

RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. We determine the radius of p -valent strongly starlike of order γ for certain polynomials of the form $F(z) = f(z) \cdot [Q(z)]^{\frac{\gamma}{n}}$.

1. Introduction

Let A_p (p is fixed integer ≥ 1) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Let Ω denote the class of bounded function $w(z)$ analytic in D and satisfying the conditions $w(0) = 0$ and $|w(z)| \leq |z|$, $z \in D$. We use P to denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ which are analytic in D and a positive real part there.

For $0 \leq \alpha < p$ and $|\lambda| < \frac{\pi}{2}$, we denote by $S_p^\lambda(\alpha)$, the family of functions $g(z) \in A_p$ which satisfy

$$(1.1) \quad \frac{zg'(z)}{g(z)} \prec \frac{p + \{2(p - \alpha) \cos \lambda \cdot \exp(-i\lambda) - p\}z}{1 - z}, \quad z \in D$$

where $\prec$ means subordination. From the definition of subordination it follows that $g(z) \in A_p$ has a representation

$$\frac{zg'(z)}{g(z)} = \frac{p + \{2(p - \alpha) \cos \lambda \cdot \exp(-i\lambda) - p\}w(z)}{1 - w(z)}$$

where $w(z) \in \Omega$. Clearly, $S_p^\lambda(\alpha)$ is subclass of p -valent λ -spiral functions of order α . For $\lambda = 0$, we have the class $S_p^*(\alpha)$, $0 \leq \alpha < p$, of p -valent starlike functions of order α , investigated by Goluzina [3].

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As noted in a function is p -valent strongly starlike of order γ , $0 < \gamma \leq 1$ if

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leq \frac{\pi}{2} \gamma.$$

Basgöze(1969) has obtained sharp inequalities of univalence(starlikeness) for certain polynomials of the form $F(z) = f(z) \cdot [Q(z)]^{\frac{\beta}{n}}$, where β is real and $Q(z)$ is a polynomial of degree $n > 0$ all of whose zeros are outside or on the unit circle $\{z \in D : |z| = 1\}$. Rajasekaran [5] extended Basgöze's results for certain classes of analytic functions of the form. Recently, J. Patel [4] generalized some of the work of Rajasekaran and Basgöze for functions belonging to the class $S_p^\lambda(\alpha)$. That is, determine the radius of starlikeness for some classes of p -valent analytic functions of the polynomial form $F(z)$.

In the present paper, we will extend the results of J. patel. Thus, we determine the radius of p -valent strongly starlike of order γ for the polynomials of the form $F(z)$ in the such problems.

2. Some Lemmas

Before proving our next results, we need the following Lemmas.

Lemma 1 (A. Gangadharan [2]). For $|z| \leq r < 1$, $|z_k| = R > r$, we have

$$\left| \frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2} \right| \leq \frac{Rr}{R^2 - r^2}.$$

Lemma 2 (Ratti [6]). If $\phi(z)$ is analytic in D and $|\phi(z)| \leq 1$ for $z \in D$, then for $|z| = r < 1$,

$$\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{1}{1 - r}.$$

Lemma 3 (Causey and Merke's [1]). If $p(z) = 1 + c_1z + c_2z + \dots \in P$, then for $|z| = r < 1$,

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2}.$$

This estimate is sharp.

Lemma 4 (J. Patel [4]). Suppose $g(z) \in S_p^\lambda(\alpha)$. Then for $|z| = r < 1$,

$$\left| \frac{zg'(z)}{g(z)} - \left\{ p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos \lambda}{1-r^2} \right\} \right| \leq \frac{2(p-\alpha)r \cos \lambda}{1-r^2}.$$

The result is sharp.

Lemma 5 (A. Gangadharan [2]). If $R_a \leq (\operatorname{Re} a) \sin\left(\frac{\pi}{2}\gamma\right) - (\operatorname{Im} a) \cos\left(\frac{\pi}{2}\gamma\right)$, $\operatorname{Im} a \geq 0$, the disk $|w - a| \leq Ra$ is contained in the sector $|\arg w| \leq \frac{\pi}{2}\gamma$, $0 < \gamma \leq 1$.

3. Main Theorem

Theorem 1. Suppose

$$(3.1) \quad F(z) = f(z)[Q(z)]^{\frac{\beta}{n}}$$

where β is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$, $R \geq 1$. If $f(z) \in A_p$ satisfies

$$(3.2) \quad \operatorname{Re} \left[\left(\frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}} \right] > 0, \quad 0 < \delta \leq 1, \quad z \in D$$

and

$$(3.3) \quad \operatorname{Re} \left[\frac{g(z)}{h(z)} \right] > 0, \quad z \in D$$

for some $g(z) \in A_p$ and $h(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p -valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$(3.4) \quad \begin{aligned} & r^4 \left[(p + \beta) \sin \frac{\pi}{2}\gamma + 2(p - \alpha) \cos \lambda \sin(\lambda - \frac{\pi}{2}\gamma) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2(\delta + 1)] \\ & - r^2 \left[(p(1 + R^2) + \beta) \sin \frac{\pi}{2}\gamma + 2(p - \alpha)R^2 \cos \lambda \sin(\lambda - \frac{\pi}{2}\gamma) \right] \\ & - r [|\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2(\delta + 1)R^2] \\ & + pR^2 \sin \frac{\pi}{2}\gamma. \end{aligned}$$

Proof. We choose a suitable branch of $(f(z)/g(z))^{\frac{1}{\delta}}$ so that $(f(z)/g(z))^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z = 0$. Thus from (3.2) and (3.3), we have

$$(3.5) \quad F(z) = p_1^\delta(z) p_2 h(z) [Q(z)]^{\frac{\beta}{n}}$$

where $p_j(z) \in P$ ($j = 1, 2$).

Then we have

$$(3.6) \quad \frac{zF'(z)}{F(z)} = \delta \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{zh'(z)}{h(z)} + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}.$$

Since $h(z) \in S_p^\lambda(\alpha)$, by Lemma 4, we have

$$(3.7) \quad \left| \frac{zh'(z)}{h(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right\} \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}.$$

Using (3.6) and (3.7) and Lemma 1, 3, we get

$$(3.8) \quad \left| \frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right\} \right| \leq \frac{2\{(p - \alpha)r \cos \lambda + r(\delta + 1)\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}.$$

Using Lemma 5, we get that the about disk is contained in the sector $|\arg w| < \frac{\pi}{2}\gamma$ provided the inequality

$$\begin{aligned} & \frac{2\{(p - \alpha)r \cos \lambda + r(\delta + 1)\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2} \\ & \leq \left\{ p + \frac{2(p - \alpha)r^2 \cos^2 \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right\} \sin \frac{\pi}{2}\gamma - \frac{2(p - \alpha)r^2 \sin \lambda \cos \lambda}{1 - r^2} \cos \frac{\pi}{2}\gamma \end{aligned}$$

is satisfied. The above inequality simplifies to $T(r) \geq 0$, where

$$\begin{aligned} T(r) = & r^4 \left[(p - 2(p - \alpha) \cos^2 \lambda + \beta) \sin \frac{\pi}{2}\gamma + (p - \alpha) \sin 2\lambda \cos \frac{\pi}{2}\gamma \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2(\delta + 1)] \\ & + r^2 \left[(-pR^2 - p + 2(p - \alpha)R^2 \cos^2 \lambda - \beta) \sin \frac{\pi}{2}\gamma - (p - \alpha)R^2 \sin 2\lambda \cos \frac{\pi}{2}\gamma \right] \\ & - r [|\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2(\delta + 1)R^2] + pR^2 \sin \frac{\pi}{2}\gamma \end{aligned}$$

Since $T(0) > 0$ and $T(1) < 1$, there exists a real root of $T(r) = 0$ in $(0, 1)$. Let $R(\gamma)$ be the smallest positive root of $T(r) = 0$ in $(0, 1)$. Then F is p -valent strongly starlike of order γ in $|z| < R(\gamma)$.

Remark. For $R = 1$ and $\gamma = 1$, the above theorem reduces to a result of J. Patel.

Theorem 2. Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies (3.2) for some $g(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p -valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned}
 & r^4 \left[(p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\
 & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2\delta] \\
 (3.9) \quad & - r^2 \left[(p(1 + R^2) + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha)R^2 \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\
 & - r[|\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2\delta R^2] \\
 & + pR^2 \sin \frac{\pi}{2} \gamma.
 \end{aligned}$$

Proof. If $f(z) \in A_p$ satisfies (3.2) for some $g(z) \in S_p^\lambda(\alpha)$, then

$$(3.10) \quad \frac{zF'(z)}{F(z)} = \delta \cdot \frac{zp'(z)}{p(z)} + \frac{zg'(z)}{g(z)} + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}.$$

Using Lemma 4, we get

$$(3.11) \quad \left| \frac{zg'(z)}{g(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right\} \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}.$$

By (3.10) and (3.11) and Lemma 1, 3, we have

$$\begin{aligned}
 & \left| \frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right\} \right| \\
 & \leq \frac{2\{(p - \alpha)r \cos \lambda + r\delta\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}.
 \end{aligned}$$

The remaining parts of the proof can be proved by similar method given in the Theorem 1.

With $\lambda = 0$, $\beta = 0$, $\delta = 1$, $R = 1$ and $\gamma = 1$, Theorem 2 gives

Corollary 1. Suppose $f(z)$ is in A_p . If $\operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0$ for $z \in D$ and $g(z) \in S_p^*(\alpha)$, then $f(z)$ is p -valent starlike for

$$|z| < \frac{p}{(p + 1 - \alpha) + \sqrt{\alpha^2 - 2\alpha + 2p + 1}}.$$

Theorem 3. Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies

$$(3.12) \quad \left| \left(\frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}} - 1 \right| < 1, \quad 0 < \delta \leq 1, \quad p \sin \frac{\pi}{2} \gamma > \delta$$

and

$$\operatorname{Re} \left(\frac{g(z)}{h(z)} \right) > 0, \quad z \in D$$

for some $g(z) \in A_p$ and $h(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p -valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$(3.13) \quad \begin{aligned} & r^4 \left[(p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2 + \delta] \\ & - r^2 \left[(p(1 + R^2) + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) R^2 \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) + \delta \right] \\ & - r [|\beta|R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2] + p R^2 \sin \frac{\pi}{2} \gamma - \delta R^2. \end{aligned}$$

Proof. We choose a suitable branch of $\left(\frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}}$ so that $\left(\frac{f(z)}{g(z)} \right)^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z = 0$. From (3.12), we deduce that

$$f(z) = g(z) \cdot (1 + w(z))^\delta, \quad \text{where } w(z) \in \Omega.$$

So that

$$F(z) = p(z) \cdot h(z) \cdot (1 + z\phi(z))^\delta [Q(z)]^{\frac{\beta}{n}}$$

where $\phi(z)$ is analytic in D and satisfies $|\phi(z)| \leq 1$ and $p \in P$ for $z \in D$.

We have

$$(3.14) \quad \frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)} + \delta \left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right) + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}.$$

Using Lemma 4 and (3.14), we have

$$(3.15) \quad \begin{aligned} & \left| \frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right\} \right| \\ & \leq \frac{2\{(p - \alpha)r \cos \lambda + r\} + \delta(1 + r)}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2} \end{aligned}$$

So, using Lemma 5 and (3.15), the result can be proved by similar method given in the Theorem 1.

Theorem 4. Suppose $F(z)$ is given by (3.1). If $f(z) \in A_p$ satisfies (3.12) for some $g(z) \in S_p^\lambda(\alpha)$, then $F(z)$ is p -valent strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is smallest positive root of the equation

$$\begin{aligned}
 & r^4 \left[(p + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) \cos \lambda \sin(\lambda - \frac{\pi}{2} \gamma) \right] \\
 & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + \delta] \\
 (3.16) \quad & - r^2 \left[(p(1 + R^2) + \beta) \sin \frac{\pi}{2} \gamma + 2(p - \alpha) R^2 \cos \lambda \sin(\lambda - \frac{\pi}{2} \gamma) + \delta \right] \\
 & - r [|\beta|R + 2(p - \alpha) R^2 \cos \lambda + \delta R^2] \\
 & + p R^2 \sin \frac{\pi}{2} \gamma - \delta R^2.
 \end{aligned}$$

Proof. We choose a suitable of $(f(z)/g(z))^{\frac{1}{\delta}}$ so that $(f(z)/g(z))^{\frac{1}{\delta}}$ is analytic in D and takes the value 1 at $z = 0$. Since $f(z) \in A_p$ (3.12) for some $g(z) \in S_p^\lambda(\alpha)$, we have

$$F(z) = g(z)(1 + z\phi(z))[Q(z)]^{\frac{\beta}{n}}$$

where $\phi(z)$ is analytic in D and satisfies the condition $|\phi(z)| \leq 1$ for $z \in D$. Thus, we have

$$(3.17) \quad \frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \delta \left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right) + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}.$$

Using Lemma 4 and (3.17), we get

$$\begin{aligned}
 (3.18) \quad & \left| \frac{zF'(z)}{F(z)} - \left\{ p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right\} \right| \\
 & \leq \frac{2(p - \alpha)r \cos \lambda + \delta(1 + r)}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}
 \end{aligned}$$

Using Lemma 5 and (3.18) and similar method in the Theorem 1, we get the Theorem 4.

Remark. Some of the results of J. Patel can be obtained from the Theorem 4 by taking $R = 1$, $\gamma = 1$.

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