864 & 0 & -(7/72) & 0 \\ 0 & 0 & 0 & 0 & 0 & 19/864 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(5/5184) & 0 & 19/864 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1/216) & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 72 \end{pmatrix}$$

are linearly independent of a vector from $\text{span}_{\mathbb{C}}\left\{\frac{\partial \iota}{\partial t}, \frac{\partial \iota}{\partial x_1}, \frac{\partial \iota}{\partial x_2}\right\}_{(0, -1/2, 1/2)}$.

This means that $(0, -1/2, 1/2)$ is an isolated point on $LW \subset \iota^{-1}(D_\varphi)$ with the property (5.6). Upshot is the almost freeness of the large wave front germ at the focal point after Proposition 4.3.

In summary we established

Proposition 5.1. *The germ of the large wave front LW at the focal point $(x_1, x_2, t) = (0, -1/2a, 1/2a)$ defines a free divisor if $a \neq 1$. If $a = 1$ it defines an almost free divisor germ at the focal point $(x_1, x_2, t) = (0, -1/2, 1/2)$.*

2. Wave propagation in the 3 dimensional space

Now we consider the following initial wave front in the 3-dimensional space, $Y := \{(z, u) \in \mathbb{C}^2 : -\frac{1}{2}(k_1 z_1^2 + k_2 z_2^2) + u = 0\}$, i.e. $F(z) = -\frac{1}{2}(k_1 z_1^2 + k_2 z_2^2)$ for $0 < k_1 < k_2$. In this case our phase function has the following expression

$$\begin{aligned} \Psi(x, t, z) &= (-x_3 + k_1(x_1 - z_1)z_1 + k_2(x_2 - z_2)z_2 + 1/2(k_1 z_1^2 + k_2 z_2^2))^2 - t^2(1 + k_1^2 z_1^2 + k_2^2 z_2^2), \\ &= -t^2 + x_3^2 - k_1^2 x_1 z_1^3 + (k_1^2 z_1^4)/4 - 2k_2 x_3(x_2 - z_2)z_2 \\ &\quad - k_2^2 t^2 z_2^2 - k_2 x_3 z_2^2 + k_2^2 (x_2 - z_2)^2 z_2^2 + k_2^2 (x_2 - z_2)z_2^3 + (k_2^2 z_2^4)/4 \\ &\quad + z_1^2(-k_1^2 t^2 + k_1^2 x_1^2 + k_1 x_3 - k_1 k_2 (x_2 - z_2)z_2 - 1/2 k_1 k_2 z_2^2) \\ &\quad + z_1(-2k_1 x_1 x_3 + 2k_1 k_2 x_1(x_2 - z_2)z_2 + k_1 k_2 x_1 z_2^2) \end{aligned} \quad (5.7)$$

It is easy to see that the point $(x_1, x_2, x_3, t) = (0, 0, 1/k_1, 1/k_1)$ is a focal point with a singular point $(z, u) = (0, 0)$ and the Milnor number $\mu(0) = 3$. We have the following tame polynomial,

$$\Psi(0, 0, 1/k_1, 1/k_1, z) = (k_1^4 z_1^4 + 4k_1 k_2 z_2^2 - 4k_2^2 z_2^2 + 2k_1^3 k_2 z_1^2 z_2^2 + k_1^2 k_2^2 z_2^4)/4k_1^2.$$

As a matter of fact, the polynomial $\Psi(0, 0, 1/k_1, 1/k_1, z)$ satisfies the criterion on the presence of A_3 singularity at the origin mentioned in [7], Theorem 2.2, (2). The situation is the same at another focal point $(x_1, x_2, x_3, t) = (0, 0, 1/k_2, 1/k_2)$. The quotient ring (1.5) for this $\Psi(0, 0, 1/k_1, 1/k_1, z)$ has dimension $\mu = 5$.

We can choose

$$\{e_1, e_2, e_3, e_4, e_5\} = \{1, z_1, z_1^2, z_2, z_2^2\}$$

as the basis (2.7). In view of (5.7), we introduce additional deformation monomials $e_6 = z_1 * z_2, e_7 = z_2^3, e_8 = z_1^3, e_9 = z_1^2 * z_2, e_{10} = z_1 * z_2^2$ together with the entries of the mapping ι ,

$$\begin{aligned} s_1 &= -t^2 + x_3^2, s_2 = -2k_1 x_1 x_3, s_3 = -k_1^2 t^2 + k_1^2 x_1^2 + k_1 x_3, s_4 = -2k_2 x_2 x_3 \\ s_5 &= -(k_2/k_1) + k_2^2/k_1^2 - k_2^2 t^2 + k_2^2 x_2^2 + k_2 x_3, s_6 = 2k_1 k_2 x_1 x_2 \\ s_7 &= -k_2^2 x_2, s_8 = -k_1^2 x_1, s_9 = -k_1 k_2 x_2, s_{10} = -k_1 k_2 x_1. \end{aligned}$$

It turns out that the image of the mapping $\iota(\mathbb{C}^4) \subset \mathbb{C}^{10}$ is contained in a constructible set $\mathbb{C} \times U$ where the value of the matrix $\Sigma(s)$ is well-defined at each point $s \in \mathbb{C} \times U$. Therefore

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(x, t, z))\mathbb{C}[z]} = 5,$$

for every $(x, t) \in \mathbb{C}^4$. This means that the Assumption I,(ii) is satisfied. By direct calculation with the aid of SINGULAR, we can verify that $\dim U = 5$. This can be seen from the fact that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(0, 0, 1/k_1, 1/k_1, z) + \sum_{i=1}^6 s_i e_i) \mathbb{C}[z]} = 5,$$

while

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[z]}{d_z(\Psi(0, 0, 1/k_1, 1/k_1, z) + \sum_{i=1}^6 s_i e_i + s_j e_j) \mathbb{C}[z]} = 7,$$

for $j = 7, 8, 9, 10$. This implies that the Assumption I,(i) is satisfied with $\nu = 6$.

At the focal point $(x_1, x_2, x_3, t) = (0, 0, 1/k_1, 1/k_1)$ the matrix $\iota^*(\Sigma)$ has the following form with rank 3

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -k_2(k_1 - k_2)/2k_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (k_1 - k_2)^2/k_1^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & (k_1 - k_2)^2/k_1^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Together with the data

$$d_{x,t}\iota(0, 0, 1/k_1, 1/k_1) = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 & 0 & 0 & -k_1^2 & 0 & -k_1 k_2 \\ 0 & 0 & 0 & -2k_2/k_1 & 0 & 0 & -k_2^2 & 0 & -k_1 k_2 & 0 \\ 2/k_1 & 0 & k_1 & 0 & k_2 & 0 & 0 & 0 & 0 & 0 \\ -2/k_1 & 0 & -2k_1 & 0 & -2k_2^2/k_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

we see that the $\text{rank } T(0, 0, 1/k_1, 1/k_1) = 7 \geq \nu$. By virtue of the Proposition 4.3, we see that the wave front defines a free divisor germ in the neighbourhood of the focal point $(0, 0, 1/k_1, 1/k_1)$.

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