

Type II_1 -factors with property T

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The property T for von Neumann algebras are defined by Connes-Jones in [5]. It is a reformulation of property T for groups due to Kazhdan [8]. The distinguished property of II_1 -factors with property T is that the fundamental group of them are countable, and the group von Neumann algebra of a countable group G has property T if and only if G has property T [5]. Here I will discuss on type II_1 -factors with property T.

In 2, I shall give some characterization of property T of type II_1 -factors.

In 3, I shall investigate strongly ergodic outer actions of a group with property T on the hyperfinite II_1 -factor R . In the case of an amenable discrete countable group K , for every outer action α of G on R , the crossed product $W^*(K, R, \alpha)$ is again a hyperfinite II_1 -factor [4]. More precisely, in [9], Ocneanu proved that all outer actions of such a group on R are cocycle conjugate. In [7], Jones showed that the converse of this result is not true, that is, there are two non cocycle conjugate outer actions of a given non amenable group G on R . His second action is given as the tensor product $\beta \otimes \text{id}$ of the so-called non-commutative Bernoulli shift action β and the trivial action id . Obviously, the action $\beta \otimes \text{id}$ is not ergodic. My first question is the following ; are there two ergodic non cocycle conjugate outer actions of a given non amenable group on R ? If two actions are cocycle conjugate, then the crossed products

associated with them are isomorphic [10]. However, in general, the converse of this does not hold. So my second question is the following ; are there two ergodic outer actions of a group on R , which construct non isomorphic factors ? The result in §3 gives an answer for this second question.

2. Characterizations of property T for II_1 -factors.

Let N be a type II_1 -factor. A Hilbert space H is a correspondence from N to N if there are commuting both side actions of N on H which are continuous, that is, if (x_λ) in N converges to $x \in N$ in the $*$ -strong topology, then $(x_\lambda \xi)$ and (ξx_λ) converge to $x \xi$ weakly for all $\xi \in H$. The set of all correspondences from N to N is a topological space with some topology and N is said to have property T if the trivial correspondence $L^2(N, tr)$ is an isolated point in the set, which is equivalent to the following ; there are an $\varepsilon > 0$ and a finite subset $\{x_1, \dots, x_n\}$ of N such that if a correspondence H from N to N contains a norm 1 vector ξ with $\|x_i \xi - \xi x_i\| < \varepsilon$ ($i = 1, \dots, n$), then there is a non zero vector $\eta \in H$ with $x \eta = \eta x$ for all $x \in N$ ([5]). Such a vector η is called a central vector of N in H .

In [1], Akemann-Walter give a characterization of property T for groups using positive definite functions. First, I shall give a characterization of property T for II_1 -factors by complete positive maps.

Let N be a II_1 -factor and N_* the predual of N . Then N_* is a partially ordered space with the order defined by positivity for linear functionals on N . For $f \in N_*$ and $x \in N$, let

$$x \circ f(y) = f(yx) \quad f \circ x(y) = f(xy) \quad (X \in N).$$

Let consider the *-strong topology for N and norm topology as linear functionals for N_* . Let N° be the opposite algebra of N

Theorem 1. A type II_1 -factor N has property T if and only if each net $\{\varphi_\nu\}$ of continuous completely positive linear maps from N to N_*° , such that $\varphi_\nu(1)$ is a state for each ν , satisfies the following ; if $\varphi_\nu(x) - \varphi_\nu(1) \circ x$ converges to 0 for all $x \in N$, then $\varphi_\nu(x) - \varphi_\nu(1) \circ x$ converges to 0 uniformly on the unit ball N_1 of N .

Proof. Assume that N has property T. Let $\{\varphi_\nu\}$ be a net of continuous completely positive linear maps from N to N_*° such that $\varphi_\nu(1)$ is a state for all ν . A positive sesquilinear form on the algebraic tensor product $N \otimes N^\circ$ is defined by

$$(a \otimes b, c \otimes d) = (\varphi_\nu(c^* a))(bd^*) \quad (a, c \in N, b, d \in N^\circ)$$

because φ_ν is a completely positive linear map of N to N_*° .

Let H be the Hilbert space obtained by this form from $N \otimes N^\circ$.

Let N act on the left and right sides on H by the way ;

$$x(a \otimes b) = xa \otimes b, \quad (a \otimes b)x = a \otimes bx \quad (x, a, b \in N).$$

Since φ_ν is continuous, this actions are normal, so that H

is a correspondence from N to N . Put $\xi_\nu = 1 \otimes 1 \in H_\nu$. Then

$\|\xi_\nu\|^2 = (\varphi_\nu(1))(1) = 1$. Assume that $\|\varphi_\nu(x) - \varphi_\nu(1) \circ x\| \rightarrow 0$ for all $x \in N$. Then, for any unitary $u \in N$,

$$\|u\xi_\nu - \xi_\nu u\|^2 = 2 - 2\operatorname{Re}(\varphi_\nu(u))(u^*) \rightarrow 0.$$

Since a element in N is decomposed into a linear combination of unitaries, by [5], we have a net $\{\eta_\nu\}$ such that

$$0 \neq \eta_\nu \in H, \quad \|\xi_\nu - \eta_\nu\| \rightarrow 0, \quad x\eta_\nu = \eta_\nu x \quad (x \in N).$$

We may assume that $\|\eta_n\| = 1$. Since the functional $(\cdot, \eta_n, \eta_n)$ is the canonical trace on N ,

$$\begin{aligned} & |(\varphi_n(x))(y) - (\varphi_n(1) \circ x)(y)| \\ &= |(\xi_n y, x^* \xi_n) - (\xi_n y, \xi_n x^*)| \\ &\leq |(\xi_n y, x^* \xi_n - x^* \eta_n)| + |(\xi_n y, \eta_n x^* - \xi_n x^*)| \\ &\leq 2 \|\xi_n - \eta_n\| \end{aligned}$$

for all x and y in N_1 . Hence

$$\sup \{ \|\varphi_n(x) - \varphi_n(1) \circ x\| ; x \in N_1 \} \longrightarrow 0.$$

Conversely, assume that N does not have property T. Then there exists a net $\{H_n, \xi_n\}$ of correspondences H_n without non trivial central vectors and unit vectors $\xi_n \in H_n$ such that $\{x \xi_n - \xi_n x\}$ converges to 0 for all $x \in N$. Let take and fix an $x \in N$. Put

$$(\varphi_n(x))(y) = (x \xi_n y, \xi_n).$$

Since the action of N on correspondences is normal, $\varphi_n(x)$ is a σ -weakly continuous linear functional on N and φ_n is a continuous linear map from N to N_*° . Since ξ_n is a unit vector, $\varphi_n(1)$ is a state. Identifying $M_n(N^*)$ with the dual $(M_n(N))^*$ by

$$f(a) = \sum_{i,j} f_{ij}(a_{ij}),$$

for $f = [f_{ij}] \in M_n(N^*)$, $a = (a_{ij}) \in M_n(N)$, in order to show the complete positivity of φ_n , it is sufficient to prove

$$\sum \varphi_n(x_i^* x_j)(a_j a_i^*) = \|\sum_{i=1}^n x_i \xi_n a_i\|^2 \geq 0.$$

Hence each φ_n is completely positive. For an $x \in N$,

$$\begin{aligned} & \|\varphi_n(x) - \varphi_n(1) \circ x\| \\ &= \sup \{ |(x \xi_n y, \xi_n) - (\xi_n y x, \xi_n)| ; y \in N_1 \} \end{aligned}$$

$$\leq \| x^* \tilde{f}_n - \tilde{f}_n x^* \| \rightarrow 0.$$

Suppose that $\{ \varphi_n(x) - \varphi_n(1) \cdot x \}$ converges to 0 uniformly on N_1 . Then for any $\varepsilon > 0$, there are n such that

$$\| \varphi_n(x) - \varphi_n(1) \cdot x \| < \varepsilon \quad \text{for all } x \in N_1.$$

So there exists a $\tilde{f}_n$ such that

$$| (u \tilde{f}_n u^*, \tilde{f}_n) - 1 | < \varepsilon < 1 \quad \text{for all unitary } u \in N.$$

Hence there exists a non zero $\eta \in H_n$ such that $u\eta = \eta u$ for all unitary $u \in N$ (cf, [6]). This contradicts that H_n has no non zero central vectors. Hence $\{ \varphi_n(x) - \varphi_n(1) \cdot x \}$ does not converge to 0 uniformly on N_1 .

Corollary 2. Assume that a II_1 -factor has property T. Let $\{ \varphi_n \}$ be a net of continuous completely positive linear maps on N such that $\varphi_n(1) = 1$. If $\| \varphi_n(x) - x \|_2 \rightarrow 0$ for all $x \in N$, then

$$\sup \{ \| \varphi_n(x) - x \|_2 ; x \in N, \| x \| \leq 1 \} \rightarrow 0.$$

Let M be a von Neumann algebra and H a correspondence from M to M . A linear map δ from M to H is called a derivation if $\delta(ab) = a\delta(b) + \delta(a)b$ for all a and b in the domain $D(\delta)$ of δ . A linear map δ from M to H is called bounded if $\sup \{ \| \delta(a) \| ; a \in M, \| a \| \leq 1 \} < +\infty$.

Theorem 3. If a type II_1 -factor N does not have property T, there exists a unbounded derivation from N to a correspondence, the domain of which is a weakly dense *-subalgebra of N containing the identity.

Proof. Let S_0 be a countable subset of N_1 such that S_0 is dense in N_1 with respect to $\|\cdot\|_2$ defined by the canonical trace tr of N . We may assume that S_0 contains the identity of N . Let S be the set of products of finite elements in $S_0 \cup S_0^*$. Then S is a countable $\|\cdot\|_2$ -dense subset of N_1 . Put $S = \{x_1, x_2, \dots, x_n, \dots\}$.

If N does not have property $\mathcal{T}$, then for each integer n , there exist a correspondence H_n from N to N without non zero central vector for N and a $\xi_n \in H_n$ such that $\|\xi_n\| = 1$ and that $\|x_j \xi_n - \xi_n x_j\| < 1/4^n$. Assume that for each $\varepsilon > 0$ there exists a integer m such that $\|x_i \xi_m - \xi_m x_i\| < \varepsilon$ for all i . Since S is a $\|\cdot\|_2$ -dense in N_1 , for any $\varepsilon > 0$, there exists an m such that $\|u \xi_m u^* - \xi_m\| < \varepsilon$ for all unitary $u \in N$. Let ζ be $\|\zeta\| = \inf \{\|\xi\|; \xi \text{ is in the norm closure of the convex hull of } u \xi_m u^*, u \text{ is a unitary in } N\}$. Then ζ is a non zero central vector in H_m . This is a contradiction. Hence we have a sequence $\{H_n, \xi_n \in H_n, \alpha > 0, a_n \in S\}$ such that $\|\xi_n\| = 1, \|x_j \xi_n - \xi_n x_j\| < 1/4^n (1 \leq j \leq n), \|a_n \xi_n - \xi_n a_n\| \geq \alpha$.

Put

$$H = \sum_{n=1}^{+\infty} \oplus H_n.$$

We shall define a new action of N on H by

$$x \zeta = (x \zeta_n), \quad \zeta x = (\zeta_n x) \quad (x \in N, \zeta_n \in H_n, \zeta = (\zeta_n) \in H).$$

Then H is an again correspondence from N to N . For each n , let $\widetilde{\xi}_n$ be $\zeta = (\zeta_i) \in H$ such that $\zeta_n = \xi_n$ and $\zeta_i = 0$ if $i \neq n$. Put

$$\delta(x) = \sum_{n=1}^{+\infty} 2^n (x \widetilde{\xi}_n - \widetilde{\xi}_n x), \quad (x \in S).$$

Let take an $x \in S$, then $x = x_i$ for some i . Since

$$\|x_i \xi_n - \xi_n x_i\| < 1/4^n \text{ for all } n > i, \text{ the series converges.}$$

Hence δ is well defined on S and we shall extend δ to the set $D(\delta)$ of finite combinations of elements in S by the natural method. Then δ is a derivation of N to the correspondence H . By the following inequality, δ is unbounded ;

$$\|\delta(a_m)\|^2 = \sum_{n=1}^{+\infty} 2^{2n} \|a_m \xi_n - \xi_n a_m\|^2 \geq 2^{2m} \alpha^2.$$

A type II_1 -factor N with property T does not have property $\overline{T}$ of Murray and von Neumann. A. Connes gives some characterization of property $\overline{T}$ ([41]). Next, we shall give a characterization of a negation of property T .

Theorem 4. A II_1 -factor N does not have property T if and only if, for any finite subset $\{x_1, \dots, x_n\}$ of N , there are a correspondence H from N to N with a non trivial central vector and a sequence (ξ_k) of unit vectors in H such that $\|x_j \xi_k - \xi_k x_j\| \xrightarrow[k \rightarrow \infty]{} 0$ for all $j = 1, \dots, n$ but that (ξ_k) does not tend to any central vector when $k \rightarrow \infty$.

Proof. Assume that N does not have property T . Let take any finite subset $\{x_1, \dots, x_n\}$ of N . Then for any integer k , there is a correspondence H_k from N to N with a unit vector η_k such that $\|\eta_k x_j - x_j \eta_k\| < 1/k$ for all $j = 1, \dots, n$ but H_k does not contain any non zero central vector. Let tr be the canonical trace. Let

$$H = L^2(N, \text{tr}) \oplus \sum_{k=1}^{+\infty} \oplus H_k.$$

For an $x \in N$ and $\xi = (\xi_j) (\xi_0 \in L^2(N, \text{tr}), \xi_k \in H_k \text{ for } k \geq 1)$,

let $x \zeta = (x \zeta_j)$, and $\zeta x = (\zeta_j x)$. Then H is a correspondence from N to N . Since all H_k 's have no non zero central vectors, central vectors in H are contained in $\mathbb{C}1 \subset L^2(N, \text{tr})$. Let ζ_k be the vector in H such that $\zeta_k = (0, \dots, 0, \gamma_k, 0, \dots)$, where γ_k is in the k -component. Then the sequence (ζ_k) satisfies the properties of the statement.

Conversely, assume that N has property T . Then by [5], there exist $\varepsilon > 0$, $y_1, \dots, y_m \in N$, $K > 0$ such that for any correspondence H from N to N and a vector $\zeta \in H$, $\|\zeta\| = 1$ with $\|y_j \zeta - \zeta y_j\| < \varepsilon' < \varepsilon$, there is a central vector γ with $\|\gamma - \zeta\| < K \varepsilon'$. Let take $\{y_1, \dots, y_m\}$ as a finite subset $\{x_1, \dots, x_n\}$. If there are such a correspondence H from N to N and a sequence (ζ_k) as in the statement, (ζ_k) must converge to some central vector. This is a contradiction.

In the case where we can take the trivial correspondence as the correspondence H in Theorem, it is a characterization of property T .

Let N be a II_1 -factor and H a correspondence from N to N . Let α be the unitary representation of the unitary group $U(N)$ of N defined by $\rho(u)\zeta = u \zeta u^*$ for all $u \in U(N)$ and $\zeta \in H$.

In [1], Akemann-Walter gave a characterization of property T for locally compact groups using some distinguished projection. Next we shall give a characterization of property T for factors by some projection.

Let H be a correspondence from N to N with a non zero central vector. Let P_c be the projection of H onto the subspace

spanned by all central vectors.

Theorem 5. A II_1 -factor N has property T if and only if,
for every correspondence with non zero central vectors, the pro-
jection P_c is contained in the C^* -algebra $C^*(\mathcal{F}(U(N)))$
generated by $\mathcal{F}(U(N))$.

Proof. Assume that N has property T. Then by Theorem 5,
 there exists a finite subset $\{u_1, \dots, u_n\}$ of N such that,
 for any correspondence H from N to N and a sequence (ξ_k)
 of unit vectors in H , if $\|x_j \xi_k - \xi_k x_j\| \rightarrow 0$ for all j ,
 then (ξ_k) must tend to some central vector when $k \rightarrow \infty$.
 We may assume the set is a set of unitaries in N which contains
 the identity 1 and $*$ -closed. Let take a correspondence H
 from N to N with non zero central vectors. Put

$$a = \sum_{j=1}^n \mathcal{F}(u_j),$$

then a is an self adjoint operator in $C^*(\mathcal{F}(U(N)))$ such that
 $a\xi = n\xi$ for all central vectors $\xi \in H$. Let $K = H \ominus P_c(H)$.
 If the restriction of $a - n1$ to K is not invertible, then
 there exists a sequence (ξ_k) of unit vectors in K such that
 $\|a\xi_k - n\xi_k\| \rightarrow 0$ when $k \rightarrow \infty$. Since

$$| \|a\xi_k\| - n | \leq \|a\xi_k - n\xi_k\| \xrightarrow[k \rightarrow \infty]{} 0,$$

$$\|\mathcal{F}(u_j)\xi_k\| = 1 \quad \text{and} \quad \|a\xi_k\| \leq n \quad \text{for all } k, j,$$

we have that

$$\|u_j \xi_k - \xi_k u_j\| = \|\mathcal{F}(u_j)\xi_k - \mathcal{F}(1)\xi_k\| \xrightarrow[k \rightarrow \infty]{} 0 \quad (1 \leq j \leq n).$$

Hence the sequence (ξ_k) of unit vectors in K must converge to
 a vector in $P_c(H)$. This is a contradiction. Therefore the proper

value n of a is an isolated point of spectrum of a . Hence the projection P_c is contained in the C^* -subalgebra of $C^*(\mathcal{F}(U(N)))$ generated by a .

Conversely, assume that N does not have property T. Then there is a net $\{H_\lambda, \xi_\lambda\}$ of correspondences H_λ without non zero central vectors and an $\xi_\lambda \in H_\lambda$ such that $\|\xi_\lambda\| = 1$ and $\|x_j \xi_\lambda - \xi_\lambda x_j\| \rightarrow 0$ for every finite subset $\{x_1, \dots, x_n\}$ of N . Let H be the direct sum of $L^2(N, \text{tr})$ and the direct sum of all H 's. Let ξ_λ be the vector in H such that the λ -component is only one non zero vector ξ_λ . Let N act on H from left and right sides by the diagonal form. Then H is a correspondence from N to N with the central vectors $\mathbb{C}1$ in $L^2(N, \text{tr})$. The projection P_c is the 1-dimensional projection on 1. If P_c is contained in $C^*(\mathcal{F}(U(N)))$, then for any $\varepsilon > 0$, there are a finite set of unitaries $\{u_1, \dots, u_n\}$ and complex numbers $\{c_1, \dots, c_n\}$ such that

$$\|P_c - \sum_j c_j \mathcal{F}(u_j)\| < \varepsilon.$$

Then

$$| |\sum_{j=1}^n c_j| - 1 | \leq \|(\sum_j c_j \mathcal{F}(u_j))(1) - P_c(1)\| < \varepsilon$$

and

$$\begin{aligned} | \sum_{j=1}^n c_j | &= \| \sum_j c_j \xi_\lambda \| \\ &\leq \| \sum_j c_j \xi_\lambda - \sum_j c_j \mathcal{F}(u_j) \xi_\lambda \| + \| \sum_j c_j \mathcal{F}(u_j) \xi_\lambda - P_c \xi_\lambda \| \\ &\leq 2\varepsilon \end{aligned}$$

for sufficiently large λ . This is a contradiction.

3. Two strongly ergodic actions of $Sp(n, \mathbb{Z})$ on R .

Here I shall give an answer for second question in §1.

Non compact groups with property T are non amenable. Special linear groups with the orders larger than 3 are typical groups with property T. In [3], we showed the existence of two strongly ergodic outer actions of $SL(n, \mathbb{Z})$ on the hyperfinite II_1 -factor R , one of which construct a II_1 -factor with property T and the other gives a II_1 -factor without property T. Another examples of groups with property T is symplectic groups. In this place, I shall give two strongly ergodic actions of $Sp(n, \mathbb{Z})$ on R . The group $SL(n, \mathbb{Z})$ can be naturally imbedded into the group $Sp(n, \mathbb{Z})$. Then under the inclusion, the restricted actions of $Sp(n, \mathbb{Z})$ to $SL(n, \mathbb{Z})$ coincides with the actions in [3].

Let take and fix an integer n . Let $B(a, b)$ be the symplectic form of a and b in the vector space $\mathbb{Z}^{2n}$ ($\mathbb{Z}$ is the set of all integers) :

$$B(a, b) = \sum_{i=1}^n a_i b_{n+i} - \sum_{i=1}^n a_{n+i} b_i,$$

where $a = (a_1, \dots, a_n, a_{n+1}, \dots, a_{2n}) \in \mathbb{Z}^{2n}$.

Let consider two groups

$$K = Sp(n, \mathbb{R}) \times_S \mathbb{R}^{2n} \quad \text{and} \quad G = Sp(n, \mathbb{Z}) \times_S \mathbb{Z}^{2n};$$

the semidirect product, the product of whose elements are defined by

$$(S, a)(T, b) = (ST, T^{-1}(a) + b), \quad (S, T \in Sp(n, \mathbb{R}), a, b \in \mathbb{R}^{2n}),$$

where the action of $Sp(n, \mathbb{R})$ on $\mathbb{R}^{2n}$ is defined by the obvious manner.

Lemma 6. The groups K and G have property T.

Proof. In [12], it is proved that the semidirect product $SL(n, R) \times_s R^n$ has property T, where the product is defined by the same manner as K. The proof is due to the following :

The semidirect product $\Gamma \times_s A$, of a locally compact group A by a locally compact group Γ with property T under an action α , has property T if

- (i) Γ has a subgroup L such that for every continuous unitary representation u of Γ , there is a non zero fixed point under u when the restriction u to L has one and that A is generated by the set $\{ a \in A; \text{ there is a } g \in \Gamma \text{ with } \alpha_g(a) = \alpha_{hg}(a) \text{ for all } h \in L \}$,
- (ii) the group generated by $\{ \alpha_g(a)a^{-1} ; a \in A, g \in \Gamma \}$ is dense in A.

In the case of $\Gamma = Sp(n, R)$, let L be the subgroup of matrices $\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}$, where E is the identity n by n matrix

and B is an n by n matrix with $B = {}^t B$ (the transposed matrix of B). Then L satisfies the conditions in (i) ([6]).

The group $Sp(n, R)$ contains the matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and $\begin{pmatrix} D & 0 \\ 0 & {}^t D^{-1} \end{pmatrix}$.

Hence the natural action of $Sp(n, R)$ on R^{2n} satisfies the conditions in (i) and (ii). Therefore K has property T. The group G has a finite covolume in K, so that G has property T.

Let take an irrational s in $[0, \pi/2] \bmod. 2\pi$. Let

$$\mu_s((S, a), (T, b)) = \exp(isB(T^{-1}a, b)),$$

for $S, T \in Sp(n, Z)$ and $a, b \in Z^{2n}$. Then μ_s is a normalized 2-cocycle of $G \times G$ to the torus T :

$$(1) \quad \mu_s(g, h) = 1$$

$$(2) \quad \mu_s(1, g) = \mu_s(h, 1) = \mu_s(g, g) = 1$$

$$(3) \quad \mu_s(f, h) \mu_s(fh, k) = \mu_s(h, k) \mu_s(f, hk)$$

for all f, g, h, k in G . In fact, it is clear that the condition (2) is satisfied because $B(., .)$ is the symplectic form. The relation (3) is proved by the following :

$$\begin{aligned} & \mu_s((S, a), (Tb)) \mu_s((S, a)(T, b), (S', c)) \\ &= \exp(i s (B(T^{-1}(a), b) + B(T^{-1}(a), S'(c)) + B(b, S'(c)))) \\ &= \mu_s((T, b), (S', c)) \mu_s((S, a), (T, b)(S', c)). \end{aligned}$$

The left μ_s -regular representation of G on $l^2(G)$ is defined by the following :

$$(\lambda^s(g) \xi)(h) = \mu_s(h^{-1}, g) \xi(g^{-1}h), \quad (g, h \in G, \xi \in l^2(G)).$$

Then λ^s is a unitary representation of G with the cocycle μ_s , that is,

$$\lambda^s(g) \lambda^s(h) = \mu_s(g, h) \lambda^s(gh), \quad (g, h \in G).$$

Lemma 7. The von Neumann algebra R_s generated by $\lambda^s(1, Z^{2n})$ is a hyperfinite II_1 -factor.

Proof. Let $\{e_j ; 1 \leq j \leq 2n\}$ be the natural basis in Z^{2n} . Then the unitaries $\{\lambda^s(e_j) ; 1 \leq j \leq 2n\}$ satisfies

$$\{\lambda^s(e_j) ; 1 \leq j \leq n\} \text{ are commutative,}$$

$$\{\lambda^s(e_{n+j}) ; 1 \leq j \leq n\} \text{ are commutative}$$

$$\lambda^s(e_j) \lambda^s(e_{n+k}) = \exp(\delta_{j,k} 2is) \lambda^s(e_{n+k}) \lambda^s(e_j),$$

for $j, k = 1, \dots, n$. Let A_j be the von Neumann algebra generated

by $\{\lambda^s(e_j), \lambda^s(e_{n+j})\}$. Then A_j is a hyperfinite II_1 -factor because s is irrational. Since A_j and A_k commute for $j \neq k$, the algebra R_s is a hyperfinite II_1 -factor.

Next we shall define an action α^s of $Sp(n, \mathbb{Z})$ on R_s by

$$\alpha^s(T)(a) = \lambda^s(T, 0) a \lambda^s(T, 0)^* \quad (a \in R_s).$$

Lemma 8. The action α^s is an ergodic outer action of $Sp(n, \mathbb{Z})$ on the hyperfinite II_1 -factor R_s .

Proof. BY the definition,

$$\alpha^s(T)(\lambda^s(1, a)) = \lambda^s(1, T(a)) \quad (T \in Sp(n, \mathbb{Z}), a \in \mathbb{Z}^{2n}).$$

Hence α^s is an action of $Sp(n, \mathbb{Z})$ on R_s . The ergodicity and outeriness of the action α^s is proved by a similar method as in [3].

Lemma 9. The von Neumann algebra $N(s, G)$ generated by $\lambda^s(G)$ is isomorphic to the crossed product $W^*(Sp(n, \mathbb{Z}), R_s, \alpha^s)$.

Proof. The algebra is generated by R_s and the unitary group $\lambda^s(Sp(n, \mathbb{Z}), 0)$. Since $N(s, G)$ is finite and the action $\text{Ad } \lambda^s(T, 0)$ is outer on R_s for all $T \in Sp(n, \mathbb{Z})$, $N(s, G)$ is isomorphic to $W^*(Sp(n, \mathbb{Z}), R_s, \alpha^s)$ by [13].

Lemma 10. The crossed product $W^*(Sp(n, \mathbb{Z}), R_s, \alpha^s)$ is a type II_1 -factor with property T.

Proof. It is clear that $W^*(Sp(n, \mathbb{Z}), R_s, \alpha^s)$ is a II_1 -factor. By Lemma 6, the group G has property T. Hence the group

von Neumann algebra $N(s, G)$ defined by the normalized 2-cocycle $\mu_s(.,.)$ has property T. Therefore by Lemma 9, we have Lemma 10.

Let j be the imbedding of $GL(n, R)$ into $Sp(n, R)$ defined by

$$j(T) = \begin{pmatrix} T & 0 \\ 0 & t_T^{-1} \end{pmatrix} \quad (T \in GL(n, R))$$

Put $\gamma^s = \alpha^s \circ j$, then γ^s is an action of $GL(n, Z)$ on the hyperfinite II_1 -factor R_s . This action γ^s is the action α^s of $GL(n, Z)$ on the hyperfinite II_1 -factor R , which is discussed in the previous paper [3]. In [3], we proved that the crossed product $W^*(SL(n, Z), R, \gamma^s)$ is a II_1 -factor with property T. Now we have another proof of it as follows.

Lemma 11. The sedirect products $j(SL(n, R)) \times_s R^{2n}$ and $j(SL(n, Z)) \times_s Z^{2n}$ have property T for $n \geq 3$.

Proof. Let consider the subgroup $\{j(T) ; T = \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix}, x \in R^{n-1}\}$. Then by a similar way as the proof of Lemma 6, it is proved that these two groups have property T.

Lemma 12. The crossed product $W^*(SL(n, Z), R, \gamma^s)$ has property T for $n \geq 3$.

Proof. The algebra $W^*(SL(n, Z), R, \gamma^s)$ is isomorphic to the algebra $(\lambda^s(j(SL(n, Z)) \times_s Z^{2n}))''$, which has property T by Lemma 11.

Let G be a countable group. Let A be a finite von Neumann algebra with a faithful normal normalized trace tr . Let M be the infinite tensor product $\bigotimes_{g \in G} (A_g, tr_g)$, where $A_g = A$ and $tr_g = tr$ for all $g \in G$. Let j_g be the canonical injection of A to $\dots \otimes 1 \otimes A_g \otimes 1 \otimes \dots \subset M$. The Bernoulli shift action of G for

(A, tr) is the action β of G on M such that $\beta_g(j_h(x)) = j_{gh}(x)$ for all $x \in A$. Then in [3], we proved that for every group G the Bernoulli shift action β constructs by the crossed product a von Neumann algebra which does not have property T. Hence we have the following :

Theorem 13. The symplectic group $Sp(n, \mathbb{Z})$ and the special linear group $SL(n, \mathbb{Z})$ ($n \geq 3$) have two kind of ergodic outer actions on the hyperfinite II_1 -factor R , one of which constructs a II_1 -factor with property T and the other gives a full II_1 -factor without property T.

In the last I denote some remarks. A II_1 -factor N is full if N does not have property $\mathcal{T}$ of Murray and von Neumann. Then the crossed products $W^*(GL(n, \mathbb{Z}), R, \gamma^S)$, $W^*(GL(n, \mathbb{Z}), R, \beta)$, $W^*(F_2, R, \gamma^S)$ and $W^*(F_2, R, \beta)$ are all full II_1 -factors.

Let α (resp. β) be an action of a group G on a von Neumann algebra A (resp. B). Then α and β are conjugate if there exists an isomorphism θ of A onto B such that $\beta_g \theta = \theta \alpha_g$ for all $g \in G$. The actions $\{\alpha^S\}$ are non conjugate actions of $Sp(n, \mathbb{Z})$ on the hyperfinite II_1 -factor R .

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