

Hysteretic Finite Element Analysis Using a Semi-implicit Method Combined with the Predictor–Corrector Scheme

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A semi-implicit (SI) method is proposed as a new time-marching scheme for transient finite element analysis, in which the magnetic hysteresis characteristics are considered at each time step. The magnetic field is updated based on a linear approximation to avoid iterations, and the differential reluctivity is corrected based on the predictor–corrector scheme to improve and evaluate the accuracy. Moreover, we investigate a variable time-step SI method, where the local error is estimated from the predictor–corrector difference, and the time step is adjusted to maintain the error below the prescribed error tolerance.

Index Terms—eddy current field, finite element analysis, magnetic hysteresis, play model, predictor–corrector scheme.

I. INTRODUCTION

A precise model that represents the magnetic hysteresis should be employed to accurately estimate the performance of electric machines. In particular, empirical models find it difficult to accurately evaluate the minor loops when the input waveform is distorted by pulse-width modulation (PWM). Therefore, in finite element (FE) analysis, the magnetic characteristics of the iron core should be determined at each FE and time step using a precise hysteresis model. Phenomenological models, such as the Jiles–Atherton (J–A) model [1], Preisach model, and play model [2] [3], are often used in transient FE analysis. The play model is mathematically equivalent to, but more computationally efficient than, the Preisach model [2], which models minor loops more accurately than the J–A model [4].

To solve the time evolution in FE analysis, an implicit time-marching scheme is generally adopted to achieve numerical stability. The Newton–Raphson (NR) method is a widely employed iteration method; however, it can suffer from poor convergence owing to the discontinuous derivative of the hysteretic curve, particularly when the Preisach or play model is used.

To avoid iterative computations, we propose a semi-implicit (SI) method, in which the magnetic field is computed using a linear approximation. Although the linear approximation limits the accuracy of the SI method, its combination with the predictor–corrector (PC) scheme can address this problem. The PC method was developed to improve accuracy, evaluate the error, and adjust the time step of ordinary differential equations [5] [6]. By employing the PC scheme for hysteresis magnetic field analysis, this study aims not only to improve the accuracy but also to estimate the error from the predictor–corrector difference and construct a variable time-step method. The formulations introduced in this paper are expected to be applicable to both 2D and 3D problems; 2D analysis results of an inductor are presented to validate the method.

II. SEMI-IMPLICIT METHOD COMBINED WITH THE PREDICTOR–CORRECTOR SCHEME

A. Solution of weak formulation

In this study, the eddy current problem is solved using FE analysis. The implicit Euler method is commonly employed to solve weak formulations as follows:

$$F(\mathbf{A}_{m+1}) \approx (\mathbf{H}_{m+1}, \nabla \times \mathbf{w})_{\Omega} + \left(\mathbf{w}, \sigma \frac{\mathbf{A}_{m+1} - \mathbf{A}_m}{\Delta t} \right)_{\Omega_1} - (\mathbf{w}, \mathbf{J}_{s,m+1})_{\Omega} = 0, \quad (1)$$

where $(\mathbf{u}, \mathbf{v})_{\Omega} = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\Omega$ and $\mathbf{H}$, $\mathbf{A}$, and $\mathbf{J}_s$ represent the magnetic field, magnetic vector potential, and source current density vectors, respectively, defined in the FE analysis domain Ω . Subscript m denotes the vector of the m -th time step. Conductivity σ is defined in the conductor domain Ω_1 , $\mathbf{w}$ represents the edge basis function, and Δt indicates the time step. The magnetic field vector $\mathbf{H}_{m+1}$ is determined from the magnetic flux density vector $\mathbf{B}_{m+1} = \nabla \times \mathbf{A}_{m+1}$ based on the hysteretic magnetic characteristics of the material. In this study, the vector play model [3] is used to determine $\mathbf{H}$.

The NR method frequently exhibits poor convergence when the differential reluctivity matrix $\nu_{d,m} = \nu_d(\mathbf{B}_m) = \partial \mathbf{H}_m / \partial \mathbf{B}_m$ is discontinuous. To solve this problem, we introduce a linear approximation of the magnetic field

$$\mathbf{H}_{m+1} \approx \mathbf{H}_m + \nu_{d,m} (\mathbf{B}_{m+1} - \mathbf{B}_m). \quad (2)$$

By substituting (2) into (1), we obtain:

$$F_{SI}(\mathbf{A}_{m+1}) = (\mathbf{H}_m + \nu_{d,m} (\mathbf{B}_{m+1} - \mathbf{B}_m), \nabla \times \mathbf{w})_{\Omega} + \left(\mathbf{w}, \sigma \frac{\mathbf{A}_{m+1} - \mathbf{A}_m}{\Delta t} \right)_{\Omega_1} - (\mathbf{w}, \mathbf{J}_{s,m+1})_{\Omega} = 0. \quad (3)$$

The SI method is defined as a time-marching scheme whose formulation is given by (3). This is a linear equation for $\mathbf{A}_{m+1}$. Hence, iterations for nonlinear solutions are not required.

As (3) employs a linear approximation, its accuracy is expected to be insufficient; hence, we introduce the PC scheme to improve and evaluate the accuracy. The PC scheme is a multi-step algorithm. First, an integration is executed using an explicit method to obtain a prediction, based on which

a nonlinear function is updated, and then a more accurate solution is obtained using an implicit method.

We propose a three-step procedure, as shown in Fig. 1, for combining the PC scheme with the SI method. Detailed descriptions are provided below.

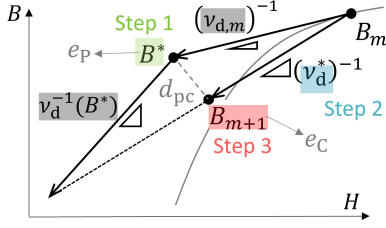


Fig. 1. Procedure of the SI method combined with the PC scheme.

Step 1: The prediction of $\mathbf{A}$ at the $(m+1)$ -th time step is derived from (3) as a predictor $\mathbf{A}^*$:

$$F_{SI}(\mathbf{A}^*) = (\mathbf{H}_m + \nu_{d,m}(\mathbf{B}^* - \mathbf{B}_m), \nabla \times \mathbf{w})_{\Omega} + \left(\mathbf{w}, \sigma \frac{\mathbf{A}^* - \mathbf{A}_m}{\Delta t} \right)_{\Omega_1} - (\mathbf{w}, \mathbf{J}_{s,m+1})_{\Omega} = 0, \quad (4)$$

where $\mathbf{B}^* = \nabla \times \mathbf{A}^*$.

Step 2: ν_d is corrected based on the predictor as

$$\nu_d^* = 0.5 \times (\nu_{d,m} + \nu_d(\mathbf{B}^*)). \quad (5)$$

Step 3: Equation (3) is solved again using the updated reluctivity ν_d^* as

$$F_{SI}(\mathbf{A}_{m+1}) = (\mathbf{H}_m + \nu_d^*(\mathbf{B}_{m+1} - \mathbf{B}_m), \nabla \times \mathbf{w})_{\Omega} + \left(\mathbf{w}, \sigma \frac{\mathbf{A}_{m+1} - \mathbf{A}_m}{\Delta t} \right)_{\Omega_1} - (\mathbf{w}, \mathbf{J}_{s,m+1})_{\Omega} = 0. \quad (6)$$

The solution to (6) is defined as the corrector.

B. Numerical Results with a fixed time step

The 2D wound-core inductor shown in Fig. 2(a) was analyzed. The iron core contained ten steel sheets. Their hysteretic characteristics were computed using the vector play model [3] identified by the measurement data of the non-oriented silicon steel sheet JIS 50A470. The created mesh shown in Fig. 2(b) consists of 10486 triangular elements. The linear triangular element was used for the vector potential with its degree of freedom being 5337.

To evaluate the error of the SI method, we analyzed a problem with good convergence and used $\mathbf{B}_{NR} = \nabla \times \mathbf{A}_{NR}$ as the reference solution. Here, $\mathbf{A}_{NR}$ denotes the solution to (1), which was solved iteratively using the NR method.

1) Sinusoidal input

A cosine voltage input with an amplitude of 1.5 V and frequency of 50 Hz, as shown in Fig. 3(a), was applied to the coil. One period (0–0.02 s) was analyzed, and the number of time steps was 120.

Fig 4(a) shows H_y – B_y locus at probe A (see Fig. 2(a)). The locus of the SI method, which is the corrector obtained from (6), conforms with that of the NR method. The magnetic flux density at the probe is plotted in Fig. 3(b), which is also in good agreement.

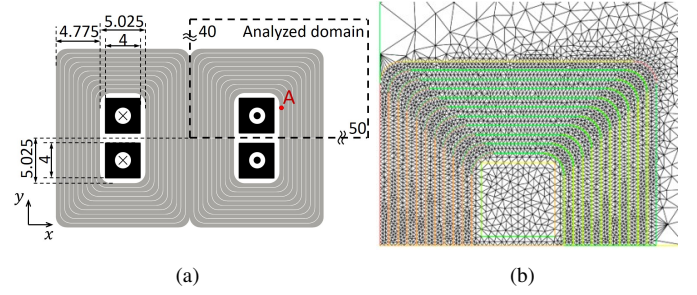


Fig. 2. (a) Iron-core inductor used in the simulation. Point A indicates the probe for obtaining data from the local field. The unit of the dimension is mm. (b) The mesh created in (a).

Fig. 3(c) shows the differences among $\mathbf{B}^*$, $\mathbf{B}_{m+1}$, and $\mathbf{B}_{NR}$, where $\mathbf{B}^*$ and $\mathbf{B}_{m+1}$ are obtained from (4) and (6), respectively. The dotted line indicates the PC difference d_{PC} , which is the difference between the predictor and corrector, calculated as $d_{PC} = \text{ave}(|\mathbf{B}^* - \mathbf{B}_{m+1}|)$. The function ave is defined as $\text{ave}(\mathbf{x}) = \sum_i x_i / N_f$, where N_f and x_i indicate the number of finite elements and value of the i -th element in the iron core region, respectively. The black and red solid lines correspond to the errors of the predictor $e_p = \text{ave}(|\mathbf{B}^* - \mathbf{B}_{NR,m+1}|)$ and corrector $e_c = \text{ave}(|\mathbf{B}_{m+1} - \mathbf{B}_{NR,m+1}|)$, respectively. The PC scheme effectively reduces the error, as evidenced by the lower value of e_c compared with e_p . The relationship $d_{PC} \approx e_c$ is generally satisfied, indicating that e_c can be estimated approximately by observing d_{PC} . As a practical application of this finding, we explore a variable time-step approach in Section III.

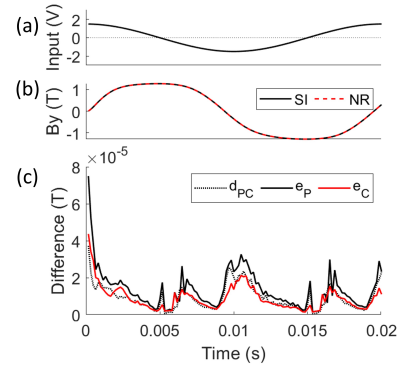


Fig. 3. Results of the SI method with a sinusoidal input voltage, which is plotted in (a). (b) The magnetic flux density at the probe. (c) PC difference d_{PC} and errors, e_p and e_c .

2) PWM input

Subsequently, a PWM voltage input, shown in Fig. 5(a), was tested. The amplitude is 3 V, and the fundamental and carrier frequencies are 50 Hz and 2.25 kHz, respectively.

Fig. 4(b) shows the H_y – B_y locus at the probe. The SI method accurately reproduces the minor loops, which is also confirmed in the waveform of B_y , shown in Fig. 5(b).

Error evaluation is presented in Fig. 5(c). When the input voltage is switched, the differential reluctivity is discontinuous, resulting in an inaccurate prediction and an increase in e_p . However, in such instances, the modification provided

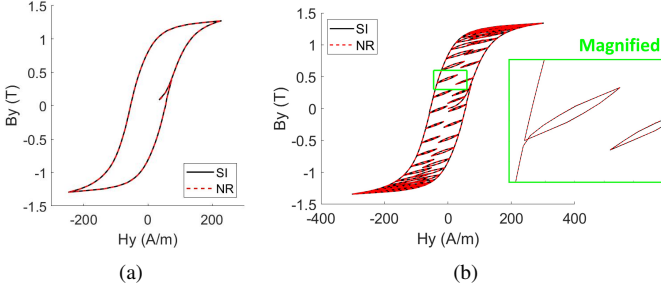


Fig. 4. H_y - B_y loci of the SI and NR method with (a) sinusoidal and (b) PWM inputs. A magnified figure is shown in (b).

by (5) is more effective, as indicated by an increase in d_{PC} . Consequently, e_C is reduced effectively.

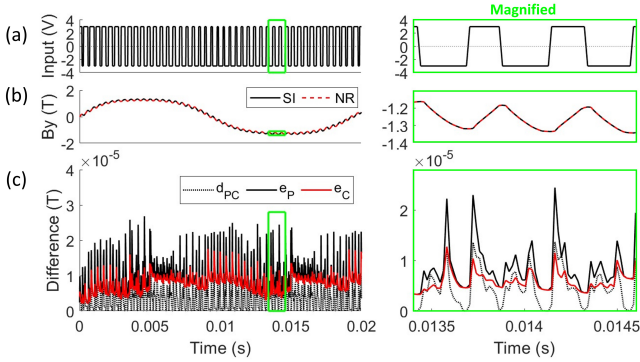


Fig. 5. Results of the SI method with the PWM input shown in (a). Magnified figures are shown on the right. The plotted parameters in (b) and (c) are the same as those in Fig. 3.

3) Comparison of the computation time

The computation was performed using the finite element software FreeFem++ [7] on a 12th Gen Intel(R) Core(TM) i7-12700 2.10 GHz. For the PWM input with the same number of time steps, the computation times of the SI and NR methods were 8880 s and 20650 s, respectively. The speed-up ratio of the proposed method was 2.33.

III. VARIABLE-STEP SEMI-IMPLICIT METHOD

In the previous section, it was observed that $d_{PC} \approx e_C$ is generally satisfied. We exploit it to propose a method for controlling e_C by adjusting Δt and maintaining d_{PC} below a certain value. This method is referred to as the variable-step semi-implicit (VSSI) method.

A. Step size determination

Suppose that B^* in (4) is a solution of the implicit Euler method and B_{m+1} in (6) is significantly closer to the true value than B^* . Then, the PC difference $d_{PC,m} = \text{ave}(|B^* - B_{m+1}|)$ represents the local discretization error of the implicit Euler method at the $(m+1)$ -th step. The local discretization error of the implicit Euler method is $O(\Delta t^2)$ for ordinary differential equations [6]. If the same error evaluation is valid for the SI method, $d_{PC,m} \approx K_m(\Delta t_m)^2$ should be satisfied, where Δt_m denotes the step size between the m -th

and $(m+1)$ -th steps, and K_m is a constant independent of the step size.

Let e_{tol} be the error tolerance that is determined in advance. Moreover, we assume that the step size is sufficiently small, such that $K_m \approx K_{m+1}$ is satisfied. Subsequently, $d_{PC,m+1} \leq e_{tol}$ is expected to be satisfied when the step size is determined by

$$\Delta t_{m+1} = r_m \Delta t_m, \quad (7)$$

where $r_m = \alpha \sqrt{e_{tol}/d_{PC,m}}$. The constant $0 < \alpha < 1$ is an empirically determined safety factor, typically chosen as approximately 0.8 [6].

The VSSI method is performed using the following procedure with an initial step size Δt_0 .

Step 1: Equations (4)–(6) are solved using the step size Δt_m to obtain B^* and B_{m+1} .

Step 2: $d_{PC,m}$ and r_m are computed.

Step 3: If $d_{PC,m} > e_{tol}$, Δt_m is updated as $\Delta t_m \leftarrow r_m \Delta t_m$, and Steps 1, 2 are repeated.

Step 4: When $d_{PC,m} \leq e_{tol}$ is satisfied, Δt_{m+1} is determined by (7) and the computation is repeated from Step 1 with $m \leftarrow m+1$.

We introduce $n_{PC,m}$ as the number of executions in Steps 1 and 2. When the step size is appropriately adjusted, Steps 1 and 2 need to be implemented only once, and $n_{PC,m}$ should be one.

B. Numerical results of VSSI method

We employed the safety factor $\alpha = 0.8$ and initial step size $\Delta t_0 = 1.0 \times 10^{-6}$ s. To prevent a sudden increase in the step size, which can lead to $n_{PC,m} > 1$, we introduced an upper limit on the coefficient r_m as $r_m = \min(2, \alpha \sqrt{e_{tol}/d_{PC,m}})$. Moreover, to tune the timing of the sign changes in the input voltage, the step size was decreased as $\Delta t_m \leftarrow 0.5 \Delta t_m$ when $E_m \cdot E_{m+1} < 0$ was satisfied, where E_m indicates the input voltage at the m -th step.

1) Sinusoidal input

We assume the same sinusoidal voltage input shown in Fig. 3(a), and $e_{tol} = 1.0 \times 10^{-6}$ T was used for the error tolerance.

Fig. 6(a) shows B_y at the probe, where the result of the VSSI method conforms with that of the NR method. Figs. 6(b), (c), and (d) show the PC difference, step size, and number of PC computations, respectively. The condition $d_{PC} \leq e_{tol}$ is satisfied and $n_{PC,m} = 1$ is observed for 97.0% of the steps, indicating that the step size was appropriately adjusted by (7).

To evaluate the error in the VSSI method, we introduce $e_B = |B_{SI} - B_{NR}|$, where B_{SI} and B_{NR} denote the magnetic flux densities at the probe obtained using the VSSI and NR methods, respectively. The step size of the NR method is 2.5×10^{-6} s, which is approximately equal to the minimum step size of the VSSI method and is sufficiently small to use the result as a reference solution.

Fig. 6(e) shows the time evolution of e_B , which is maximized when the change in B is almost the largest. The VSSI method was also tested with $e_{tol} = 1.0 \times 10^{-5}$ and 10^{-7} T, and the relationship between the global error

$e_{\text{global}} = \max(e_B)$ and e_{tol} is shown in Fig. 7(a). The dashed line shows a fitted line whose coefficients were determined using the least-squares method. Its slope is 0.49, indicating that $e_{\text{global}} \propto \sqrt{e_{\text{tol}}}$ is approximately satisfied. The result indicates that the global error can be controlled using e_{tol} .

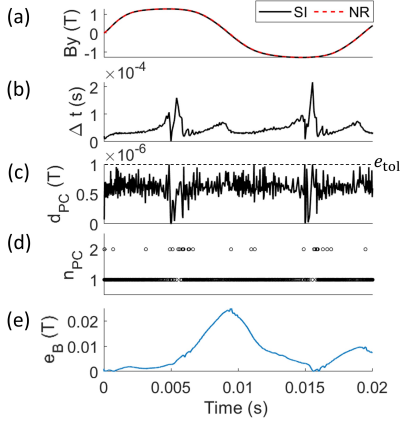


Fig. 6. Results of the VSSI method for a sinusoidal input. (a) The magnetic flux density B_y at the probe. (b) Step size Δt . (c) PC difference d_{PC} . (d) Number of PC computations n_{PC} . (e) The error of the magnetic flux density e_B .

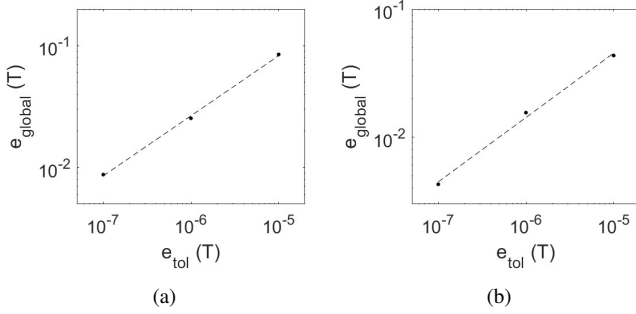


Fig. 7. Relationship between e_{tol} and the global error e_{global} for (a) sinusoidal and (b) PWM inputs. The dashed lines are fitted using the least-squares method.

2) PWM input

We used the PWM voltage input shown in Fig. 5(a). The result with $e_{\text{tol}} = 1.0 \times 10^{-6}$ T is shown in Fig. 8. Similar to the case for a sinusoidal input, the time step was adjusted such that $d_{\text{PC}} \leq e_{\text{tol}}$ was satisfied and $n_{\text{PC},m}$ was one for 93.4 % of the steps. The error e_B shown in Fig. 8(e) was also computed using $e_{\text{tol}} = 1.0 \times 10^{-5}$ and 10^{-7} T, and the relationship between e_{tol} and e_{global} is depicted in Fig. 7(b). The slope of the fitted line is 0.50, which means $e_{\text{global}} \propto \sqrt{e_{\text{tol}}}$ also holds for the PWM input.

IV. CONCLUSION

An SI method combined with a PC scheme was proposed for the transient FE analysis of eddy-current fields. The magnetic hysteresis characteristics were considered at each time step. A linear approximation was introduced to solve the time evolution of the weak formulation without an iterative procedure, and the differential reluctivity matrix was corrected to evaluate

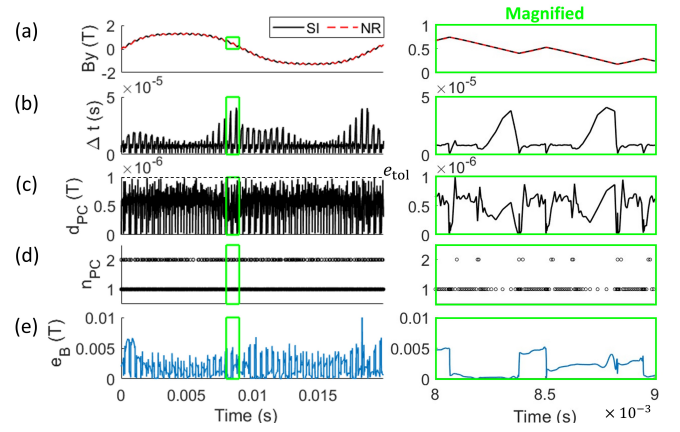


Fig. 8. Results of the VSSI method with a PWM input. The same parameters are plotted as those shown in Fig. 6. Magnified figures are shown on the right.

and improve the accuracy. In the numerical analyses using a two-dimensional inductor model, the BH loci of the SI method conformed with those of the NR method. The results also indicated that the error in the SI method can be evaluated based on the PC difference. The proposed SI method reduced the computation time by a factor of 2.33 compared to the NR method with the same number of timesteps.

Furthermore, we investigated a variable time-step method in which the step size was determined from the PC difference. The global error was controlled by the prescribed error tolerance.

The proposed method will be applied to motor analysis in future studies.

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REFERENCES

- [1] J. Gyselinck, P. Dular, N. Sadowski, J. Leite, and J. P. A. Bastos, "Incorporation of a Jiles-Atherton vector hysteresis model in 2D FE magnetic field computations: Application of the Newton-Raphson method," *COMPEL - The International Journal for Computation and Mathematics in Electrical and Electronic Engineering*, vol. 23, no. 3, pp. 685–693, 2004.
- [2] S. Bobbio, G. Miano, C. Serpico, and C. Visone, "Models of magnetic hysteresis based on play and stop hysterons," *IEEE Trans. Magn.*, vol. 33, no. 6, pp. 4417–4426, 1997.
- [3] R. Mitsuoka, T. Mifune, T. Matsuo, and C. Kaido, "A vector play model for finite-element eddy-current analysis using the Newton-Raphson method," *IEEE Trans. Magn.*, vol. 49, no. 5, pp. 1689–1692, 2013.
- [4] A. Benabou, S. Clénet, and F. Piriou, "Comparison of Preisach and Jiles-Atherton models to take into account hysteresis phenomenon for finite element analysis," *J. Magn. Magn. Mater.*, vol. 261, no. 1, pp. 139–160, 2003.
- [5] E. Hairer, G. Wanner, and S. P. Nørsett, "Solving Ordinary Differential Equations I," vol. 8. Berlin, Heidelberg: Springer, 1993.
- [6] J. C. Butcher, "Numerical methods for ordinary differential equations," John Wiley & Sons, 2016.
- [7] F. Hecht, "New development in freefem++,", *J. Numer. Math.*, vol. 20, no. 3–4, pp. 251–266, 2012.