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Saturation of the approximation by spectral decompositions

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1 Introduction.

Let Ω be an open domain in the n dimensional Euclidean space $\boldsymbol{R}^n$. Consider the operator $A = -\Delta$ in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$, where $\Delta = \partial^2/\partial x_1{}^2 + \cdots + \partial^2/\partial x_n{}^2$ is the Laplacian. Denote by $\hat{A}$ a nonnegative selfadjoint extension of A. Let $\{ k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0,\infty)$. Suppose we have two constants $\kappa_1, \kappa_2 > 0$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0,\infty)$, $(k_\lambda(t)-1)/\lambda^{-\kappa_1} t^{\kappa_2}$ are uniformly bounded in λ and $t \in [0,\infty)$, and $(k_\lambda(t)-1)/\lambda^{-\kappa_1} t^{\kappa_2}$ converge to a nonzero constant as $\lambda \to \infty$ for any $t \in [0,\infty)$. Let

$$I_\lambda(r) = \int_0^\infty k_\lambda\left(t^2\right) J_\nu\left(rt\right) t^{\nu+1} dt,$$

where $\nu = n/2 - 2\kappa_2 + 1$ and J_ν is the Bessel function of order ν. We assume, furthermore, the following conditions

$$\int_0^R s^{2\kappa_2-1} ds \left| \int_s^R r^{n/2-2\kappa_2+2} I_\lambda(r)\, dr \right| = O\left(\lambda^{-\kappa_1}\right), \tag{1.1}$$

$$\left| \int_R^\infty r^{\nu+1} I_\lambda(r)\, dr \right| = o\left(\lambda^{-\kappa_1}\right), \tag{1.2}$$

and

$$\left(\sum_{T=0}^\infty T^{4\kappa_2-3} \max_{T\le s\le T+1} \left| \int_R^\infty J_\nu(sr)\, I_\lambda(r)\, r\, dr \right|^2 \right)^{1/2} = o\left(\lambda^{-\kappa_1}\right) \tag{1.3}$$

as $\lambda \to \infty$ for any small $R > 0$.

We shall consider the approximation operator $k_\lambda\left(\hat{A}\right)$ for $f \in L^2(\Omega)$. We say $\Delta f \in L^\infty_{\mathrm{loc}}(\Omega)$ if for every compact set K in Ω there is a constant C_K such that

$$\left|\int_K f(x)\,\Delta g(x)\,dx\right| \le C_K \|g\|_{L^1(K)}$$

for any infinitely differentiable function g whose support is contained in K. Let $\{\varphi_\varepsilon\}$ be an infinitely differentiable approximate identity with supports contained in $\{x\,;\,|x| < \varepsilon\}$. For a function f on Ω and $x \in \Omega$, f is said to be regulated at x if $f * \varphi_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$.

In 1970, Igari proved the following Theorem in [5].

Theorem A. *Suppose that there exist a complete orthonormal system $\{u_j\}$ of smooth functions in $L^2(\Omega)$ and a numerical sequence $\{\lambda_j\}$ for which $-\Delta u_j = \lambda_j u_j$ in Ω. Let*

$$f_j = \int_\Omega f(x)\,\overline{u_j(x)}\,dx, \qquad f \in L^2(\Omega)$$

and

$$s_\lambda^\delta f = \sum_{\lambda_j \le \lambda} \left(1 - \frac{\lambda_j}{\lambda}\right)^\delta f_j\, u_j, \qquad f \in L^2(\Omega).$$

Let $\delta \ge (n+3)/2$ and $f \in L^2(\Omega)$ be regulated in Ω. Then the following hold.

(i) *The following conditions are equivalent.*

(ia)

$$\left\| s_\lambda^\delta f - f \right\|_{L^\infty(K)} = O\left(\lambda^{-1}\right)$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^\infty_{\mathrm{loc}}(\Omega)$.

(ii) *The following conditions are equivalent.*

(ii a)

$$\left\| s_\lambda^\delta f - f \right\|_{L^\infty(K)} = o\left(\lambda^{-1}\right)$$

as $\lambda \to \infty$ *for every compact set* K *in* Ω.

(ii b) Δf *vanishes in* Ω.

Our aim is to give a generalization of Theorem A. Let $\{ k_\lambda(t)\}$ be a family of bounded Borel functions on $[0,\infty)$. We can define the bounded operator $k_\lambda\left(\hat{A}\right)$ in $L^2(\Omega)$.

Example 1. Suppose that there exist a complete orthonormal system $\{u_j\}$ of smooth functions in $L^2(\Omega)$ and a sequence $\{\lambda_j\}$ such that $-\Delta u_j = \lambda_j u_j$ in Ω. Let

$$f_j = \int_\Omega f(x)\,\overline{u_j(x)}\,dx, \qquad f \in L^2(\Omega).$$

Let $\hat{A}$ be the selfadjoint extension of $-\Delta$ defined by

$$D\left(\hat{A}\right) = \left\{ f \in L^2(\Omega)\,;\ \sum_{j=1}^{\infty} \lambda_j^2 \,|\, f_j\,|^2 < \infty \right\}$$

and

$$\hat{A} f = \sum_{j=1}^{\infty} \lambda_j f_j u_j, \qquad f \in D\left(\hat{A}\right).$$

For any $f \in L^2(\Omega)$ the spectral decomposition of f is given by

$$E((-\infty,t])f = \sum_{\lambda_j \le t} f_j u_j$$

and $k_\lambda\left(\hat{A}\right)$ is defined by

$$k_\lambda\left(\hat{A}\right) f = \sum_{j=1}^{\infty} k_\lambda(\lambda_j)\, f_j u_j, \qquad f \in L^2(\Omega).$$

Example 2. Let $\Omega = \boldsymbol{R}^n$. Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} f(x)\, e^{-i\xi\cdot x}\, dx, \qquad f \in L^2(\boldsymbol{R}^n).$$

In this case, there is a unique nonnegative selfadjoint extension $\hat{A}$ of $-\Delta$ defined by

$$D\left(\hat{A}\right) = \left\{ f \in L^2(\boldsymbol{R}^n);\ |\xi|^2 \hat{f}(\xi) \in L^2(\boldsymbol{R}^n) \right\}$$

and

$$\hat{A} f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} |\xi|^2 \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi, \qquad f \in D\left(\hat{A}\right).$$

Then the spectral decomposition of $f \in L^2(\boldsymbol{R}^n)$ is given by

$$E((-\infty, t])\, f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{|\xi|^2 \le t} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$

and $k_\lambda\left(\hat{A}\right)$ is defined by

$$k_\lambda\left(\hat{A}\right) f(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\boldsymbol{R}^n} k_\lambda\left(|\xi|^2\right) \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi, \qquad f \in L^2(\boldsymbol{R}^n).$$

For $\kappa_2 > 0$ and $1 < p \le \infty$, we say $(-\Delta)^{\kappa_2} f$ belongs to $L^p_{\mathrm{loc}}(\Omega)$ if for every bounded open set G in Ω with the closure $\overline{G}$ contained in Ω, there is a constant C_G such that

$$\left| \int_{\overline{G}} f(x)\, (-\Delta)^{\kappa_2} g(x)\, dx \right| \le C_G \|g\|_{L^{p'}(\overline{G})}$$

for any infinitely differentiable function g with support contained in $\overline{G}$, where $1/p + 1/p' = 1$.

Our results are stated as follows.

Main theorem. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Let $\{k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0,\infty)$ and $\kappa_1, \kappa_2 > 0$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0,\infty)$, $\lambda^{\kappa_1} t^{-\kappa_2}[k_\lambda(t)-1]$ are uniformly bounded in λ and $t \in [0,\infty)$, and $\lambda^{\kappa_1} t^{-\kappa_2}[k_\lambda(t)-1]$ converge to a nonzero constant as $\lambda \to \infty$ for any $t \in [0,\infty)$.*

Suppose that $\{k_\lambda(t)\}$ satisfies the conditions (1.1), (1.2) and (1.3) *as $\lambda \to \infty$. Let f be a regulated function in $L^2(\Omega)$. Furthermore, suppose that $1 < p \leq \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following two conditions are equivalent.*

(ia)

$$\left\| k_\lambda(\hat{A})\, f - f \right\|_{L^p(K)} = O\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $(-\Delta)^{\kappa_2} f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(iia) *Suppose that $(-\Delta)^{\kappa_2} f$ vanishes in G. Then*

$$\left\| k_\lambda(\hat{A})\, f - f \right\|_{L^p(K)} = o\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$ for any compact set $K \subset G$.

(iib) *If*

$$\left\| k_\lambda(\hat{A})\, f - f \right\|_{L^p(K)} = o\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$ for any compact set $K \subset G$, then $(-\Delta)^{\kappa_2} f$ vanishes in G.

If $\delta > (n+3)/2$ and $k_\lambda(t) = \left(1 - t/\lambda^2\right)_+^\delta$, then the conditions (1.1), (1.2) and (1.3) are satisfied. Therefore we have the following:

Corollary 1. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Let $s_\lambda^\delta = \left(1 - \hat{A}/\lambda^2\right)_+^\delta$ and $\delta > (n+3)/2$. Let f be a regulated function in $L^2(\Omega)$. Suppose that $1 < p \leq \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K)} = O\left(\lambda^{-2}\right)$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(iia) *Suppose that Δf vanishes in G. Then*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K)} = o\left(\lambda^{-2}\right)$$

as $\lambda \to \infty$ for any compact set $K \subset G$.

(iib) *If*

$$\left\| s_\lambda^\delta f - f \right\|_{L^p(K)} = o\left(\lambda^{-2}\right)$$

as $\lambda \to \infty$ for any compact set $K \subset G$, then Δf vanishes in G.

Our main theorem follows from Theorem 1 in §2 and Theorem 2 in §3. Corollary 1 is proved in §4.

2 Saturation of the approximation.

Let Ω be an open domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Let

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \tag{2.1}$$

be a differential operator, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $D^\alpha = (-i)^{|\alpha|} (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$ and $a_\alpha \in C^\infty(\Omega)$. We consider A as an operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Suppose that A is formally selfadjoint and semibounded. If $\hat{A}$ is a selfadjoint extension of A with the same lower bound c, then $\hat{A}$ can be represented in the form of

$$\hat{A} = \int_c^\infty t\, E(dt).$$

Let $\{k_\lambda(t)\}$ be a family of bounded Borel functions on $[c, \infty)$, $\kappa_1, \kappa_2 > 0$ and

$$\psi_\lambda(t) := \frac{k_\lambda(t) - 1}{\lambda^{-\kappa_1} t^{\kappa_2}}. \tag{2.2}$$

Suppose that

(1) $\psi_\lambda(t)$ are uniformly bounded in λ and $t \in [c, \infty)$, and

(2) $\psi_\lambda(t)$ converge to a nonzero constant C as $\lambda \to \infty$ for any $t \in [c, \infty)$.

Lemma 1. *If $f \in L^2(\Omega)$ and $g \in D\left(\hat{A}^{\kappa_2}\right)$, then $\lambda^{\kappa_1}\left(k_\lambda\left(\hat{A}\right) f - f, g\right) \longrightarrow C\left(f, \hat{A}^{\kappa_2} g\right)$ as $\lambda \to \infty$.*

Proof. By the definition of $k_\lambda\left(\hat{A}\right)$, we have

$$\begin{aligned}
\lambda^{\kappa_1}\left(k_\lambda\left(\hat{A}\right)f - f, g\right) &= \lambda^{\kappa_1}\int_c^\infty [k_\lambda(t)-1]\,(E(dt)f,g) \\
&= \int_c^\infty \frac{k_\lambda(t)-1}{\lambda^{-\kappa_1}}\,(f,E(dt)\,g) \\
&= \int_c^\infty \frac{k_\lambda(t)-1}{\lambda^{-\kappa_1}\,t^{\kappa_2}}\,t^{\kappa_2}\,(f,E(dt)g) \\
&= \int_c^\infty \psi_\lambda(t)\,t^{\kappa_2}(f,E(dt)g) \\
&= \left(f,\overline{\psi_\lambda}\left(\hat{A}\right)\hat{A}^{\kappa_2}g\right) \\
&= \int_c^\infty \psi_\lambda(t)\left(f,E(dt)\,\hat{A}^{\kappa_2}g\right).
\end{aligned}$$

Let $\rho = \left(f,E(\cdot)\,\hat{A}^{\kappa_2}g\right)$ and $|\rho|$ be the total variation of ρ. Then

$$\int_c^\infty |\rho|(dt) \le ||f||_{L^2(\Omega)}\left\|\hat{A}^{\kappa_2}g\right\|_{L^2(\Omega)} < \infty.$$

Therefore, by Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned}
\lim_{\lambda\to\infty}\lambda^{\kappa_1}\left(k_\lambda\left(\hat{A}\right)f - f, g\right) &= \lim_{\lambda\to\infty}\int_c^\infty \psi_\lambda(t)\,\rho(dt) \\
&= \int_c^\infty \lim_{\lambda\to\infty}\psi_\lambda(t)\,\rho(dt) = C\int_c^\infty \rho(dt) = C\left(f,\hat{A}^{\kappa_2}g\right).
\end{aligned}$$

Thus Lemma 1 is proved.

Let G be an open subset in Ω with compact closure $\overline{G}$ and $1 < p \le \infty$. We say $A^{\kappa_2}f \in L^p\left(\overline{G}\right)$ if

$$\|A^{\kappa_2}f\|_{L^p(\overline{G})} := \sup_{\|g\|_{L^{p'}\left(\overline{G}\right)}=1}\left|\int_\Omega f(x)\,\overline{\hat{A}^{\kappa_2}g\,(x)}\,dx\right| < \infty,$$

where $1/p + 1/p' = 1$ and g is an infinitely differentiable function whose support is contained in $\overline{G}$.

Theorem 1. *Let Ω be an open domain in $\boldsymbol{R}^n$ and A be a formally selfadjoint semi-bounded differential operator with coefficients in $C^\infty(\Omega)$ given by (2.1). Suppose that $\hat{A}$ is a selfadjoint extension of A with the same lower bound c. Let $\{k_\lambda(t)\}$ be a family of bounded Borel functions on $[c,\infty)$ and $\kappa_1, \kappa_2 > 0$ such that the sequence $\{\psi_\lambda(t)\}$ of Borel functions on $[c,\infty)$ given by (2.2) satisfies (1) and (2). Let $f \in L^2(\Omega)$, $1 < p \leq \infty$ and G be any open set in Ω with compact closure $\overline{G}$. Then the following hold.*

(i) *If*

$$\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p\left(\overline{G}\right)} = O\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$, then $A^{\kappa_2} f \in L^p\left(\overline{G}\right)$.

(ii) *If*

$$\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p\left(\overline{G}\right)} = o\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$, then $A^{\kappa_2} f$ vanishes in $\overline{G}$.

Proof. Let g be an infinitely differentiable function and $\operatorname{supp} g$ be the support of g. Suppose that $\operatorname{supp} g \subset \overline{G}$. Then by Lemma 1

$$\lambda^{\kappa_1}\left(k_\lambda\left(\hat{A}\right) f - f, g\right) \longrightarrow C\left(f, \hat{A}^{\kappa_2} g\right) \qquad \text{as} \quad \lambda \to \infty. \tag{2.3}$$

On the other hand, we have

$$\left| \lambda^{\kappa_1}\left(k_\lambda\left(\hat{A}\right) f - f, g\right)\right| \leq \lambda^{\kappa_1} \left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p\left(\overline{G}\right)} \| g \|_{L^{p'}\left(\overline{G}\right)}. \tag{2.4}$$

If $\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p\left(\overline{G}\right)} = O\left(\lambda^{-\kappa_1}\right)$ as $\lambda \to \infty$, then by (2.4) for any λ

$$\left| \lambda^{\kappa_1} \left(k_\lambda\left(\hat{A}\right) f - f, g \right) \right| \le C' \| g \|_{L^{p'}\left(\overline{G}\right)}$$

with some constant $C' > 0$. Therefore, by (2.3), we have

$$\left| \int_\Omega f(x)\, \overline{\hat{A}^{\kappa_2} g\,(x)}\, dx \right| = \left| \left(f, \hat{A}^{\kappa_2} g \right) \right| \le C^{-1} C' \| g \|_{L^{p'}\left(\overline{G}\right)}$$

for any g. Thus (i) is proved.

If $\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p\left(\overline{G}\right)} = o\left(\lambda^{-\kappa_1}\right)$ as $\lambda \to \infty$, then in the same way as in (i), (ii) is proved.

Examples. (1) Riesz summation: For $\kappa > 0$ and $\delta > 0$, the Riesz summation is given by the multiplier $k_\lambda(t) = \left[1 - (t/\lambda^2)^\kappa\right]_+^\delta$. In this case, $(\lambda^2/t)^\kappa\,[k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [c, \infty)$ with a constant $c > 0$ and

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2} t)^\kappa} = \lim_{s\to+0} \frac{(1 - s^\kappa)^\delta - 1}{s^\kappa} = -\lim_{s\to+0} \delta\,(1 - s)^{\delta - 1} = -\delta$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 2\kappa$, $\kappa_2 = \kappa$ and $C = -\delta$, where C is a constant in (2).

(2) Fejér-Korovkin summation is defined by

$$k_\lambda(t) = \begin{cases} \left(1 - \dfrac{t}{\lambda^2}\right) \cos \dfrac{\pi t}{\lambda^2} + \dfrac{1}{\lambda^2} \cot \dfrac{\pi}{\lambda^2} \sin \dfrac{\pi t}{\lambda^2} & t < \lambda^2, \\ 0 & t \ge \lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2\,[k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2} t)^2} = \lim_{s\to+0} \frac{\cos \pi s - 1}{s^2} = \lim_{s\to+0} \frac{\cos^2 \pi s - 1}{s^2 (\cos \pi s + 1)} = -\lim_{s\to+0} \frac{\sin^2 \pi s}{s^2 (\cos \pi s + 1)} = -\frac{\pi^2}{2}$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -\pi^2/2$.

(3) Rogosinski summation is given by

$$k_\lambda(t) = \begin{cases} \cos\dfrac{\pi t}{2\lambda^2} & t < \lambda^2, \\ 0 & t \geq \lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2[k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t) - 1}{(\lambda^{-2}t)^2} = \lim_{s\to+0} \frac{\cos\frac{\pi}{2}s - 1}{s^2} = -\lim_{s\to+0} \frac{\sin^2\frac{\pi}{2}s}{s^2\left(\cos\frac{\pi}{2}s + 1\right)} = -\left(\frac{\pi}{2}\right)^2 \cdot \frac{1}{2} = -\frac{\pi^2}{8}$$

for any $t \in [c, \infty)$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -\pi^2/8$.

(4) Jackson summation is given by

$$k_\lambda(t) = \begin{cases} 1 - \dfrac{3}{2}\left(\dfrac{t}{\lambda^2}\right)^2 + \dfrac{3}{4}\left(\dfrac{t}{\lambda^2}\right)^3 & t < \lambda^2, \\ \dfrac{1}{4}\left(2 - \dfrac{t}{\lambda^2}\right)^3 & \lambda^2 \leq t < 2\lambda^2, \\ 0 & t \geq 2\lambda^2. \end{cases}$$

In this case, $(\lambda^2/t)^2[k_\lambda(t) - 1]$ are uniformly bounded in λ and $t \in [c, \infty)$ and $\lim_{\lambda\to\infty} (\lambda^2/t)^2[k_\lambda(t) - 1] = -3/2$. Thus $\kappa_1 = 4$, $\kappa_2 = 2$ and $C = -3/2$.

(5) Gauss-Weierstrass summation: We consider the multiplier $k_\lambda^W(t) = \exp(-t/\lambda)$. The function of t $(\lambda/t)[k_\lambda(t) - 1]$ is bounded uniformly in λ, and we have

$$\lim_{\lambda\to\infty} \frac{k_\lambda(t) - 1}{\lambda^{-1}t} = \lim_{s\to+0} \frac{e^{-s} - 1}{s} = -\lim_{s\to+0} e^{-s} = -1.$$

Thus $\kappa_1 = \kappa_2 = 1$ and $C = -1$. Poisson summation is given by the function $k_\lambda^P(t) = \exp\left(-\sqrt{t}/\lambda\right)$, and we have $\kappa_1 = 1$ and $\kappa_2 = 1/2$.

3 Estimates of $k_\lambda(\hat{A})f - f$.

The aim of this section is to prove the following theorem.

Theorem 2. *Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω. Suppose that K is a compact set in Ω and K' is a closed subset of K with* $\operatorname{dist}(K', K^c) > 0$. *Let $\{k_\lambda(t)\}$ be a family of bounded piecewise smooth functions on $[0,\infty)$ such that $k_\lambda(t)\sqrt{t}^{\,n/2-2\kappa_2+1/2} \in L^1(0,\infty)$ with a constant $\kappa_2 > 0$ and $k_\lambda(0)=1$ for any λ.*

Suppose that $\{k_\lambda(t)\}$ satisfies the conditions (1.1), (1.2) *and* (1.3) *with a constant $\kappa_1 > 0$ and $0 < R < \operatorname{dist}(K', K^c) > 0$. Let f be a regulated function in $L^2(\Omega)$. Suppose that $1 < p \le \infty$ and $f \in L^p(K)$. Then the following hold.*

(i) *If $(-\Delta)^{\kappa_2} f \in L^p(K)$, then*

$$\left\| k_\lambda(\hat{A})\, f - f \right\|_{L^p(K')} = O\left(\lambda^{-\kappa_1}\right) \qquad as \quad \lambda \to \infty.$$

(ii) *If $(-\Delta)^{\kappa_2} f$ vanishes in K, then*

$$\left\| k_\lambda(\hat{A})\, f - f \right\|_{L^p(K')} = o\left(\lambda^{-\kappa_1}\right) \qquad as \quad \lambda \to \infty.$$

3.1 Generalized eigenfunction system.

In order to prove Theorem 2, we shall use the generalized eigenfunction system corresponding to an ordered representation of $L^2(\Omega)$ associated with the Laplace operator.

We shall begin with several definitions. We consider $A = -\Delta$ as an operator in $L^2(\Omega)$ with the domain of definition $D(A) = C_c^\infty(\Omega)$. Let $\hat{A}$ be a nonnegative selfadjoint extension of A. Let $\mathfrak{B}$ be the Borel field on $\boldsymbol{R}$ and E be the unique spectral measure corresponding to $\hat{A}$. For $h \in L^2(\Omega)$, we define the following closed subspace of $L^2(\Omega)$:

$$\begin{aligned} H(h) &:= \left\{ F\left(\hat{A}\right) h;\ F \text{ is a Borel function on } \boldsymbol{R} \text{ and } h \in D\left(F\left(\hat{A}\right)\right) \right\} \\ &= \left\{ F\left(\hat{A}\right) h;\ F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)\, h, h)) \right\}. \end{aligned}$$

If $f \in H(h)$, then we can write uniquely $f = F\left(\hat{A}\right) h$, where $F \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ and

$$||f||_{L^2(\Omega)} = \left(\int_{\boldsymbol{R}} |F(t)|^2 (E(dt)\, h, h) \right)^{1/2}.$$

Therefore we can define an isomorphism U_h from $H(h)$ onto $L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot)h, h))$ by $U_h f := F$, which preserves inner products.

There exist a sequence of functions $\{h_j\} \subset L^2(\Omega)$ and a sequence of sets $\{e_j\} \subset \mathfrak{B}$, called the set of multiplicity, with the following properties (see [3, XII.3.16] or [4, Chap.14]):

(I)

$$L^2(\Omega) = \bigoplus_j H(h_j).$$

That is, $H(h_j)$ are mutually orthogonal and span $L^2(\Omega)$.

(II) $\quad \boldsymbol{R} = e_1 \supseteq e_2 \supseteq \cdots.$

(III) $\quad (E(e)\, h_j, h_j) = (E(e \cap e_j)\, h_1, h_1) \quad$ for any $\; e \in \mathfrak{B}$.

By (I), for $f \in L^2(\Omega)$ we can write uniquely

$$f = \sum_j F_j\left(\hat{A}\right) h_j,$$

where $F_j \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot) h_j, h_j))$ and

$$\left(\sum_j \int_{\boldsymbol{R}} |F_j(t)|^2 \, (E(dt) h_j, h_j)\right)^{1/2} = \left(\sum_j \left\| F_j\left(\hat{A}\right) h_j \right\|^2_{L^2(\Omega)}\right)^{1/2} = \|f\|_{L^2(\Omega)} < \infty.$$

Therefore we can define an isometry U from $L^2(\Omega)$ onto $\bigoplus_j L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot) h_j, h_j))$, which is equivalent to say

$$L^2(\Omega) \leftrightarrow \left\{ \{F_j\}\,; F_j \in L^2(\boldsymbol{R}, \mathfrak{B}, (E(\cdot) h_j, h_j)) \text{ and } \sum_j \int_{\boldsymbol{R}} |F_j(t)|^2 \, (E(dt) h_j, h_j) < \infty \right\},$$

and the correspondence is given by $Uf := \{F_j\}$. We denote $F_j =: (Uf)_j$.

By (III) we have

$$\bigoplus_j L^2(\boldsymbol{R}, (E(\cdot) h_j, h_j)) = \bigoplus_j L^2(e_j, (E(\cdot) h_1, h_1)).$$

Let $\rho := (E(\cdot) h_1, h_1)$. Then U is an isomorphism from $L^2(\Omega)$ onto $\bigoplus_j L^2(e_j, \rho)$ which preserves inner products, that is, for any $f, g \in L^2(\Omega)$ it holds that

$$(f, g)_{L^2(\Omega)} = \sum_j \int_{e_j} (Uf)_j(t) \, \overline{(Ug)_j(t)} \, \rho(dt). \tag{3.1}$$

U is called an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$.

With these understood, there exists a sequence of functions $\{u_j(x, t)\}$ defined on the product space of $\Omega \times \boldsymbol{R}$ such that the following conditions are satisfied (see [3, XII.3 and XIV.6] or [4, Chap.15]):

(i) The functions $u_j(x, t)$ are $dx \times d\rho(t)$-measurable and vanish outside $\Omega \times e_j$, where dx is the Lebesgue measure.

(ii) For any fixed $t \in \boldsymbol{R}$, each $u_j(x,t)$ belongs $C^\infty(\Omega)$ and satisfies

(3.2) $$-\Delta u_j(x,t) = t\, u_j(x,t), \qquad x \in \Omega.$$

(iii) For each compact subset K of Ω and each bounded Borel set e in $\boldsymbol{R}$

$$\operatorname*{ess\,sup}_{x \in K} \int_e |u_j(x,t)|^2 \rho(dt) < \infty.$$

(iv) For each $f \in L^2(\Omega)$

(3.3) $$(Uf)_j(t) = \int_\Omega f(x)\, \overline{u_j(x,t)}\, dx,$$

where the integral exists in the sense of $L^2(e_j, \rho)$.

(v) For each $f \in L^2(\Omega)$ and each $e \in \mathfrak{B}$

(3.4) $$E(e)\, f(x) = \sum_j \int_e (Uf)_j(t)\, u_j(x,t)\, \rho(dt),$$

where the integral exists and the series converges in the sense of $L^2(\Omega)$.

$\{u_j\}$ is called the generalized eigenfunction system of $\hat{A}$ corresponding to U. By (v), for $f \in L^2(\Omega)$ we have

(3.5) $$f(x) = \sum_j \int_{\boldsymbol{R}} (Uf)_j(t)\, u_j(x,t)\, \rho(dt)$$

and

(3.6) $$k_\lambda(\hat{A})\, f(x) = \sum_j \int_{\boldsymbol{R}} k_\lambda(t)\, (Uf)_j(t)\, u_j(x,t)\, \rho(dt).$$

3.2 Decomposition of $k_\lambda\left(\hat{A}\right) f - f$.

Throughout what follows, Ω denotes an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ is a nonnegative selfadjoint extension of $-\Delta$. Let U denote an ordered representation of $L^2(\Omega)$ with respect to $\hat{A}$, $\{u_j\}$ the generalized eigenfunction system and ρ the measure associated with U. We denote the gamma function by Γ, the unit sphere in $\boldsymbol{R}^n$ by S^{n-1}, the Lebesgue measure on the unit sphere S^{n-1} by σ and the surface area $2\sqrt{\pi}^n/\Gamma(n/2)$ of S^{n-1} by ω_n. Let κ_2 be a constant in (1.1), (1.2) and (1.3), and $\nu = n/2 - 2\kappa_2 + 1$.

Lemma 2. *Let $f \in L^2(\Omega)$, $x \in \Omega$ and $R > 0$. Then*

$$
\begin{aligned}
&k_\lambda\left(\hat{A}\right) f(x) - f(x) \\
&= -\sum_j \int_0^\infty t\,(Uf)_j(t)\, u_j(x,t)\,\rho(dt) \int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r \frac{J_{\nu+1}\left(\sqrt{t}\,s\right)}{\left(\sqrt{t}\,s\right)^{\nu+1}}\, s\,ds \\
&\quad + \sum_j \int_0^\infty \frac{(Uf)_j(t)\, u_j(x,t)}{\sqrt{t}^{\nu}}\,\rho(dt) \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\,r\right)\, r\,dr \\
&\quad - f(x) \times \frac{1}{2^\nu \Gamma(\nu+1)} \int_R^\infty I_\lambda(r)\, r^{\nu+1} dr,
\end{aligned}
$$

where

$$
I_\lambda(r) = \int_0^\infty k_\lambda\left(s^2\right) J_\nu(rs)\, s^{\nu+1} ds.
$$

Proof. First observe that the function $k_\lambda(t)$ is piecewise smooth on $[0,\infty)$ and $k_\lambda(t)\sqrt{t}^{\nu-1}$ is integrable on $(0,\infty)$. By Hankel's integral formula ([2, p.73,(60)]), we have

$$
\begin{aligned}
k_\lambda(t) &= \frac{1}{\sqrt{t}^{\nu}} \int_0^\infty J_\nu\left(\sqrt{t}\,r\right) r\,dr \int_0^\infty k_\lambda\left(s^2\right) J_\nu(rs)\, s^{\nu+1} ds \\
&= \frac{1}{\sqrt{t}^{\nu}} \int_0^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\,r\right) r\,dr.
\end{aligned}
$$

Then, by (3.5), (3.6) and the fact that $k_\lambda(0)=1$, we have

$$
\begin{aligned}
&k_\lambda\big(\hat{A}\big)\, f(x) - f(x)\\
&\quad = \sum_j \int_0^\infty \{\, k_\lambda(t) - k_\lambda(0)\} \,(\,Uf)_j(t)\, u_j(x,t\,)\,\rho(dt)\\
&\quad = \sum_j \int_0^\infty (\,Uf)_j(t)\, u_j(x,t\,)\,\rho(dt) \int_0^\infty \left\{ \frac{J_\nu\left(\sqrt{t}\, r\right)}{\sqrt{t}^{\,\nu}} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda(r)\, r\, dr\\
&\quad = \sum_j \int_0^\infty (\,Uf)_j(t)\, u_j(x,t\,)\,\rho(dt) \int_0^R \left\{ \frac{J_\nu\left(\sqrt{t}\, r\right)}{\sqrt{t}^{\,\nu}} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda\,(r)\, r\, dr\\
&\quad + \sum_j \int_0^\infty (\,Uf)_j(t)\, u_j(x,t\,)\,\rho(dt) \int_R^\infty \left\{ \frac{J_\nu\left(\sqrt{t}\, r\right)}{\sqrt{t}^{\,\nu}} - \frac{r^\nu}{2^\nu \Gamma(\nu+1)} \right\} I_\lambda\,(r)\, r\, dr.
\end{aligned}
$$

Now apply the formula ([7, p.45])

$$
\frac{J_\nu\left(\sqrt{t}\, r\right)}{\sqrt{t}^{\,\nu}} - \frac{r^\nu}{2^\nu \Gamma(\,\nu+1)} = -\,t\, r^\nu \int_0^r \frac{J_{\nu+1}\left(\sqrt{t}\, s\right)}{\left(\sqrt{t}\, s\right)^{\nu+1}}\, s\, ds.
$$

Note that for the second term, we have

$$
\begin{aligned}
&\sum_j \int_0^\infty (\,Uf)_j(t)\, u_j(x,t\,)\,\rho(dt) \int_R^\infty \left\{ \frac{J_\nu\left(\sqrt{t}\, r\right)}{\sqrt{t}^{\,\nu}} - \frac{r^\nu}{2^\nu \Gamma(\,\nu+1)} \right\} I_\lambda(r)\, r\, dr\\
&\quad = \sum_j \int_0^\infty \frac{(\,Uf)_j(t)\, u_j(x,t\,)}{\sqrt{t}^{\,\nu}}\,\rho(dt) \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\, r\right)\, r\, dr\\
&\quad\ - f(x) \times \frac{1}{2^\nu \Gamma(\,\nu+1)} \int_R^\infty I_\lambda(r)\, r^{\nu+1} dr.
\end{aligned}
$$

Thus we get Lemma 2.

3.3 Proof of Theorem 2.

Let f be a regulated function in $L^2(\Omega)$. Let K be a compact set in Ω and K' be a closed set in K with $\operatorname{dist}(K', K^c) > 0$. We choose $0 < R < \operatorname{dist}(K', K^c)$. Let κ_1 and κ_2 be constants in (1.1), (1.2) and (1.3). Let $\nu = n/2 - 2\,\kappa_2 + 1$ and $1 < p \le \infty$.

Suppose that $f \in L^p(K)$ and $(-\Delta)^{\kappa_2} f \in L^p(K)$. By Lemma 2, we have

$$\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p(K')} \leq \|f\|_{L^p(K')} \times \frac{1}{2^\nu \Gamma(\nu+1)} \left| \int_R^\infty I_\lambda(r)\, r^{\nu+1} dr \right|$$
$$+ \left\| \int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r s\, ds \sum_j \int_0^\infty t\, (Uf)_j(t)\, u_j(\cdot, t)\, \frac{J_{\nu+1}\left(\sqrt{t}\, s\right)}{\left(\sqrt{t}\, s\right)^{\nu+1}}\, \rho(dt) \right\|_{L^p(K')}$$
$$(3.7) \qquad + \left\| \sum_j \int_0^\infty \frac{(Uf)_j(t)\, u_j(\cdot, t)}{\sqrt{t}^{\nu}}\, \rho(dt) \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\, r\right) r\, dr \right\|_{L^\infty(K')}.$$

Lemma 3. *We have*

$$\left\| \int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r s\, ds \sum_j \int_0^\infty t\, (Uf)_j(t)\, u_j(\cdot, t)\, \frac{J_{\nu+1}\left(\sqrt{t}\, s\right)}{\left(\sqrt{t}\, s\right)^{\nu+1}}\, \rho(dt) \right\|_{L^p(K')}$$
$$\leq C\, \lambda^{-\kappa_1} \left\| (-\Delta)^{\kappa_2} f \right\|_{L^p(K)}.$$

Proof. Let $x \in K'$ and $0 < s < R$. Put

$$g_s(y) = \frac{1}{s^{\nu+1} |y|^{n/2-1}} \int_0^\infty J_{\nu+1}(s\, r) J_{n/2-1}(|y| r) dr,$$
$$g_s^x(y) = g_s(x-y).$$

If $|y| > s$, then $g_s(y) = 0$ ([7, p.404,(6)]). Therefore supp $g_s^x \subset K \subset \Omega$. Then, by (3.3), we have

$$\begin{aligned}
(U g_s^x)_j(t) &= \int_\Omega g_s^x(y)\, \overline{u_j(y,t)}\, dy \\
&= \int_\Omega g_s(y)\, \overline{u_j(x-y,t)}\, dy \\
&= \frac{1}{s^{\nu+1}} \int \overline{u_j(x-y,t)}\, dy\, \frac{1}{|y|^{n/2-1}} \int_0^\infty J_{\nu+1}(s\, r)\, J_{n/2-1}(|y| r)\, dr \\
&= \frac{1}{s^{\nu+1}} \int_0^\infty q^{n/2} dq \int_{S^{n-1}} \overline{u_j(x-q\, w,t)}\, \sigma(d\, w) \int_0^\infty J_{\nu+1}(s\, r) J_{n/2-1}(qr)\, dr.
\end{aligned}$$

On the other hand, by (3.2), $u_j(y,t) \in C^\infty(\Omega)$, and we have $-\Delta u_j(y,t) = t\, u_j(y,t)$ for $y \in \Omega$. Therefore, by the mean-value formula, we have

$$\int_{S^{n-1}} u_j(x - q\,w, t)\,\sigma(dw) = \sqrt{2\,\pi}^{\,n} \frac{J_{n/2-1}\left(\sqrt{t}\,q\right)}{\left(\sqrt{t}\,q\right)^{n/2-1}} u_j(x,t).$$

Thus, by Hankel's formula, we have

$$\begin{aligned}
(U g_s^x)_j(t) &= \frac{\sqrt{2\pi}^{\,n}}{\sqrt{t}^{\,n/2-1} s^{\nu+1}} \overline{u_j(x,t)} \int_0^\infty J_{n/2-1}\left(\sqrt{t}\,q\right) q\,dq \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr)dr \\
&= \frac{\sqrt{2\,\pi}^{\,n} J_{\nu+1}\left(\sqrt{t}\,s\right)}{\sqrt{t}^{\,n/2} s^{\nu+1}} \overline{u_j(x,t)}.
\end{aligned}$$

We can assume that $f \in C_c^\infty(\Omega)$ by approximation. Then, by (3.1), we have

$$\begin{aligned}
&\sum_j \int_0^\infty t\,(Uf)_j(t)\, u_j(x,t) \frac{J_{\nu+1}\left(\sqrt{t}\,s\right)}{\left(\sqrt{t}\,s\right)^{\nu+1}} \rho(dt) \\
&= \frac{1}{\sqrt{2\,\pi}^{\,n}} \sum_j \int_{e_j} t^{\kappa_2} (Uf)_j(t) \overline{(U g_s^x)_j(t)}\, \rho(dt) \\
&= \frac{1}{\sqrt{2\,\pi}^{\,n}} \int_\Omega [(-\Delta)^{\kappa_2} f(y)]\, g_s^x(y) dy \\
&= \frac{1}{\sqrt{2\,\pi}^{\,n}} \int_\Omega [(-\Delta)^{\kappa_2} f(y)]\, g_s(x-y) dy.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r s\,ds \sum_j \int_0^\infty t\,(Uf)_j(t)\, u_j(x,t) \frac{J_{\nu+1}\left(\sqrt{t}\,s\right)}{\left(\sqrt{t}\,s\right)^{\nu+1}} \rho(dt) \\
&= \frac{1}{\sqrt{2\,\pi}^{\,n}} \int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r s\,ds \int_{|y|<s} [(-\Delta)^{\kappa_2} f(x-y)]\, g_s(y)\, dy \\
&= \frac{1}{\sqrt{2\,\pi}^{\,n}} \int_0^R s\,ds \int_s^R I_\lambda(r)\, r^{\nu+1} dr \int_{|y|<s} [(-\Delta)^{\kappa_2} f(x-y)]\, g_s(y)\, dy.
\end{aligned}$$

Applying successively Minkowski's inequality for integral, we have

$$\begin{aligned}
&\left\| \int_0^R I_\lambda(r)\, r^{\nu+1} dr \int_0^r s\, ds \sum_j \int_0^\infty t\,(Uf)_j(t)\, u_j(\cdot, t)\, \frac{J_{\nu+1}\left(\sqrt{t}\, s\right)}{\left(\sqrt{t}\, s\right)^{\nu+1}}\, \rho(dt) \right\|_{L^p(K')} \\
&\leq \frac{1}{\sqrt{2\pi}^n} \int_0^R s\, ds \left\| \int_s^R I_\lambda(r)\, r^{\nu+1} dr \int_{|y|<s} [(-\Delta)^{\kappa_2} f(\cdot - y)]\; g_s(y)\, dy \right\|_{L^p(K')} \\
&= \frac{1}{\sqrt{2\pi}^n} \int_0^R s\, ds \left| \int_s^R I_\lambda(r)\, r^{\nu+1} dr \right| \left\| \int_{|y|<s} [(-\Delta)^{\kappa_2} f(\cdot - y)]\, g_s(y)\, dy \right\|_{L^p(K')} \\
&\leq \frac{1}{\sqrt{2\pi}^n} \int_0^R s\, ds \left| \int_s^R I_\lambda(r)\, r^{\nu+1} dr \right| \int_{|y|<s} \|(-\Delta)^{\kappa_2} f(\cdot - y)\, \|_{L^p(K')}\, |g_s(y)|\, dy \\
&\leq \frac{1}{\sqrt{2\pi}^n} \|(-\Delta)^{\kappa_2} f\, \|_{L^p(K)} \int_0^R s\, ds \left| \int_s^R I_\lambda(r)\, r^{\nu+1} dr \right| \int_{|y|<s} |g_s(y)|\, dy.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{|y|<s} |g_s(y)|\, dy &= \frac{1}{s^{\nu+1}} \int_{|y|<s} \frac{1}{|y|^{n/2-1}}\, dy \left| \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(|y|r)\, dr \right| \\
&= \frac{\omega_n}{s^{\nu+1}} \int_0^s q^{n/2} dq \left| \int_0^\infty J_{\nu+1}(sr) J_{n/2-1}(qr)\, dr \right| \\
&= \frac{\omega_n\, \Gamma((2\nu+n+2)/4)}{\Gamma(n/2)\, \Gamma((2\nu-n+6)/4) s^{\nu+n/2+1}} \\
&\quad \times \int_0^s \left| {}_2F_1\left((2\nu+n+2)/4, -(2\nu-n+2)/4; n/2; q^2/s^2\right) \right| q^{n-1} dq \\
&\leq \frac{C_{\kappa_2}}{s^{\nu-n/2+1}},
\end{aligned}$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is Gauss' hypergeometric function. Therefore the last term is bounded by

$$C_{\kappa_2} \|(-\Delta)^{\kappa_2} f\, \|_{L^p(K)} \int_0^R s^{2\kappa_2 - 1} ds \left| \int_s^R I_\lambda(r)\, r^{\nu+1} dr \right|.$$

By the condition (1.1), we get the bound $C\, \lambda^{-\kappa_1} \|(-\Delta)^{\kappa_2} f\, \|_{L^p(K)}$ for the last term. Thus Lemma 3 is proved.

We shall use the following lemma ([1, p.655]).

Lemma 4. *Under the assumptions above, if* K *is a compact set contained in* Ω, *then*

$$\left(\sum_j \int_{T \leq \sqrt{t} \leq T+1} |u_j(x,t)|^2 \rho(dt) \right)^{1/2} \leq C_K (T+1)^{(n-1)/2},$$

where C_K *is a constant independent of* $T \geq 0$ *and* $x \in K$.

Lemma 5. *We have*

$$\left\| \sum_j \int_0^\infty \frac{(Uf)_j(t)\, u_j(\cdot,t)}{\sqrt{t}^{\nu}} \rho(dt) \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\, r\right) r\, dr \right\|_{L^\infty(K)} = o\left(\lambda^{-\kappa_1}\right)$$

as $\lambda \to \infty$.

Proof. We have, by Schwarz's inequality,

$$\begin{aligned} &\left| \sum_j \int_0^\infty \frac{(Uf)_j(t)\, u_j(x,t)}{\sqrt{t}^{\nu}} \rho(dt) \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\, r\right) r\, dr \right| \\ &\quad \leq \left(\sum_j \int_{e_j} |(Uf)_j(t)|^2 \rho(dt) \right)^{1/2} \\ &\qquad \times \left(\sum_j \int_0^\infty \frac{|u_j(x,t)|^2}{t^\nu} \rho(dt) \left| \int_R^\infty I_\lambda(r)\, J_\nu\left(\sqrt{t}\, r\right) r\, dr \right|^2 \right)^{1/2}. \end{aligned}$$

Now, by (3.1), we have

$$\left(\sum_j \int_{e_j} |(Uf)_j(t)|^2 \rho(dt) \right)^{1/2} = \| f \|_{L^2(\Omega)}.$$

By Lemma 4, there exists a constant C_K such that

$$\left(\sum_j \int_0^\infty \frac{|u_j(x,t)|^2}{t^\nu}\,\rho(dt)\left|\int_R^\infty I_\lambda(r)\,J_\nu\left(\sqrt{t}\,r\right)\,r\,dr\right|^2\right)^{1/2}$$
$$\leq C_K\left(\sum_{T=0}^{\infty} T^{4\kappa_2-3}\max_{T\leq s\leq T+1}\left|\int_R^\infty I_\lambda(r)\,J_\nu(s\,r)\,r\,dr\right|^2\right)^{1/2}$$

uniformly in $x \in K$. Therefore, by (1.3), we have

$$\left|\sum_j \int_0^\infty \frac{(Uf)_j(t)\,u_j(x,t)}{\sqrt{t}^{\,\nu}}\,\rho(dt)\int_R^\infty I_\lambda(r)\,J_\nu\left(\sqrt{t}\,r\right)\,r\,dr\right| = o\left(\lambda^{-\kappa_1}\right)$$

uniformly in $x \in K$ as $\lambda \to \infty$. Thus Lemma 5 is proved.

We remark that $\left|\int_R^\infty I_\lambda(r)\,r^{\nu+1}dr\right| = o\left(\lambda^{-\kappa_1}\right)$ by the assumption (1.2).

By (3.7) together with Lemmas 3 and 5, $\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p(K')} = O\left(\lambda^{-\kappa_1}\right)$ as $\lambda \to \infty$.

If $(-\Delta)^{\kappa_2} f$ vanishes in K, then by Lemma 3

$$\left\| \int_0^R I_\lambda(r)\,r^{\nu+1}dr \int_0^r s\,ds \sum_j \int_0^\infty t\,(Uf)_j(t)\,u_j(\cdot,t)\,\frac{J_{\nu+1}\left(\sqrt{t}\,s\right)}{\left(\sqrt{t}\,s\right)^{\nu+1}}\,\rho(dt)\right\|_{L^p(K')} = 0.$$

Therefore, by (3.7) and Lemma 5, we have $\left\| k_\lambda\left(\hat{A}\right) f - f \right\|_{L^p(K')} = o\left(\lambda^{-\kappa_1}\right)$ as $\lambda \to \infty$.

Consequently, Theorem 2 is proved.

4 Applications of main theorem.

4.1 Proof of Corollary 1.

Let $k_\lambda(t) = (1 - t/\lambda^2)_+^\delta$. Then we have the formula (see [2, p.92,(34)])

$$k_\lambda(t) = \frac{2^\delta\,\Gamma(\delta+1)}{\lambda^{\delta-n/2}\sqrt{t}^{\,n/2-1}} \int_0^\infty \frac{J_{n/2+\delta}(\lambda r)\,J_{n/2-1}\left(\sqrt{t}\,r\right)}{r^\delta}\,dr,$$

and can take $\kappa_2 = 1$. We have

$$I_\lambda(r) = \int_0^\infty k_\lambda\left(t^2\right) J_{n/2-1}(r\,t)\,t^{n/2}dt = 2^\delta\,\Gamma(\delta+1)\,\lambda^{n/2-\delta}\,J_{n/2+\delta}(\lambda r)\,r^{-\delta-1}.$$

To check the conditions (1.1), (1.2) and (1.3), let $R > 0$ and $\delta > (n-3)/2$. Then we have

$$\left|\int_R^\infty I_\lambda(r)\,r^{n/2}\,dr\right| = 2^\delta\,\Gamma(\delta+1)\,\lambda^{n/2-\delta}\left|\int_R^\infty \frac{J_{n/2+\delta}(\lambda r)}{r^{\delta-n/2+1}}\,dr\right| \le C_{\delta,R}\,\lambda^{(n-3)/2-\delta}.$$

On the other hand, we have

$$\int_0^R s\,ds\left|\int_s^R I_\lambda(r)\,r^{n/2}\,dr\right| = 2^\delta\,\Gamma(\delta+1)\,\lambda^{n/2-\delta}\int_0^R s\,ds\left|\int_s^R \frac{J_{n/2+\delta}(\lambda r)}{r^{\delta-n/2+1}}\,dr\right|$$

$$\le \begin{cases} C_\delta\,\lambda^{(n-3)/2-\delta} & \text{if } (n-3)/2 < \delta < (n+1)/2, \\ C_\delta\,\lambda^{(n-3)/2-\delta}\log\lambda & \text{if } \delta = (n+1)/2, \\ C_\delta\,\lambda^{-2} & \text{if } \delta > (n+1)/2. \end{cases}$$

We now apply the estimates (see [6, p.202, Lemma 18.10 a])

$$\left|\int_R^\infty \frac{J_{n/2+\delta}(\lambda r)\,J_{n/2-1}(s\,r)}{r^\delta}\,dr\right|$$

$$\le \begin{cases} C_{\delta,R}\,\lambda^{-1/2}s^{-1/2} & \text{if } s,\lambda > 0, \\ C_{\delta,R}\dfrac{\lambda^{-3/2}s^{1/2}}{\lambda - s} + C_{\delta,R}\,\lambda^{-3/2}s^{-1/2} & \text{if } 0 < s < \lambda, \\ C_{\delta,R}\dfrac{\lambda^{1/2}s^{-3/2}}{s-\lambda} + C_{\delta,R}\,\lambda^{-1/2}s^{-3/2} & \text{if } 0 < \lambda < s. \end{cases}$$

Then we have

$$\begin{aligned}
&\left(\sum_{T=0}^{\infty} T \max_{T\le s\le T+1} \left| \int_R^{\infty} I_\lambda(r)\, J_{n/2-1}(s\, r)\, r\, dr \right|^2\right)^{1/2} \\
&= 2^{\delta}\,\Gamma(\delta+1)\,\lambda^{n/2-\delta} \left(\sum_{T=0}^{\infty} T \max_{T\le s\le T+1} \left| \int_R^{\infty} \frac{J_{n/2+\delta}(\lambda r)\, J_{n/2-1}(s\, r)}{r^{\delta}}\, dr \right|^2\right)^{1/2} \\
&\le C_{\delta,R}\,\lambda^{(n-1)/2-\delta}.
\end{aligned}$$

If $\delta > (n+3)/2$, then the last term is $o(\lambda^{-2})$. Thus Corollary 1 follows from Main theorem.

4.2 The Gauss-Weierstrass summation.

Let $k_\lambda^W(t) = e^{-t/\lambda}$ $(\lambda \to \infty)$. We then have

$$(4.1)\ \int_0^{\infty} k_\lambda^W\left(t^2\right) J_\nu(r\,t)\, t^{\nu+1} dt = \int_0^{\infty} e^{-t^2/\lambda} J_\nu(r\,t)\, t^{\nu+1}\, dt = \frac{\lambda^{\nu+1} r^{\nu}}{2^{\nu+1}} \exp\left(-\frac{\lambda\, r^2}{4}\right)$$

(cf. [2, 7.7.3]). Let Ω be an open domain in $\boldsymbol{R}^n$ and $\hat{A}$ be a nonnegative selfadjoint extension of $-\Delta$ in Ω.

Corollary 2. *Let f be a regulated function in $L^2(\Omega)$. Suppose that $1 < p \le \infty$ and $f \in L^p_{\mathrm{loc}}(\Omega)$. Then the following hold.*

(i) *The following are equivalent.*

(ia)

$$\left\| k_\lambda^W\left(\hat{A}\right) f - f \right\|_{L^p(K)} = O\left(\lambda^{-1}\right)$$

as $\lambda \to \infty$ for every compact set K in Ω.

(ib) $\Delta f \in L^p_{\mathrm{loc}}(\Omega)$.

(ii) *Let $G \subset \Omega$ be any open set.*

(ii a) *Suppose that Δf vanishes in G. Then*

$$\left\| k_\lambda^W\left(\hat{A}\right) f - f \right\|_{L^p(K)} = o\left(\lambda^{-1}\right)$$

as $\lambda \longrightarrow \infty$ for any compact set $K \subset G$.

(ii b) *If*

$$\left\| k_\lambda^W\left(\hat{A}\right) f - f \right\|_{L^p(K)} = o\left(\lambda^{-1}\right)$$

as $\lambda \longrightarrow \infty$ for any compact set $K \subset G$, then Δf vanishes in G.

Proof. For the Gauss-Weierstrass summation method we take $\kappa_2 = 1$. Let R be a small positive number. By (4.1), we have

$$\int_R^\infty r^{n/2}\, dr \int_0^\infty k_\lambda^W\left(t^2\right) J_\nu(r\,t)\, t^{\nu+1} dt = \left(\frac{\lambda}{2}\right)^{n/2} \int_R^\infty r^{n-1} \exp\left(-\frac{\lambda\, r^2}{4}\right) dr = o\left(\lambda^{-1}\right),$$

$$\begin{aligned} &\int_0^R s\, ds \left| \int_s^R r^{n/2}\, dr \int_0^\infty k_\lambda^W\left(t^2\right) J_\nu(r\,t)\, t^{\nu+1} dt \right| \\ &= \left(\frac{\lambda}{2}\right)^{n/2} \int_0^R s\, ds \int_s^R r^{n-1} \exp\left(-\frac{\lambda\, r^2}{4}\right) dr = O\left(\lambda^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} &\left(\sum_{T=0}^\infty T \max_{T \le s \le T+1} \left| \int_R^\infty J_{n/2-1}(s\,r)\, r\, dr \int_0^\infty k_\lambda^W\left(t^2\right) J_\nu(r\,t)\, t^{\nu+1} dt \right|^2 \right)^{1/2} \\ &= \left(\frac{\lambda}{2}\right)^{n/2} \left(\int_R^\infty r^{n-1} \exp\left(-\frac{\lambda\, r^2}{2}\right) dr \right)^{1/2} = o\left(\lambda^{-1}\right). \end{aligned}$$

Thus Corollary 2 follows from Main theorem.

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