

# Uniform stability and attractivity for linear systems with periodic coefficients

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## 1 Introduction

In this paper, we consider the linear system

$$\mathbf{x}' = A(t)\mathbf{x} = \begin{pmatrix} -r(t) & p(t) \\ -p(t) & -q(t) \end{pmatrix} \mathbf{x}, \quad (1)$$

where the prime denotes  $d/dt$ ; the coefficients  $p(t)$ ,  $q(t)$  and  $r(t)$  are continuous for  $t \geq 0$ , and  $p(t)$  is a periodic function with period  $\omega > 0$ . System (1) has the zero solution  $\mathbf{x}(t) \equiv \mathbf{0} \in \mathbb{R}^2$ . We say that the zero solution of (1) is *attractive* if every solution  $\mathbf{x}(t)$  of (1) tends to  $\mathbf{0}$  as time  $t$  increases.

If  $q(t)$  and  $r(t)$  are also periodic functions with period  $\omega$ , Floquet's theorem is available. Let  $\Phi(t)$  be the fundamental matrix of (1) with  $\Phi(0) = E$ , the  $2 \times 2$  identity matrix. Then  $\Phi(\omega)$  is called the *monodromy matrix* of (1). Let  $\mu_1$  and  $\mu_2$  be the eigenvalues of the monodromy matrix  $\Phi(\omega)$ . The eigenvalues  $\mu_1$  and  $\mu_2$  are often called the Floquet multipliers of (1). By Abel's formula,

$$\det \Phi(\omega) = \det \Phi(0) \exp \left( - \int_0^\omega (q(s) + r(s)) ds \right) = \exp \left( - \int_0^\omega (q(s) + r(s)) ds \right).$$

Thus, the Floquet multipliers  $\mu_1$  and  $\mu_2$  are the roots of the equation

$$\mu^2 - \text{tr} \Phi(\omega) \mu + \exp \left( - \int_0^\omega (q(s) + r(s)) ds \right) = 0.$$

It is well-known that the zero solution of (1) is attractive if and only if the Floquet multipliers  $\mu_1$  and  $\mu_2$  have magnitudes strictly less than 1. Hence, in the case where  $p(t)$ ,  $q(t)$  and  $r(t)$  are periodic, necessary and sufficient conditions for the zero solution of (1) to be attractive are that

$$|\text{tr} \Phi(\omega)| < 1 + \exp \left( - \int_0^\omega (q(s) + r(s)) ds \right)$$

and

$$\exp \left( - \int_0^\omega (q(s) + r(s)) ds \right) < 1.$$

For example, we can find Floquet's theorem in the books [2, 3, 5, 8, 16].

Although the above conditions are necessary and sufficient for the zero solution of (1) to be attractive, it is difficult to estimate the absolute value of the trace of  $\Phi(\omega)$ , because it is impossible to find a fundamental matrix of (1) in general. Of course, Floquet's theorem is useless when  $q(t)$  or  $r(t)$  is not periodic. Then, without knowledge of a fundamental matrix of (1), can we decide whether the zero solution is attractive? What kind of condition on  $A(t)$  will guarantee the attractivity of the zero solution of (1)?

## 2 The main theorem

To give an answer to the above question, we prepare some notations. Let

$$R(t) = \int_0^t r(s)ds \quad \text{and} \quad \psi(t) = 2(q(t) - r(t))$$

for  $t \geq 0$  and denote a positive part and a negative part of  $\psi(t)$  by

$$\psi_+(t) = \max\{0, \psi(t)\} \quad \text{and} \quad \psi_-(t) = \max\{0, -\psi(t)\},$$

respectively. Note that  $\psi(t) = \psi_+(t) - \psi_-(t)$  and  $|\psi(t)| = \psi_+(t) + \psi_-(t)$ .

We introduce an important concept here. A nonnegative function  $\phi(t)$  is said to be *weakly integrally positive* if

$$\int_I \phi(t)dt = \infty$$

for every set  $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$  such that  $\tau_n + \delta < \sigma_n < \tau_{n+1} < \sigma_n + \Delta$  for some  $\delta > 0$  and  $\Delta > 0$ . For example,  $1/(1+t)$  and  $\sin^2 t/(1+t)$  are weakly integrally positive functions (see [6, 7, 13–15]).

Our main result is stated as follows:

**Theorem 1.** *Suppose that  $q(t)$  and  $R(t)$  are bounded for  $t \geq 0$ . Suppose also that*

(i)  $\psi_+(t)$  *is weakly integrally positive;*

(ii)  $\int_0^{\infty} \psi_-(t)dt < \infty$ .

*Then the zero solution of (1) is attractive.*

To prove Theorem 1, we need some lemmas. We present the lemmas without the proofs.

**Lemma 2.** *Suppose that assumption (ii) in Theorem 1 holds. Let  $v(t)$  be nonnegative and continuously differentiable on  $[t_0, \infty)$  for some  $t_0 > 0$ . If*

$$v'(t) \leq \psi_-(t)v(t) \quad \text{for } t \geq t_0, \tag{2}$$

then  $v'(t)$  is absolutely integrable, and therefore  $v(t)$  has a nonnegative limiting value.

**Lemma 3.** Suppose that  $R(t)$  is bounded for  $t \geq 0$ . If assumption (ii) in Theorem 1 holds, then all solutions of (1) are uniformly stable and uniformly bounded.

Recall that  $p(t)$  is a periodic function with period  $\omega > 0$ . Let

$$\bar{p} = \max_{t \in [0, \omega]} p(t) \quad \text{and} \quad \underline{p} = \min_{t \in [0, \omega]} p(t).$$

Taking  $\bar{p} \geq \underline{p}$  into account, we see that if  $\bar{p} + \underline{p} \geq 0$ , then  $\bar{p} > 0$ ; if  $\bar{p} + \underline{p} < 0$ , then  $\underline{p} < 0$ . Since  $p(t)$  is continuous for  $t \geq 0$ , we see that  $p(t)$  has the following property (we omit the proof).

**Lemma 4.** Let  $m$  be any integer. If  $\underline{p} + \bar{p} \geq 0$ , then there exist numbers  $a$  and  $b$  with  $0 \leq a < b \leq \omega$  such that

$$p(t) \geq \frac{1}{2}\bar{p} > 0 \quad \text{for } m\omega + a \leq t \leq m\omega + b.$$

If  $\underline{p} + \bar{p} < 0$ , then there exist numbers  $a$  and  $b$  with  $0 \leq a < b \leq \omega$  such that

$$p(t) \leq \frac{1}{2}\underline{p} < 0 \quad \text{for } m\omega + a \leq t \leq m\omega + b.$$

### 3 Proof of the main theorem

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Let  $x(t; t_0, x_0)$  be a solution of (1) passing through  $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^2$ . It follows from Lemma 3 that for any  $\alpha > 0$ , there exists a  $\beta(\alpha) > 0$  such that  $t_0 \geq 0$  and  $\|x_0\| < \alpha$  imply that

$$\|x(t; t_0, x_0)\| < \beta \quad \text{for } t \geq t_0. \quad (3)$$

For the sake of brevity, we write  $(x(t), y(t)) = x(t; t_0, x_0)$  and

$$v(t) = V(t, x(t), y(t)).$$

Then, we have

$$v(t) = \frac{1}{2}e^{2R(t)}(x^2(t) + y^2(t)) \quad (4)$$

and

$$v'(t) = -(q(t) - r(t))e^{2R(t)}y^2(t) \leq \psi_-(t)v(t) \quad (5)$$

for  $t \geq t_0$ . Hence, from Lemma 2, we see that  $v(t)$  has a limiting value  $v_0 \geq 0$ . If  $v_0 = 0$ , then by (4) the solution  $(x(t), y(t))$  tends to 0 as  $t \rightarrow \infty$ . This completes the proof. Thus, we need consider only the case in which  $v_0 > 0$ . We will show that this case does not occur.

Because of (3), we see that  $|y(t)|$  is bounded for  $t \geq t_0$ . Hence,  $|y(t)|$  has an inferior limit and a superior limit. First, we will show that the inferior limit of  $|y(t)|$  is zero, and we will then show that the superior limit of  $|y(t)|$  is also zero.

Suppose that  $\liminf_{t \rightarrow \infty} |y(t)| > 0$ . Then, there exist a  $\gamma > 0$  and a  $T_1 \geq t_0$  such that  $|y(t)| > \gamma$  for  $t \geq T_1$ . It follows from (5) and Lemma 2 that

$$\begin{aligned} \infty &> \int_{t_0}^{\infty} |v'(s)| ds = \frac{1}{2} \int_{t_0}^{\infty} |\psi(s)| e^{2R(s)} y^2(s) ds \\ &\geq \frac{1}{2} \gamma^2 \int_{T_1}^{\infty} \psi_+(s) e^{2R(s)} ds \geq \frac{1}{2} \gamma^2 e^{-2L} \int_{T_1}^{\infty} \psi_+(s) ds, \end{aligned}$$

where  $L$  is the number given in the proof of Lemma 3. This contradicts assumption (i). Thus, we see that  $\liminf_{t \rightarrow \infty} |y(t)| = 0$ .

Suppose that  $\limsup_{t \rightarrow \infty} |y(t)| > 0$ . Let  $\nu = \limsup_{t \rightarrow \infty} |y(t)|$ . Since  $q(t)$  is bounded, we can find a  $\bar{q} > 0$  such that

$$|q(t)| \leq \bar{q} \quad \text{for } t \geq 0. \quad (6)$$

Since  $v(t)$  tends to a positive value  $v_0$  as  $t \rightarrow \infty$ , there exists a  $T_2 \geq t_0$  such that

$$0 < \frac{1}{2} v_0 < v(t) < \frac{3}{2} v_0 \quad \text{for } t \geq T_2. \quad (7)$$

Let  $\varepsilon$  be so small that

$$0 < \varepsilon < \min \left\{ \frac{1}{2} \nu, \sqrt{\frac{\bar{p}^2 e^{-2L} v_0}{4(\bar{q} + 2/(b-a))^2 + \bar{p}^2}}, \sqrt{\frac{\underline{p}^2 e^{-2L} v_0}{4(\bar{q} + 2/(b-a))^2 + \underline{p}^2}} \right\}, \quad (8)$$

where  $a$  and  $b$  are the numbers given in Lemma 4. Then, since  $\liminf_{t \rightarrow \infty} |y(t)| = 0$ , we can select two intervals  $[\tau_n, \sigma_n]$  and  $[t_n, s_n]$  with  $[t_n, s_n] \subset [\tau_n, \sigma_n]$ ,  $T_2 < \tau_n$  and  $\tau_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $|y(\tau_n)| = |y(\sigma_n)| = \varepsilon$ ,  $|y(t_n)| = \nu/2$ ,  $|y(s_n)| = 3\nu/4$  and

$$|y(t)| \geq \varepsilon \quad \text{for } \tau_n < t < \sigma_n, \quad (9)$$

$$|y(t)| \leq \varepsilon \quad \text{for } \sigma_n < t < \tau_{n+1}, \quad (10)$$

$$\frac{1}{2} \nu < |y(t)| < \frac{3}{4} \nu \quad \text{for } t_n < t < s_n. \quad (11)$$

By (4), (7) and (10), we have

$$|x(t)| = \sqrt{2e^{-2R(t)} v(t) - y^2(t)} \geq \sqrt{e^{-2L} v_0 - \varepsilon^2} \quad (12)$$

for  $\sigma_n \leq t \leq \tau_{n+1}$ .

*Claim.* The sequences  $\{\tau_n\}$  and  $\{\sigma_n\}$  satisfy  $\tau_{n+1} - \sigma_n \leq 2\omega$  for any integer  $n$ .

Suppose that there exists an  $n_0 \in \mathbb{N}$  such that  $\tau_{n_0+1} - \sigma_{n_0} > 2\omega$ . We can choose an  $m \in \mathbb{N}$  such that  $(m-1)\omega < \sigma_{n_0} \leq m\omega$ . Hence, we have

$$\tau_{n_0+1} > \sigma_{n_0} + 2\omega > (m-1)\omega + 2\omega = (m+1)\omega,$$

and therefore  $[m\omega, (m+1)\omega] \subset [\sigma_{n_0}, \tau_{n_0+1}]$ . There are two cases to consider: (a)  $\bar{p} + \underline{p} \geq 0$  and (b)  $\bar{p} + \underline{p} < 0$ . In case (a), by Lemma 4,  $p(t) \geq \bar{p}/2 > 0$  for  $t \in [a + m\omega, b + m\omega] \subset$

$[m\omega, (m+1)\omega]$ . Hence, using the second equation in system (1) with (6), (10) and (12), we have

$$|y'(t)| \geq |p(t)||x(t)| - |q(t)||y(t)| \geq \frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \quad (13)$$

for  $a + m\omega < t < b + m\omega$ . It follows from (8) that

$$\frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon > \frac{2}{b-a}\varepsilon. \quad (14)$$

From (10) and (13), we can estimate that

$$\begin{aligned} 2\varepsilon &\geq |y(b + m\omega)| + |y(a + m\omega)| \geq \left| \int_{a+m\omega}^{b+m\omega} y'(s) ds \right| \\ &= \int_{a+m\omega}^{b+m\omega} |y'(s)| ds \geq (b-a) \left( \frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \right). \end{aligned}$$

This contradicts (14). In case (b), by Lemma 4,  $p(t) \leq \underline{p}/2 < 0$  for  $t \in [a + m\omega, b + m\omega] \subset [m\omega, (m+1)\omega]$ . Hence, combining this with (6), (10) and (12), we obtain

$$|y'(t)| \geq |p(t)||x(t)| - |q(t)||y(t)| \geq -\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \quad (15)$$

for  $a + m\omega < t < b + m\omega$ . It follows from (8) that

$$-\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon > \frac{2}{b-a}\varepsilon. \quad (16)$$

From (10) and (15), we can estimate that

$$\begin{aligned} 2\varepsilon &\geq |y(b + m\omega)| + |y(a + m\omega)| \geq \left| \int_{a+m\omega}^{b+m\omega} y'(s) ds \right| \\ &= \int_{a+m\omega}^{b+m\omega} |y'(s)| ds \geq (b-a) \left( -\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \right). \end{aligned}$$

This contradicts (16). Thus, the claim is proved.

Let  $I = \bigcup_{n=1}^{\infty} [\tau_n, \sigma_n]$ . Then, by means of Lemma 2 with (5) and (9), we get

$$\begin{aligned} \infty &> \int_{t_0}^{\infty} |v'(s)| ds = \frac{1}{2} \int_{t_0}^{\infty} |\psi(s)| e^{2R(s)} y^2(s) ds \\ &\geq \frac{1}{2} e^{-2L} \int_{t_0}^{\infty} \psi_+(s) y^2(s) ds \geq \frac{1}{2} \varepsilon^2 e^{-2L} \int_I \psi_+(s) ds. \end{aligned}$$

Hence, it follows from assumption (i) and the claim that  $\liminf_{n \rightarrow \infty} (\sigma_n - \tau_n) = 0$ . Since  $[t_n, s_n] \subset [\tau_n, \sigma_n]$ , it follows that

$$\liminf_{n \rightarrow \infty} (s_n - t_n) = 0. \quad (17)$$

By (4), (7) and (11), we have

$$|x(t)| = \sqrt{2e^{-2R(t)}v(t) - y^2(t)} \leq \sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}}$$

for  $t_n \leq t \leq s_n$ . Let  $K = \max\{|\bar{p}|, |p|\}$ . Then, from (6) and (11), we see that

$$|y'(t)| \leq |p(t)||x(t)| + |q(t)||y(t)| < K\sqrt{3e^{2L}v_0 - \frac{\nu^2}{4}} + \frac{3}{4}\bar{q}\nu$$

for  $t_n \leq t \leq s_n$ . Letting  $N = K\sqrt{3e^{2L}v_0 - \nu^2/4} + 3\bar{q}\nu/4$  and integrating this inequality from  $t_n$  to  $s_n$ , we obtain

$$\begin{aligned} \frac{1}{4}\nu &= |y(s_n)| - |y(t_n)| \leq |y(s_n) - y(t_n)| \\ &= \left| \int_{t_n}^{s_n} y'(s) ds \right| \leq \int_{t_n}^{s_n} |y'(s)| ds \leq N(s_n - t_n). \end{aligned}$$

This contradicts (17). We therefore conclude that  $\limsup_{t \rightarrow \infty} |y(t)| = \nu = 0$ .

In summary,  $y(t)$  tends to zero as  $t \rightarrow \infty$ . Hence, there exists a  $T_3 \geq T_2$  such that

$$|y(t)| < \varepsilon \quad \text{for } t \geq T_3. \quad (18)$$

Let  $l$  be an integer satisfying  $l\omega > T_3$ . Using (18) instead of (10) and following the same process as in the proof of the claim, we see that if  $\bar{p} + p \geq 0$ , then

$$\begin{aligned} 2\varepsilon &\geq |y(b + l\omega)| + |y(a + l\omega)| \geq \left| \int_{a+l\omega}^{b+l\omega} y'(s) ds \right| \\ &= \int_{a+l\omega}^{b+l\omega} |y'(s)| ds \geq (b - a) \left( \frac{1}{2}\bar{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \right) > 2\varepsilon, \end{aligned}$$

which is a contradiction; if  $\bar{p} + p < 0$ , then

$$\begin{aligned} 2\varepsilon &\geq |y(b + l\omega)| + |y(a + l\omega)| \geq \left| \int_{a+l\omega}^{b+l\omega} y'(s) ds \right| \\ &= \int_{a+l\omega}^{b+l\omega} |y'(s)| ds \geq (b - a) \left( -\frac{1}{2}\underline{p}\sqrt{e^{-2L}v_0 - \varepsilon^2} - \bar{q}\varepsilon \right) > 2\varepsilon, \end{aligned}$$

which is again a contradiction. Thus, the case of  $v_0 > 0$  cannot happen.

The proof of Theorem 1 is thus complete. □

## 4 Examples

We illustrate our main result with simple examples in which  $p(t)$ ,  $q(t)$  and  $r(t)$  are periodic. It is well-known that if the zero solution of a linear periodic system is attractive, then it is uniformly asymptotically stable (for example, see [5, 18]).

**Example 1.** Let  $\lambda > 0$ . Consider system (1) with

$$p(t) = \cos t, \quad q(t) = \frac{\lambda}{2 - \sin t} \quad \text{and} \quad r(t) = 0. \quad (19)$$

Then the zero solution is attractive.

Since  $\lambda/3 \leq q(t) \leq \lambda$  and  $R(t) \equiv 0$ , it is clear that  $q(t)$  and  $R(t)$  are bounded for  $t \geq 0$ . Also, assumptions (i) and (ii) are satisfied. In fact, we have

$$\psi(t) = 2(q(t) - r(t)) = \frac{2\lambda}{2 - \sin t},$$

and therefore

$$\psi_+(t) = \frac{2\lambda}{2 - \sin t} \quad \text{and} \quad \psi_-(t) = 0$$

for  $t \geq 0$ . Hence,  $\psi_+(t)$  is weakly integrally positive and

$$\int_0^\infty \psi_-(t) dt = 0.$$

Thus, by means of Theorem 1, we conclude that the zero solution is attractive.

Figure 1(a) shows a positive orbit of (1) with (19) and  $\lambda = 0.1$ . The starting point  $\mathbf{x}_0$  is  $(-1, 0)$  and the initial time  $t_0$  is 0. The positive orbit moves around the origin 0 in a clockwise and a counter-clockwise direction alternately, because  $p(t)$  changes its sign. The positive orbit approaches the origin 0 as it goes up and down.

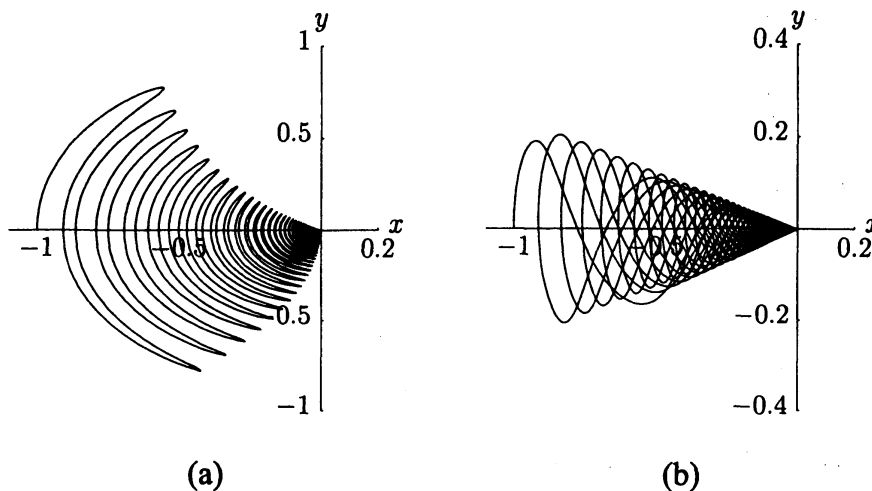


Figure 1: (a) A positive orbit of (1) with (20); (b) a positive orbit of (1) with (21)

**Example 2.** Let  $\lambda \geq 1$ . Consider system (1) with

$$p(t) = \cos \lambda t, \quad q(t) = \cos^2 t + \sin t \quad \text{and} \quad r(t) = \sin t. \quad (20)$$

Then the zero solution is attractive.

It is easy to check that  $q(t)$  and  $R(t)$  are bounded for  $t \geq 0$  and that assumptions (i) and (ii) are satisfied. We omit the details.

In Figure 1(b), we show a positive orbit of (1) with (20) and  $\lambda = 4$ . The positive orbit starts from the point  $(-1, 0)$  at the initial time 0. The positive orbit goes to the right and then goes to the left, and it repeats such a movement regularly. Although the positive orbit displays intricate behavior, it approaches the origin 0 ultimately.

In Examples 1 and 2, all coefficients of (1) are periodic functions with period  $2\pi$ . However, we cannot find the monodromy matrix  $\Phi(2\pi)$ . It is particularly hard to estimate the absolute value of the trace of  $\Phi(2\pi)$ . For this reason, we cannot apply Floquet's theorem to Examples 1 and 2 directly. Theorem 1 has the advantage of being applicable to cases where the monodromy matrix of (1) cannot be found and cases where  $q(t)$  or  $r(t)$  is not periodic.

Fortunately, in Examples 1 and 2 the Floquet multipliers  $\mu_1$  and  $\mu_2$  can be calculated by a numerical scheme. As shown in Tables 1 and 2,  $|\mu_1| < 1$  and  $|\mu_2| < 1$ . Hence, we see that the zero solution of (1) is attractive.

| $\lambda$ | $\mu_1$         | $\mu_2$         |
|-----------|-----------------|-----------------|
| 1         | 0.3351718550789 | 0.0793024028529 |
| 0.1       | 0.8888872982404 | 0.7827240687567 |
| 0.01      | 0.9882826823640 | 0.9758079535053 |
| 0.001     | 0.9988220356864 | 0.9975540561378 |

Table 1: Floquet multipliers of (1) with (20)

| $\lambda$ | $\mu_1$         | $\mu_2$         |
|-----------|-----------------|-----------------|
| 1         | 0.5569470757759 | 0.0775907086028 |
| 10        | 0.9845517600942 | 0.0438919719768 |
| 100       | 0.9998429464892 | 0.0432207062297 |
| 1000      | 0.9999986933319 | 0.0432139974342 |

Table 2: Floquet multipliers of (1) with (21)

**Remark.** The zero solution of system (1) with (19) is attractive if and only if  $\lambda > 0$ . In fact, if  $\lambda \leq 0$ , then

$$\exp\left(-\int_0^\omega (q(s) + r(s))ds\right) = \exp\left(-\int_0^\omega \frac{\lambda}{2 - \sin s} ds\right) \geq 1.$$

Hence, as mentioned in Section 1, the zero solution is not attractive in this case.

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