

Ergodic theory of diffeomorphisms

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1. A "almost everywhere" stable manifold theorem

Theorem. Let M be a compact differentiable manifold and
 $f : M \rightarrow M$ a diffeomorphism of class $C^{r,\theta}$ (r integer ≥ 1 ,
 $\theta \in (0,1]$). Let d be a Riemann metric on M .

There is a Borel set $\Gamma \subset M$ with the following properties

(a) $f\Gamma \subset \Gamma$ and $\sigma(\Gamma) = 1$ for every f -invariant proba-
bility measure σ on M

(b) Let $x \in \Gamma$ and $\lambda_x^{(1)} < \dots < \lambda_x^{(r)}$ be the strictly
negative characteristic exponents of Tf . Define $\gamma_x^{(1)} \subset \dots \subset \gamma_x^{(r)}$
by

$$\gamma_x^{(p)} = \{y \in M : \limsup_{n \rightarrow \infty} \frac{1}{n} \log d(f^n x, f^n y) \leq \lambda_x^{(p)}\}$$

for $p = 1, \dots, r$. Then $\gamma_x^{(p)}$ is the image of $V_x^{(p)}$ by an
injective $C^{r,\theta}$ immersion tangent to the identity at x .

- Characteristic exponents will be explained later.

- This is an "almost everywhere" stable manifold theorem,
 where several stable manifolds with different rates of convergence
 may be present at a point.

Corollary. If ρ is ergodic and all the characteristic
exponents of Tf are < 0 a.e., then ρ is carried by an
attracting periodic orbit.

- The proof of the theorem can be reduced to proving the existence of local stable manifolds.

- Using exponential maps, the local theorem can be formulated as a theorem about invariant manifolds for a non linear vector bundle map T over $\tau : M \rightarrow M$. The differentiability of $T_x : E_x \rightarrow E_{\tau x}$ is used, but $\tau : M \rightarrow M$ is just assumed to be a measure preserving transformation.

- In particular, one can take the vector bundle to be trivial, i.e. one studies the ergodic properties of nonlinear maps F_x , $x \in M$, such that F_x maps the unit ball of $\mathbb{R}^m$, into $\mathbb{R}^m$ and $F_x 0 = 0$.

- The linear version of this problem is the multiplicative ergodic theorem which we have to study first.

2. The multiplicative ergodic theorem.

Let (M, Σ, ρ) be a fixed probability space, and $\tau : M \rightarrow M$ a measurable map preserving ρ . We denote by f^+ the positive part of a real function f .

Theorem. Let $T : M \rightarrow M_m$ be a measurable function to the real $m \times m$ matrices such that

$$\log^+ \|T(\cdot)\| \in L^1(M, \rho)$$

and write $T_x^n = T(\tau^{n-1}x) \dots T(\tau x)T(x)$.

There is $\Gamma \subset M$ such that $\tau\Gamma \subset \Gamma$ and $\rho(\Gamma) = 1$. Furthermore,
if $x \in \Gamma$, $u \in \mathbb{R}^m$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T_x^n u\| = \chi(x, u)$$

exists, finite or $-\infty$.

The values of $\chi(x, u)$ for $u \neq 0$ are called characteristic exponents. Notice that

$$V_x^\lambda = \{u \in \mathbb{R}^m : \chi(x, u) \leq \lambda\}$$

is a linear subspace of $\mathbb{R}^m$.

Complement.

Let $*$ denote matrix transposition. One may take Γ such that, if $x \in \Gamma$,

$$(a) \quad \lim_{n \rightarrow \infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x \quad \text{exists}$$

(b) the characteristic exponents $\lambda_x^{(r)}$ are the log's of the eigenvalues of Λ_x . The space V_x^λ is the sum of the eigenspaces $U_x^{(r)}$ of Λ_x corresponding to the eigenvalues $\leq \lambda$. The functions $x \mapsto \lambda_x^{(r)}$, $x \mapsto m_x^{(r)} = \dim U_x^{(r)}$ are τ -invariant.

The proof can be obtained in two steps.

I. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(T_x^n)^{\wedge q}\| \quad (*)$$

exists almost everywhere (this follows from a "subadditive ergodic theorem" and insures the existence of a limit for the sum

of the largest q eigenvalues of $(T_x^{n*} T_x^n)^{1/2n}$.

II. From

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|T(\tau^n x)\| \leq 0 \quad (**)$$

and (*) for $q = 1, \dots, m$ one obtains without further assumption the existence of the limits asserted by the multiplicative ergodic theorem.

3. Proof of a local stable manifold theorem

To prove the desired nonlinear version of the multiplicative ergodic theorem, we put

$$F_x^n = F_{\tau^{n-1}x} \circ \dots \circ F_{\tau x} \circ F_x$$

and assume that

$$\int \rho(dx) \log^+ \|F_x\|_{r,\theta} < +\infty.$$

We want to prove the existence of a measurable set $\Gamma \subset M$ with $\tau\Gamma \subset \Gamma$, $\rho(\Gamma) = 1$, and measurable functions $\beta > \alpha > 0$ on Γ such that if $x \in \Gamma$, and $\lambda < 0$ is not a characteristic exponent of $T = Tf$ at x ,

$$D_x = \{u \in \mathbb{R}^m: \|u\| \leq \alpha(x), \|F_x^n u\| \leq \beta(x)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{r,\theta}$ submanifold of the ball $\|u\| \leq \alpha(x)$, tangent at 0

to V_x^λ .

If F_x is replaced by its linear part $T(x) = T_x f$, this follows from the multiplicative ergodic theorem. The idea of the proof of the nonlinear theorem is to consider F_x as a perturbation of $T(x)$. If $u \in D_x$, $F_x^n u$ tends exponentially fast (with n) to 0, therefore the deviation of F_x^n from $T(\tau_x^n)$, at the relevant point $F_x^n u$, goes exponentially to zero. The heart of the proof reduces thus to the following fact.

If (T'_x) is a sequence of $n \times n$ matrices and

$$\sup_n \|T'_n - T(\tau_x^{n-1})\| e^{n\eta}$$

is sufficiently small (for some $\eta > 0$, and $T(\tau_x^{n-1})$ such that the limits (*) exist and (**) holds), then, if we write

$$T'^n = T'_n \cdots T'_1$$

the limit

$$\lim_{n \rightarrow \infty} (T'^n * T^n)^{1/2n} = \Lambda'_x$$

exists and has the same eigenvalues (including multiplicity) as Λ_x . The eigenspaces depend continuously on the perturbation.

4. Abstract results about matrix products

1. Theorem. Let $T = (T_n)_{n>0}$ be a sequence of real $m \times m$

matrices such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|T_n\| \leq 0$$

We write $T^n = T_n \cdots T_2 \cdot T_1$ and assume that the limits

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\|$$

exist for $q = 1, \dots, m$.

$$(a) \quad \lim_{n \rightarrow \infty} (T^{n*} T^n)^{1/2n} = \Lambda$$

exists, where $*$ denotes matrix transposition.

(b) Let $\exp \lambda^{(1)} < \dots < \exp \lambda^{(s)}$ be the eigenvalues of
 Λ [real $\lambda^{(r)}$, possibly $\lambda^{(1)} = -\infty$], and $U^{(1)}, \dots, U^{(s)}$ the
corresponding eigenspaces. Writing $V^{(0)} = \{0\}$ and $V^{(r)} =$
 $U^{(1)} + \dots + U^{(r)}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T^n u\| = \lambda^{(r)} \quad \text{when } u \in V^{(r)} \setminus V^{(r-1)} \quad \text{for}$$

$r = 1, \dots, s$.

① The eigenvalues of $(T^{n*} T^n)^{1/2n}$ send to limits

$$\lambda^{(1)} < \dots < \lambda^{(s)}$$

- Let $U_n^{(r)}$ be the space spanned by the eigenvectors of $(T^{n*} T^n)^{1/2n}$ corresponding to eigenvalues sending to $\lambda^{(r)}$.

② Lemma. Given $\delta > 0 \exists K > 0$ s.t., for all $k > 0$,

$$\begin{aligned} & \max\{|(u, u')| : u \in U_n^{(r)}, u' \in U_{n+k}^{(r')}, \|u\| = \|u'\| = 1\} \\ & \leq K \exp[-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta)] \end{aligned}$$

Equivalently: if $\lambda^{(r)} = \lambda$, $\lambda^{(r')} = \lambda'$, $U_n^{(r)} = U_n$, $U_n^{(r')} = U'_n$,
if v is the orthogonal projection in U'_{n+k} of $u \in U_n$, then

$$\|v\| \leq K \|u\| \exp[-n(|\lambda' - \lambda| - \delta)]$$

- If $\lambda < \lambda'$, $k = 1$, then for large n

$$\|v\| \exp[(n+1)(\lambda' - \frac{\delta}{4})] \leq \|T^{n+1}u\|$$

$$\leq \|T_{n+1}\| \|T^n u\| \leq \exp[C + (n+1)\frac{\delta}{2}] \cdot \|u\| \exp[n(\lambda + \frac{\delta}{4})]$$

$$\Rightarrow \|v\| \leq \exp[C - \lambda' + \frac{3}{4}\delta] \cdot \exp[-n(\lambda' - \lambda - \delta)] \cdot \|u\|$$

- Induction on k ($\lambda < \lambda'$)

- Orthogonality

③ $(U_n^{(r)})_{n>0}$ is Cauchy $\Rightarrow$ (a) $U_n^{(r)} \rightarrow U^{(r)}$

$$\Rightarrow \max\{|(u, u')| : u \in U^{(r)}, u' \in U_n^{(r')}, \|u\| = \|u'\| = 1\}$$

$$\leq K \exp[-n(|\lambda^{(r')} - \lambda^{(r)}| - \delta)]$$

$\Rightarrow$ (b)

2. Theorem. Let the notation and assumptions be as in theorem 1. Furthermore, assume that $\det \Lambda \neq 0$.

Let $\eta > 0$ be given and, for $T' = (T'_n)_{n>0}$, write

$$\|T' - T\| = \sup_n \|T'_n - T_n\| e^{3n\eta}$$

and $T'^n = T'_n \cdots T'_2 \cdot T'_1$. Then there are $\delta, A > 0$ and, given $\epsilon > 0$, there are $B_\epsilon > 0$, $B'_\epsilon > 1$ with the following properties.
If $\|T' - T\| < \delta$

$$\lim_{n \rightarrow \infty} (T'^n T'^n)^{1/2n} = \Lambda' \quad (1)$$

exists and has the same eigenvalues as Λ (including multiplicity).
Furthermore, if $P^{(r)}(T')$ denotes the orthogonal projection of Λ' corresponding to $\exp \lambda^{(r)}$, and $\|T'' - T\| < \delta$, we have

$$\|P^{(r)}(T') - P^{(r)}(T'')\| \leq A \|T' - T''\| \quad (2)$$

$$B_\epsilon \exp n(\lambda^{(r)} - \epsilon) \leq \|T'^n P^{(r)}(T')\| \leq B'_\epsilon \exp n(\lambda^{(r)} + \epsilon) \quad (3)$$

① To prove the existence of (1) and spectrum $\Lambda = \text{spectrum } \Lambda'$, it suffices to show

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|(T'^n)^{\wedge q}\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(T^n)^{\wedge q}\|$$

In fact it suffices to do this for $q = 1$. Equivalently, it suffices to find an open set $U \subset \mathbb{R}^m$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T'^n u\| = \lambda^{(s)} \quad \text{for } u \in U$$

This results from the following

② Lemma. Let $(\xi_1, \dots, \xi_m)$ be an orthonormal basis of $\mathbb{R}^m$ diagonalizing Λ , with ξ_m corresponding to the largest eigen-
value $\exp \lambda^{(s)}$. There is then $\delta > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|T'^n u\| = \lambda^{(s)}$$

whenever $0 < \alpha \leq 1$, $\|T' - T\| \leq \delta \alpha$, and $u \in U$, where

$$U = \left\{ \sum_{k=1}^{m-1} u_k \frac{\xi_k}{\alpha} + u_m \xi_m : \max_{k < m} |u_k| < |u_m| \right\}$$

③ The lemma implies $\|P^{(r)}(T') - P^{(r)}(T)\| \leq A \|T' - T\|$

④ Proof of the lemma ($\alpha = 1$ for simplicity).

Let $\xi_k^{(n)}$: unit vector $\sim T^n \xi_k$, and $\xi^{(n)}$ the matrix with columns $\xi_k^{(n)}$. Then $\|\xi_k^{(n)}\| < \sqrt{m}$ and

$$D_\varepsilon = \sup_n e^{-n\varepsilon} \|\xi^{(n)-1}\| < +\infty \quad \text{if } \varepsilon > 0$$

$$T_n \xi_k^{(n-1)} = t_k^{(n)} \xi_k^{(n)} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{j=1}^n t_k^{(j)} = \lambda^{(r(k))}$$

where we may assume $r(k)$ increasing with k .

- For any $u \in \mathbb{R}^m$, let $T'^n u = \sum_k u_k^{(n)} \xi_k^{(n)}$

Let μ be the smallest integer such that

$$(\forall n) \quad \max_{j \leq \mu} |u_j^{(n)}| \geq \max_{k > \mu} |u_k^{(n)}|$$

Assuming $\|T'-T\| \leq \delta$, we estimate the $u_k^{(n)}$ recursively

$$- \quad |u_k^{(n)}| \leq t_k^{(n)} |u_k^{(n-1)}| + D\delta e^{-2n\eta} \sum_{\ell} |u_{\ell}^{(n-1)}|$$

Replace the $t_k^{(n)}$ by $t_k^{(n)*}$ so that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r=1}^N \log t_k^{(n)*} = \lambda(r(\mu)) \quad \text{for } k \leq \mu$$

$$t_{\mu}^{(n)*} = t_{\mu}^{(n)}$$

Choose C such that (for all $v \geq 0$, $N > v$, $k, \ell \leq \mu$)

$$\prod_{n=v+1}^{N-1} t_{\ell}^{(n)*} / \prod_{n=v+1}^N t_k^{(n)*} \leq C e^{N\eta}$$

Then, if $U^{(v)} = \max_{\ell} |u_{\ell}^{(v)}|$,

$$|u_k^{(n)}| \leq \prod_{n=v+1}^N t_k^{(n)*} \cdot \prod_{n=v+1}^N (1+mCD\delta e^{-n\eta}) U^{(v)}$$

Choosing $\delta = \frac{1}{mCD} \prod_{n=1}^{\infty} (1-e^{-n\eta})^2$ yields, for $N > v$

$$\boxed{|u_k^{(N)}| \leq C' \prod_{n=v+1}^N t_k^{(n)*} \cdot \prod_{n=v+1}^N (1-e^{-n\eta}) \cdot U^{(v)}} \quad (4)$$

with $C' \leq \frac{1}{mCD\delta}$

- Choose v so that $|u_{\mu}^{(v)}| = \max_k |u_k^{(v)}| = U^{(v)}$, then

$$\boxed{|u_{\mu}^{(N)}| \geq \prod_{n=v+1}^N t_{\mu}^{(n)} \cdot \prod_{n=v+1}^N (1-e^{-n\eta}) \cdot U^{(v)}} \quad (5)$$

$$- \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|T^n u\| = \lambda^{(r(\mu))} \Rightarrow \text{lemma } (r(\mu)=s).$$

⑤ (2) and (3) be obtained from (4) and (5).

For instance the second half of (3) follows from (4).